ON SUBHARMONIC FUNCTIONS IN STRIPS

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1. Introduction

Various properties of subharmonic functions on a strip \( R^n \times (0, 1) \) have been studied by many authors (see Hardy, Ingham and Pólya [17] for \( n=1 \); Brawn [6–12], Armitage [3, 4], Armitage and Fugard [5] for \( n \geq 1 \)). Recently Yoshida [26–28] extended some of them for subharmonic functions on a cylinder \( R^1 \times D \), where \( D \) is a smooth bounded domain in \( R^m \). The purpose of this paper is to consider several properties of subharmonic functions on a (generalized) strip \( R^n \times D \) from a point of view different from Yoshida's. We shall argue especially the Phragmén-Lindelöf principle, the harmonic majorization and the properties of hyperplane mean values of subharmonic functions. In case \( m=1 \) every bounded domain in \( R^n \) is similar to \( (0, 1) \). However, there arises a regularity problem for bounded domains \( D \) in \( R^n \) if \( m>1 \). In this paper we let \( D \) be a bounded Lipschitz domain if \( m>1 \).

We denote by \( P=(X, Y) \) a point in \( R^{n+m}=R^n \times R^m \), where \( X=(x_1, \ldots, x_n) \in R^n \) and \( Y=(y_1, \ldots, y_m) \in R^m \). We write \(|P|, |X| \) and \(|Y| \) for \( (\sum_{j=1}^{n} x_j^2+\sum_{j=1}^{m} y_j^2)^{1/2} \), \( (\sum_{j=1}^{n} x_j^2)^{1/2} \) and \( (\sum_{j=1}^{m} y_j^2)^{1/2} \), respectively. We identify \( R^n \) and \( R^m \) with \( \{(X, Y) ; Y=0\} \) and \( \{(X, Y) ; X=0\} \), respectively. We denote by \( S^{n-1} \) the unit sphere \( \{x \in R^n ; |x|=1\} \) with center at the origin in \( R^n \), and by \( \tau \) the surface measure on \( S^{n-1} \). We write briefly \( L_D \) for a strip \( R^n \times D \).

Let \( s \) be a subharmonic function on \( L_D \). If

\[
\lim_{P \to Q, Q \in \partial L_D} \sup_{P \in L_D} s(P) = 0
\]

then we say that \( s \) satisfies the Phragmén-Lindelöf boundary condition. We denote by \( s^+ \) the positive part of \( s \). In view of the behavior at \( \infty \) of the Bessel function \( I_{n/2-1} \) of the third kind of order \( n/2-1 \),

\[
I_{n/2-1}(r) \sim r^{-1/2} e^r \quad \text{as} \quad r \to \infty,
\]

([23; p. 203]), Brawn's result [7; Theorem 2, Corollary] may read as follows:

**Theorem A.** Let \( m=1 \) and \( D=(0, 1) \). If \( s \) is a subharmonic function in \( L_D \)

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satisfying (PL) such that

\[ \liminf_{r \to \infty} r^{(n-1)/2} e^{-\pi r} \int_{S^{n-1}} \int_0^1 s^+(r, y) \sin(\pi y) \, dy \, d\tau(a) = 0, \]

then \( s \leq 0 \) in \( L_D \).

We note that the function \( \sin(\pi y) \) appearing in (2) is related to the eigenvalue problem

\[ (\Lambda_Y + \mu) f = 0 \quad \text{on } D, \]
\[ f = 0 \quad \text{on } \partial D, \]

where \( \Lambda_Y = \sum_{j=1}^m \partial^2/\partial y_j^2 \). If \( m = 1 \) and \( D = (0, 1) \), then the constant \( \pi^2 \) and the function \( \sin(\pi y) \) are the first positive eigenvalue of (3) and its positive eigenfunction. In general we let \( \lambda_D \) be the square root of the first positive eigenvalue of (3) and let \( f_D \) be its positive eigenfunction. In Section 3 we shall prove the following generalization of Theorem A:

**Theorem 1.** Let \( m \geq 1 \) and \( D \subseteq \mathbb{R}^m \). If \( s \) is a subharmonic function on \( L_D \) satisfying (PL) such that

\[ \liminf_{r \to \infty} r^{(n-1)/2} \exp(-\lambda_D r) \int_{S^{n-1}} \int_D s^+(r, Y) f_D(Y) \, dY \, d\tau(a) = 0, \]

then \( s \leq 0 \) on \( L_D \).

Next we shall deal with subharmonic functions which do not necessarily satisfy (PL). Let \( \mathcal{S} \) be the set of all functions \( s \) defined on the closure of \( L_D \) that are subharmonic in \( L_D \) and satisfy

\[ \limsup_{P \to Q, P \in L_D} s(P) = s(Q) < +\infty \quad \text{for every } Q \in \partial L_D. \]

Let \( \mathcal{A} \) be the set of all subharmonic functions \( s \) that have nonnegative harmonic majorants (see [11; p. 262]). In case \( m = 1 \) we can easily deduce the following sufficient condition for \( s \) to belong to \( \mathcal{A} \) in the same line as in Brawn [8; Theorem 2]:

**Theorem B.** Let \( m = 1 \) and \( D = (0, 1) \). If \( s \in \mathcal{S} \) satisfies (2) and

\[ \int_{\mathbb{R}^n} s^+(X, 0)(1 + |X|)^{(1-n)/2} e^{-\pi|X|} \, dX < \infty, \]
\[ \int_{\mathbb{R}^n} s^+(X, 1)(1 + |X|)^{(1-n)/2} e^{-\pi|X|} \, dX < \infty, \]

then \( s \in \mathcal{A} \).

In order to generalize Theorem B to the case \( m \geq 1 \), we need to define the normal derivative of \( f_D \). Let \( n_Y \) be the inward normal at \( Y \) with respect to \( \partial D \) and let \( \sigma \) be the surface measure on \( \partial D \). It is well known that \( n_Y \) exists \( \sigma \)-a.e. on \( \partial D \) (see e.g. [22; p. 242]). We shall observe in the next section that for \( \sigma \)-a.e. \( Y \) on \( \partial D \) the
normal derivative of $f_D$,
\[
\frac{\partial}{\partial n_Y} f_D(Y) = \lim_{t \to 0} \frac{\partial}{\partial n_Y} f_D(Y + t n_Y)
\]
extists, and that $\frac{\partial f_D}{\partial n_Y} > 0$ $\sigma$-a.e. and is square integrable with respect to $\sigma$. We note that if $m=1$, $D=(0, 1)$ and $f_B(y) = \sin(\pi y)$, then $\frac{\partial f_D}{\partial n_Y} = \pi$ on $\{0, 1\}$, and $\sigma = \delta_0 + \delta_1$, where $\delta_0$ and $\delta_1$ are the Dirac measures at 0 and 1. Our generalization of Theorem B is

**Theorem 2.** Let $m \geq 1$ and $D \subset R^m$. If $s \in \mathcal{S}$ satisfies (4) and

\[
\int_{R^n} \int_{\partial D} s^+(X, Y) (1 + |X|)^{(1-n)/2} \exp(-\lambda_D |X|) \frac{\partial f_D}{\partial n_Y}(Y) dX d\sigma(Y) < \infty,
\]
then $\int_{\partial D} s^+(Q) \omega(P, dQ)$ is a nonnegative harmonic majorant of $s$ and hence $s \in \mathcal{A}$ where $\omega$ is the harmonic measure.

Finally we shall consider the mean
\[
\mathcal{M} s(Y) = \int_{R^n} s(X, Y) dX
\]
of a subharmonic function $s$ on $L_D$. In case $m=1$ and $D=(0, 1)$, there are a number of studies on $\mathcal{M} s$ (see [3, 4, 7–9, 12, 15] and [17]). As a typical example we quote [8; Theorem 3]:

**Theorem C.** Let $m=1$ and $D=(0, 1)$. If $s \in \mathcal{S}$ and

(i) $\int_{R^n} |s(X, y)| dX < \infty$, \quad $0 \leq y \leq 1$,

(ii) $\lim_{|X, y| \to \infty, (X, y) \in L_D} s^+(X, y)|X|^{(n-1)/2} e^{-\pi |X|} = 0$,

then $\mathcal{M} s$ is a convex function of $y \in D$.

Since the assumption of Theorem C implies that $s \in \mathcal{A}$ by Theorem B, it may be natural to consider the properties of $\mathcal{M} s$ for $s \in \mathcal{A}$. Noting that the subharmonicity corresponds to the convexity in case $m=1$, we shall prove

**Theorem 3.** Let $m \geq 1$ and $D \subset R^m$. Let $s \in \mathcal{A}$ and let $h$ be a nonnegative harmonic majorant of $s$. If
\[
\int_{R^n} h(X, Y) dX < \infty \quad \text{for some} \quad Y \in D,
\]
then $\mathcal{M} s(Y)$ is a subharmonic function on $D$ or identically $-\infty$ on $D$.

From this theorem we shall derive

**Corollary 1.** Let $m \geq 1$ and $D \subset R^m$. If $s \in \mathcal{S}$ satisfies (4) and
\[
\int_{\partial D} \int_{R^n} s^+(X, Y) \frac{\partial f_D}{\partial n_Y}(Y) dX d\sigma(Y) < \infty,
\]
then \( A_s(Y) \) is a subharmonic function on \( D \) or identically \( -\infty \) on \( D \).

In case \( m=1 \) and \( D=(0, 1) \), the inequality in Corollary 1 reduces to

\[
\int_{\mathbb{R}^n} (s^+(X, 0) + s^+(X, 1)) \, dX < \infty.
\]

Hence Theorem C readily follows from Corollary 1.

The author would like to acknowledge that this work was inspired by Yoshida [26—28], and would like to thank Professor Yoshida for showing him the preprints of [26—28].

2. Preliminaries

We shall use the following notation: Let \( X_0=(0, \ldots, 0) \in \mathbb{R}^n \), \( Y_0=(0, \ldots, 0) \in \mathbb{R}^m \) and \( P_0=(X_0, Y_0) \in \mathbb{R}^{n+m} \). Let \( S^{n-1} \) be the unit sphere with center at \( X_0 \) in \( \mathbb{R}^n \) and let \( \tau \) be the surface measure on \( S^{n-1} \). Denote by \( B^n(X, r) \) (respectively \( B^m(Y, r), B(P, r) \)) the \( n \) (respectively \( m, n+m \))-dimensional open ball with center at \( X \) (respectively \( Y, P \)) and radius \( r \). We may assume that \( D \supset B^n(Y_0, 2) \). Let \( \pi: \mathbb{R}^{n+m} \to \mathbb{R}^n \) be the projection defined by \( \pi((X, Y))=X \) and let \( \pi_0(P)=(\pi(P), Y_0) \). We observe that \( \Omega(X)=B^n(X, 1) \cap D \) is a bounded Lipschitz domain in \( \mathbb{R}^{n+m} \). For simplicity we write \( L \) for \( L_D \) in the sequel.

Unless otherwise specified, \( A \) will stand for a positive constant depending only on \( L \), possibly changing from one occurrence to the next, even in the same string. If \( f \) and \( g \) are positive quantities such that \( A^{-1}f \equiv g \equiv Af \), then we write \( f \sim g \).

The boundary Harnack principle ([25; Theorem 1]) stated below is a useful tool.

**Lemma A.** Suppose that \( U \) is a bounded Lipschitz domain in \( \mathbb{R}^{n+m} \), \( Q \) is a point in \( U \), \( E \) is a relatively open set on \( \partial U \), \( S \) is a subdomain of \( U \) satisfying \( \partial S \cap \partial U \subset E \). Then there is a positive constant \( C \), such that whenever \( u \) and \( v \) are two positive harmonic functions in \( U \) vanishing on \( E \) and \( u(Q) \equiv v(Q) \), then \( u(P) \equiv Cv(P) \) for all \( P \in S \).

Applying Lemma A to \( U=\Omega(P), Q=\pi_0(P), E=\partial \Omega(P) \cap \partial L \) and \( S=\cap \partial L \), we obtain

**Lemma 1 ([1; Lemma 1]).** Let \( P \in L \). Let \( u \) and \( v \) be positive harmonic functions on \( \Omega(\pi(P)) \) which vanish continuously on \( \partial \Omega(\pi(P)) \cap \partial L \). If \( u(\pi_0(P)) \equiv v(\pi_0(P)) \), then \( u(P) \equiv Av(P) \).

We need the following simple Phragmén—Lindelöf principle, which will be improved by Theorem 1.

**Lemma 2 ([1; Lemma 2]).** Let \( L' \) be a subdomain of \( L \). If \( s \) is subharmonic in
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Li, bounded above in $L'$ and nonpositive on $\partial L'$, i.e.,

$$\limsup_{P \to Q} s(P) \leq 0 \text{ for any } Q \in \partial L',$$

then $s \leq 0$ in $L'$.

One of the main difficulties arising in case $m>1$ seems to be caused by the lack of the exact formulas for the Green and Poisson kernels for $L$ (see [6] in case $m=1$ and $D=(0,1)$). Instead of those formulas we shall use the Riesz—Martin representation ([20] and [19; Chapters 6 and 12]). In the previous paper [1] we determined the Martin compactification of $L$ (see [10] for $m=1$). We shall describe the Martin compactification of $L$ as follows: We denote by $L=R^n \times \bar{D}$ the Euclidean closure of $L$ in $R^{n+m}$. Let $\hat{L}=\bar{L} \cup \{M; \alpha \in \mathbb{S}^{n-1}\}$ be a compact topological space with open base $O_1 \cup O_2$, where $O_1=\{U \cap L; U$ is an open set of $R^{n+m}\}$ and $O_2=\{U(\alpha, \varepsilon, R); \alpha \in \mathbb{S}^{n-1}, 0<\varepsilon<1$ and $R>0\}$ with $U(\alpha, \varepsilon, R)=\{M_\beta; \beta \in \mathbb{S}^{n-1}, \sum_{i=1}^n \alpha_i \beta_i > 1-\varepsilon\} \cup \{(X, Y) \in \bar{L}; (1-\varepsilon)^{-1} \sum_{i=1}^n \alpha_i |X_i Y_i > |X| > R\}$. We note that $M_\alpha$ is considered to be an ideal boundary point, and that $P_j=(X_j, Y_j) \in \bar{L}$ converges to $M_\alpha$ if and only if $\lim_{j \to \infty} |X_j| = +\infty$ and $\lim_{j \to \infty} X_j/Y_j = \alpha$.

Theorem D ([1; Theorem 1]). See also [10], [16; Chapter 8, 4 Appendix]). The Martin compactification of $L$ is homeomorphic to $\hat{L}$. Every point on $\hat{L} \setminus L$ is a minimal boundary point. More precisely, let $G$ be the Green function for $L$ and let $K$ be the Martin kernel defined by

$$K(P, Q) = \begin{cases} G(P, Q)/G(P_0, Q) & \text{if } Q \in L, \\ \lim_{M \to Q, M \in L} G(P, M)/G(P_0, M) & \text{if } Q \in \hat{L} \setminus L. \end{cases}$$

Then there are a positive constant $\lambda_D^*$ and a positive continuous function $f_D^*$ on $D$, vanishing on $\partial D$ and satisfying $f_D^*(Y_0)=1$, such that

$$K(P, M_\alpha) = f_D^*(Y) \exp \left( \lambda_D^* \sum_{i=1}^n \alpha_i x_i \right)$$

for $P=(X, Y)$ and $Q=M_\alpha \in \hat{L} \setminus L$.

We write the Laplacian $\Delta$ as

$$\Delta = \Delta_x + \Delta_y = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^m \frac{\partial^2}{\partial y_j^2}.$$ 

Since $\Delta K(\cdot, M_\alpha)=0$, it follows from (6) that $(\Delta_y + \lambda_D^2)f_D^*=0$ on $D$, so that $\lambda_D^*$ is an eigenvalue of (3). Furthermore we show

Proposition 1. The constant $\lambda_D^*$ is the first positive eigenvalue of (3). Hence $\lambda_D^* \neq \lambda_*^*$ and $f_D^*=\text{const} \cdot f_D^*$.

Proof. Suppose that (3) has a positive eigenvalue $\mu=\lambda^2$ smaller than $\lambda_D^*$. Let $f \neq 0$ be an eigenfunction corresponding to $\lambda^2$. By a straightforward calculation
we see that
\[ u(X, Y) = f(Y) e^{\lambda x_1} \]
is harmonic on \( L \). Let \( A_0 = \sup_{Y \in D} |f(Y)| < \infty \), \( L' = \{X; x_1 < 1\} \times D \) and \( L'' = \{X; x_1 < 0\} \times D \). First we compare \( u \) and the bounded positive harmonic function \( v \) on \( L' \) such that
\[
\lim_{r \to 0} v(P) = \begin{cases} A_0 & \text{if } Q \in \{X; x_1 = 1\} \times D, \\ 0 & \text{if } Q \in \{X; x_1 < 1\} \times \partial D. \end{cases}
\]
Lemma 2 applied to \( s = |u| - e^\lambda v \) leads to
\[ |u(P)| \leq e^\lambda v(P) \quad \text{for } P \in L'. \]
Next we compare \( v \) and the Martin kernel \( K(\cdot, M_\beta) \) with \( \beta = (1, 0, \ldots, 0) \). We infer from Lemma 1 that there is a positive constant \( A_1 \) such that if \( P = (X, Y) \in \{X; x_1 = 0\} \times \bar{D} \), then \( v(P) \leq A_1 K(P, M_\beta) \). Hence Lemma 2 applied to \( s = v - A_1 K(\cdot, M_\beta) \) yields
\[ v(P) \leq A_1 K(P, M_\beta) \quad \text{on } L''. \]
Therefore we have from (6)
\[ |f(Y)| \leq A_1 e^{\lambda} f_D^* (Y) e^{(l_B - \lambda)x_1} \quad \text{if } x_1 < 0. \]
Since \( \lambda^*_B > \lambda \), letting \( x_1 \to -\infty \), we obtain \( f(Y) \equiv 0 \) on \( D \), a contradiction.

Remark 1. In case \( D \) is a piecewise \( C^1 \) domain it is known that if \( f \) is a positive eigenfunction for (3), then the eigenvalue corresponding to \( f \) is the first positive eigenvalue ([13; p. 458]).

Remark 2. If \( m = 1, D = (0, 1) \) and \( Y_0 = 1/2 \), then \( \lambda_B = \pi \) and \( f_D (y) = \sin (\pi y) \). If \( m = 2 \) and \( D = B^m (Y_0, r) \), then \( \lambda_B = \lambda_0 / r \) and
\[
f_D (Y) = 2^{m^2 - 1} \Gamma \left( \frac{m}{2} \right) \left( \frac{\lambda_0}{r} \right)^{1 - m/2} |Y|^{-1 - m/2} J_{m/2 - 1} \left( \frac{\lambda_0}{r} |Y| \right),
\]
where \( J_{m/2 - 1} \) is the Bessel function of the first kind of order \( m/2 - 1 \) and \( \lambda_0 \) is the least positive number such that \( J_{m/2 - 1} (\lambda_0) = 0 \) (see [23; p. 45] and [6; p. 441]).

Hereafter we let \( f_D (Y_0) = 1 \). We observe from (6) that
\[
\int_{S^n - 1} K(P, M_\alpha) d\tau (\alpha) = f_D (Y) \int_{S^n - 1} \exp \left( \lambda_B \sum_{i=1}^n \alpha_i x_i \right) d\tau (\alpha), \quad P = (X, Y),
\]
is a positive harmonic function depending only on \( |X| \) and \( Y \). On account of the formulas for Bessel functions in [23; p. 79], we see that the above integral is equal to
\[
(7) \quad c_0 f_D (Y) |X|^{l - n/2} I_{n/2 - 1} (\lambda_B |X|),
\]
where \( c_0 \) is a positive constant depending only on \( n \) and \( I_{n/2 - 1} \) is the Bessel function of the third kind of order \( n/2 - 1 \). Let \( K_{n/2 - 1} \) be another Bessel function appearing
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in [23; p. 78], which satisfies

\[ K_{n/2-1}(r) \sim r^{-1/2}e^{-r} \quad \text{as} \quad r \to \infty. \]

Since \( K_{n/2-1} \) satisfies the same second order differential equation [23; (1) on p. 77] as \( I_{n/2-1} \), it follows that

\[ f_D(Y)|X|^{1-n/2}K_{n/2-1}(\lambda_D|X|) \]

is positive and harmonic on \( \{(X, Y) \in \mathbb{L}; X \neq X_0\} \).

Let \( G^D \) be the Green function for \( D \). We see that if \( m=1 \) and \( D=(a, b) \), then

\[ G^D(y, y') = \min \left\{ \frac{b-y'}{b-a}(y-a), \frac{y'-a}{b-a}(b-y) \right\} \quad \text{for} \quad y, y' \in D. \]

We observe that \( G^D(\cdot, Y') \) is harmonic on \( D \setminus \{Y'\} \) and

\[ \Delta_y G^D(\cdot, Y') = \begin{cases} -\delta_{Y'}, & \text{if} \quad m = 1, \\ -2\pi \delta_{Y'}, & \text{if} \quad m = 2, \\ (2-m)\sigma_m \delta_{Y'}, & \text{if} \quad m \geq 3, \end{cases} \]

where \( \sigma_m \) denotes the surface area of the unit sphere \( S^{m-1} \) and \( \delta_{Y'} \) denotes the Dirac measure at \( Y' \). We have

Lemma 3. Let \( G_0(\cdot) = G(\cdot, P_0) \) and \( G^D_0(\cdot) = G^D(\cdot, Y_0). \) If \( Y \in D \setminus B^m(Y_0, 1) \), then

(i) \( G_0((X, Y)) \sim f_D(Y)(1 + |X|)^{(1-n)/2} \exp (-\lambda_D|X|), \)

(ii) \( G^D_0(Y) \sim f_D(Y). \)

Proof. Applying Lemma 1 to \( u = f_D(Y)|X|^{1-n/2}K_{n/2-1}(\lambda_D|X|) \) and \( v = G_0 \), we obtain that

\[ G_0((X, Y)) \sim f_D(Y)|X|^{1-n/2}K_{n/2-1}(\lambda_D|X|) \quad \text{for} \quad |X| = 1. \]

On account of Lemma 2 and (8) we have (i) for \( |X| \equiv 1. \) From Lemma A with \( U = B^n(X_0, 2) \times (D \setminus B^m(Y_0, 1/2)), \ u = G_0 \) and \( v = f_D(Y)|X|^{1-n/2}I_{n/2-1}(\lambda_D|X|) \) we infer (i) for \( |X| \equiv 1 \) and \( Y \in D \setminus B^m(Y_0, 1). \) We regard \( G^D_0 \) as a positive harmonic function on \( \{(X, Y) \in \mathbb{L}; Y \neq Y_0\}. \) Applying Lemma A to the same \( U \) as above, \( u = G^D_0 \) and \( v = G_0, \) we obtain

\[ G^D_0(Y) \sim G_0((X, Y)) \quad \text{for} \quad |X| = 1 \quad \text{and} \quad Y \in D \setminus B^m(Y_0, 1). \]

Hence (i) leads to (ii).

Remark 3. In view of Widman [24; Theorems 2.2 and 2.5], if \( D \) is a Liapunov domain in \( \mathbb{R}^m (m \equiv 2), \) then

\[ G^D_0(Y) \sim f_D(Y) \sim \text{dist}(Y, \partial D) \quad \text{for} \quad Y \in D \setminus B^m(Y_0, 1). \]
This relation also holds in case $m=1$.

In [14] Dahlberg studied a relationship between the harmonic measure and the normal derivative of the Green function for a bounded Lipschitz domain. Since $G^D$ and $f_D$ are comparable by Lemma 3 (ii), we can prove the next lemma in a way similar to [14; Lemma 9].

**Lemma 4.** Let $n_Y$ be the inward normal at $Y$ with respect to $\partial D$. For $\sigma$-a.e. point $Y$ on $\partial D$ the normal derivative of $f_D$,

$$\frac{\partial}{\partial n_Y} f_D(Y) = \lim_{t \to 0} \frac{\partial}{\partial n_Y} f_D(Y + tn_Y)$$

exists and is positive. The normal derivative $\frac{\partial f_D}{\partial n_Y}$ is square integrable with respect to the surface measure $\sigma$ on $\partial D$. Furthermore if $h$ is $C^2$ on a domain including $\overline{D}$, then the following Green's identity holds:

$$\int_D f_D(\Delta_Y + \lambda_D^2)h \, dY = \int_{\partial D} h \frac{\partial f_D}{\partial n_Y} \, d\sigma.$$

Let $\omega(P, E)$ be the harmonic measure at $P \in L$ of $E \subset \partial L$.

**Lemma 5.** If $E$ is a Borel measurable set on $\partial L$, then

$$\omega(P_0, E) \sim \int_E (1 + |X|)^{\frac{(1-n)^2}{2}} \exp(-\lambda_D |X|) \frac{\partial f_D}{\partial n_Y}(Y) \, dX \, d\sigma(Y).$$

**Proof.** It is sufficient to prove the lemma in case $E \subset B^n(X_1, 1) \times \partial D$ for some $X_1$. Let $P_1 = (X_1, Y_0)$ and $L' = B^n(X_1, 3) \times D$. By $\omega(\cdot, E, L')$ and $G'$ we denote the harmonic measure of $E$ and the Green function for $L'$. Applying Lemma A to $U = B^n(X_1, 2) \times (D \setminus B^n(Y_0, 1/2))$, $u = G'(\cdot, P_1)$ and $v = G(\cdot, P_1)$, we obtain that $G'((X, Y), P_1) \sim G((X, Y), P_1)$ for $X \in B^n(X_1, 1)$ and $Y \in D \setminus B^n(Y_0, 1)$. By Lemma 3 (i) and a suitable translation we have $G((X, Y), P_1) \sim f_D(Y)$ for $X \in B^n(X_1, 1)$ and $Y \in D \setminus B^n(Y_0, 1)$. Hence we infer from [14; Theorem 3 (b)] that

$$\omega(P_1, E, L') \sim \int_E \frac{\partial f_D}{\partial n_Y} \, dX \, d\sigma(Y).$$

If $|X_1 - X_0| \leq 3$, then the Harnack principle leads to

$$\omega(P_0, E) \sim \omega(P_1, E) \sim \omega(P_1, E, L') \sim \int_E \frac{\partial f_D}{\partial n_Y} \, dX \, d\sigma(Y).$$

In case $|X_1 - X_0| > 3$, using Lemma 1 for $P \in \partial B^n(X_1, 2) \times D$ and then using Lemma 2, we obtain

$$\omega(P_0, E) \sim G(P_0, P_1) \omega(P_1, E, L').$$

Hence by Lemma 3

$$\omega(P_0, E) \sim \int_E (1 + |X|)^{\frac{(1-n)^2}{2}} \exp(-\lambda_D |X|) \frac{\partial f_D}{\partial n_Y}(Y) \, dX \, d\sigma(Y).$$
Remark 4. If $D$ is a Liapunov domain, then $\partial f_D/\partial n_\gamma \sim 1$ by Remark 3, so that

$$\omega(P_0, E) \sim \int_E (1 + |X|^{(1-n)/2} \exp (-\lambda_D |X|) dX d\sigma(Y).$$

If $m=1$ and $D=(a, b)$, then

$$\omega(P_0, E) = \omega(P_0, E_a) + \omega(P_0, E_b) \sim \int_{\pi(E_a) \cup \pi(E_b)} (1 + |X|^{(1-n)/2} \exp (-\lambda_D |X|) dX,$$

where $E_a = \pi(E_a) \times \{a\} = \{(X, y) \in E; y = a\}$ and $E_b = \pi(E_b) \times \{b\} = \{(X, y); y = b\}$.

3. Proofs of Theorems 1 and 2

Brawn [7; Theorem 2, Corollary] (see also [5; Theorem 4]) proved Theorem A by using the Nevanlinna mean $\mathcal{M}(s, r)$ of $s$ defined by

$$\mathcal{M}(s, r) = \int_{S^{n-1}} d\tau(\alpha) \int_0^1 s(r\alpha, y) \sin(\pi y) dy$$

$$= \int_0^1 \sin(\pi y) dy \int_{S^{n-1}} s(r\alpha, y) d\tau(\alpha).$$

In the expressions of $\mathcal{M}(s, r)$, there are two averaging operations,

$$\int_0^1 s \sin(\pi y) dy \quad \text{and} \quad \int_{S^{n-1}} s d\tau(\alpha).$$

Naturally, the operation $\int_D s f_D(Y) dY$ is considered to be a generalization of the first. We shall observe that these operations produce symmetrized subharmonic functions from given subharmonic functions on $L$ (see Lemmas 6 and 7 below). We shall prove Theorem 1 by using this phenomenon. Let us begin with

Lemma 6. Let $s$ be a nonnegative subharmonic function on $L$ satisfying (PL). Then

$$S(X, Y) = f_D(Y) \int_D s(X, Y') f_D(Y') dY'$$

is a subharmonic function on $L$ satisfying (PL).

Proof. On account of (PL) $s$ is bounded on $B^n(X_0, r) \times D$ for every $r > 0$. Since $f_D$ is continuous on $\overline{D}$, it follows that $S$ is locally integrable on $L$. Since $s$ satisfies (PL) and is nonnegative on $L$,

$$\hat{s}(P) = \begin{cases} 
  s(P) & \text{if } P \in L, \\
  0 & \text{if } P \in \mathbb{R}^{n+m} \setminus L,
\end{cases}$$

is a subharmonic function on $\mathbb{R}^{n+m}$. On account of [19; Theorem 4.20], there is a nonincreasing sequence of $C^2$ subharmonic functions $s_j$ on $\mathbb{R}^{n+m}$ converging to $\hat{s}$. 
In view of (PL) and the construction of the sequence in [19; Theorem 4.20], we may furthermore assume that there are compact subsets $K_j$ of $D$ such that $K_{j+1}D$ and

$$s_j(P) \leq 1/j \quad \text{for} \quad P \in B^n(0, j+1) \times (R^m \setminus K_j).$$

Let

$$S_j(X, Y) = f_D(Y) \int_D s_j(X, Y') f_D(Y') dY'.$$

It follows from the monotone convergence theorem that $S_j \uparrow S$, so that $S$ is upper semicontinuous. Let $P=(X, Y) \in B^n(0, j+1) \times D$. Since $\Delta s_j \geq 0$, we have from Lemma 4 and (9)

$$\Delta S_j(P) = f_D(Y) \int_D \{ \Delta X s_j(X, Y') - \lambda_D^2 s_j(X, Y') \} f_D(Y') dY'$$

$$\equiv -f_D(Y) \int_D (\Delta Y + \lambda_D^2) s_j(X, Y') f_D(Y') dY'$$

$$= -f_D(Y) \int_{\partial D} s_j(X, Y') \frac{\partial f_D}{\partial n_Y'} (Y') d\sigma(Y')$$

$$\equiv -\frac{1}{j} f_D(Y) \int_{\partial D} \frac{\partial f_D}{\partial n_Y'} (Y') d\sigma(Y').$$

Since the last term tends to zero as $j \to \infty$, it follows from the dominated convergence theorem that if $\varphi \in C_0^\infty(L)$ and $\varphi \geq 0$, then

$$\int_L S \Delta \varphi dP = \lim_{j \to \infty} \int_L S_j \Delta \varphi dP = \lim_{j \to \infty} \int_L \varphi \Delta S_j dP \geq 0.$$

Hence $\Delta S \geq 0$ on $L$ in the distribution sense, so that $S$ is subharmonic on $L$.

In the same manner as in Armitage [2; Lemma], we can prove

Lemma 7 (cf. [28; Lemma 1]). If $s$ is subharmonic on $L$, then

$$\int_{S^{n-1}} s(|X|, Y) d\tau(z)$$

is a subharmonic function on $L$ depending only on $|X|$ and $Y$.

The next lemma is a preliminary version of Theorem 1.

Lemma 8. Let $S$ be a subharmonic function on $L$ satisfying (PL). If

$$\liminf_{r \to \infty} r^{(n-1)/2} \exp(-\lambda_D r) \sup_{|X|=r, Y \in D} S(X, Y) \equiv 0,$$

then $S \equiv 0$ on $L$.

Proof. Let $(X_1, Y_1) \in L$ and $\varepsilon > 0$ be given. We find $r=|X_1| + 2$ such that

$$\sup_{|X|=r, Y \in D} S(X, Y) \equiv \varepsilon r^{(1-n)/2} \exp(\lambda_D r).$$
Let $h$ be the bounded harmonic function on $B^n(X_0, r) \times D$ such that

$$h(P) = \begin{cases} \epsilon r^{1-n/2} I_{n/2-1}(\lambda_D r) & \text{on } \partial B^n(X_0, r) \times D, \\ 0 & \text{on } B^n(X_0, r) \times \partial D. \end{cases}$$

By (1) and the maximum principle we have $S \equiv Ah$ on $B^n(X_0, r) \times D$. Let

$$\nu(X, Y) = f_D(Y)|X|^{1-n/2} I_{n/2-1}(\lambda_D |X|)$$

and recall that this function is positive and harmonic in $L$ and vanishes on $\partial L$. Since

$$r^{1-n/2} I_{n/2-1}(\lambda_D r) \equiv A(r-1)^{1-n/2} I_{n/2-1}(\lambda_D (r-1)),$$

it follows that

$$h(X, Y) \equiv \epsilon r^{1-n/2} I_{n/2-1}(\lambda_D r) \equiv A \epsilon \nu(X, Y_0) \text{ for } |X| = r-1.$$

Hence Lemma 1 leads to

$$h \equiv A \epsilon \nu \text{ on } \partial B^n(X_0, r-1) \times D.$$

Using the maximum principle, we obtain

$$S \equiv h \equiv A \epsilon \nu \text{ on } B^n(X_0, r-1) \times D,$$

and in particular $S(X_1, Y_1) \equiv A \epsilon f_D(Y_1)|X_1|^{1-n/2} I_{n/2-1}(\lambda_D |X_1|)$. Since $\epsilon$ is arbitrary, it follows that $S(X_1, Y_1) \equiv 0$, so that $S \equiv 0$ on $L$.

**Proof of Theorem 1.** On account of Lemmas 6 and 7

$$S(X, Y) = f_D(Y) \int_{S_{n-1}} \int_D s^+(|X| \alpha, Y') f_D(Y') \, dY' \, d\tau(\alpha)$$

is a subharmonic function on $L$ satisfying (PL). It follows from (4) that $S$ satisfies (10). Hence Lemma 8 leads to $S \equiv 0$ on $L$, so that $s^+$ must identically equal zero. Thus $s \equiv 0$ on $L$.

**Proof of Theorem 2.** Let

$$h(P) = \int_{\partial L} s^+(Q) \omega(P, dQ).$$

On account of Lemma 5 and (5), $h$ is positive and harmonic on $L$. By the aid of [21; 2.24 The Vitali—Carathéodory Theorem] applied to the measure $\omega(P_0, \cdot)$, we find a nonincreasing sequence of nonnegative lower semicontinuous functions $\nu_j$ on $\partial L$ such that $s^+ \equiv v_j$ and

$$h(P_0) \equiv h_j(P_0) - 1/j,$$

where

$$h_j(P) = \int_{\partial L} v_j(Q) \omega(P, dQ).$$
We observe that

$$\lim_{j \to \infty} h_j = h \quad \text{on} \quad L.$$  

In fact, $h^* = \lim_{j \to \infty} h_j$ is a harmonic function which majorizes $h$ since $v_j$ is non-increasing. We infer from $h^*(P_0) = h(P_0)$ and the maximum principle that $h^* = h$ on $L$.

Now we claim that $h_j$ majorizes $s$ on $L$. Let $\varepsilon > 0$ and $M \in \partial L$. Since $v_j$ is lower semicontinuous, there is $r > 0$ such that

$$v_j(Q) \equiv s^+(M) - \varepsilon \quad \text{for} \quad Q \in B(M, r) \cap \partial L.$$  

Hence

$$h_j(P) \equiv (s^+(M) - \varepsilon) \omega(P, B(M, r) \cap \partial L) \quad \text{for} \quad P \in L,$$

so that

$$\lim \inf_{P \to M, P \in L} h_j(P) \equiv s^+(M) - \varepsilon.$$  

Therefore

$$\lim \sup_{P \to M, P \in L} (s(P) - h_j(P)) \equiv s^+(M) - \lim \inf_{P \to M, P \in L} h_j(P) \equiv \varepsilon.$$  

Since $\varepsilon$ and $M$ are arbitrary, $s - h_j$ satisfies (PL). Applying Theorem 1 to $s - h_j$, we obtain $s \leq h$ on $L$, and $s \leq h$ on $L$ by (11).

4. Proof of Theorem 3

The Riesz–Martin decomposition ([20] and [19; Chapters 6 and 12]) yields that $s \in \mathcal{A}$ if and only if there are a signed measure $\nu$ on $\tilde{L} \setminus L$ and a nonnegative measure $\mu$ on $L$ such that

$$s(P) = \int_{L \setminus L} K(P, Q) d\nu(Q) - \int_L G(P, Q) d\mu(Q).$$

We first treat the mean of a positive harmonic function, and then treat that of a Green potential.

**Lemma 9.** If $h$ is a positive harmonic function on $L$, then $\mathcal{A}h(Y)$ is harmonic on $D$ or identically $+\infty$ on $D$.

**Proof.** We assume that $\mathcal{A}h(Y') < \infty$ for some $Y' \in D$. Take a compact subset $K$ of $D$. Then Harnack’s inequality [19; Theorem 2.14 and Corollary 2.15] yields that

$$\sup_{Y \in K} h(X, Y) \equiv Ah(X, Y') \quad \text{for all} \quad X \in \mathbb{R}^n,$$

where $A$ is a positive constant depending only on $Y'$, $K$ and $D$. Moreover every first and second order derivative $\partial \partial h$ of $h$ satisfies

$$\sup_{Y \in K} |\partial \partial h(X, Y)| \equiv A'h(X, Y') \quad \text{for all} \quad X \in \mathbb{R}^n,$$
where $A'$ depends only on $Y'$, $K$ and $D$ (see [18; p. 37]). Since $\mathcal{M}h(Y')<\infty$, it follows that
\[
\varphi(r) = \sup_{Y \in K} \sum_{i=1}^{n} \int_{\partial B^n(x_o, r)} \left| \frac{\partial}{\partial x_i} h(X, Y) \right| d\tau_r(X)
\]
is integrable with respect to $r$, where $\tau_r$ stands for the surface measure on $\partial B^n(x_o, r)$. Hence we can choose $r_j \to \infty$ such that $\varphi(r_j) \to 0$. Using Green's formula, we obtain
\[
\Delta_Y \mathcal{M}h(Y) = \int_{R^n} \Delta_Y h(X, Y) dX = -\int_{R^n} \Delta_X h(X, Y) dX
\]
\[
= -\lim_{j \to \infty} \int_{B^n(x_o, r_j)} \Delta_X h(X, Y) dX
\]
\[
= \lim_{j \to \infty} \int_{\partial B^n(x_o, r_j)} \frac{\partial}{\partial n_X} h(X, Y) d\tau_{\partial X}(X) = 0
\]
for $Y \in K$. Since $K$ is arbitrary, $\mathcal{M}h$ is harmonic on $D$.

From Lemma 9 we have a relation between $G$ and $G^D$, which may be of some independent interest.

**Proposition 2.** There is a positive constant $c_1$ depending only on $n$ and $m$ such that
\[
G^D(Y, Y') = c_1 \int_{R^n} G((X, Y), (X', Y')) dX.
\]

**Proof.** We may assume that $X' = x_0$. Let $Y' \in D$ and put
\[
v(Y) = \int_{R^n} G((X, Y), (X_0, Y')) dX.
\]
We infer from Lemmas 3 and 9 that $v$ is harmonic on $D \setminus \{Y'\}$. It follows from Fatou's lemma that $v$ is lower semicontinuous on $D$, and from Lemma 3 and Lebesgue's dominated convergence theorem that
\[
\lim_{Y \to Y'_1} v(Y) = 0 \quad \text{for} \quad Y_1 \in \partial D.
\]
The maximum principle yields that $v(Y) \equiv v(Y') \equiv +\infty$ (actually if $m=1$, then $v(Y') < +\infty$ and if $m \geq 2$, then $v(Y') = +\infty$), so that $v$ is superharmonic on $D$. Therefore
\[
\Delta_Y v = -c(Y', D) \delta_Y,
\]
in the distribution sense, where $c(Y', D)$ is a positive constant which may depend on $Y'$ and $D$ (see [19; Theorem 5.4]).

What remains is to prove that $c(Y', D)$ does not depend on $Y'$ and $D$. Take $r > 0$ such that $B^n(Y', r) \subset D$. Let $L' = R^n \times B^n(Y', r)$ and let $G'$ be the Green function for $L'$. Since $G(\cdot, (X_0, Y')) - G'(\cdot, (X_0, Y'))$ is a positive harmonic function on $L'$ decreasing rapidly at the infinity by Lemma 3, it follows from Lemma 9 that
\[
v(Y) = \int_{R^n} G'( (X, Y), (X_0, Y')) dX
\]
is harmonic on $B^n(Y', r)$. Hence
\[ c(Y', D) = c(Y', B^n(Y', r)) = c(r). \]

We infer from the arbitrariness of $r$ that $c(r)$ is equal to a positive constant depending only on $n$ and $m$. The proof is complete.

Lemma 10. If $u$ is a Green potential on $L$, then $\mathcal{M}u(Y)$ is a Green potential on $D$ or identically $+\infty$ on $D$.

**Proof.** Let
\[ u(P) = \int_L G(P, Q) d\mu(Q), \]
where $\mu$ is a Radon measure on $L$. We have from Fubini's theorem and Proposition 2 that
\[ \mathcal{M}u(Y) = \int_{\mathbb{R}^n} \int_L G((X, Y), Q) d\mu(Q) dX = c_1^{-1} \int_D G^D(Y, Y') d\mu_0(Y'), \]
where $\mu_0$ is the measure on $D$ defined by $\mu_0(E) = \mu(R^n \times E)$. If there is a compact set $F \subset D$ such that $\mu_0(F) = \infty$, then $\mathcal{M}u \equiv \infty$ on $D$. If there is no such compact set, then $\mu_0$ is a Radon measure on $D$ and $\mathcal{M}u$ is a Green potential on $D$ or identically $+\infty$.

**Proof of Theorem 3.** Since $h-s$ is a nonnegative superharmonic function, it follows from the Riesz—Martin decomposition that
\[ h-s = u+p, \]
where $u$ is a nonnegative harmonic function on $L$ and $p$ is a Green potential on $L$. We infer from the assumption and Lemmas 9 and 10 that
\[ \mathcal{M}s(Y) = \mathcal{M}h(Y) - \mathcal{M}u(Y) - \mathcal{M}p(Y) \]
is a subharmonic function on $D$ or identically $-\infty$.

**Proof of Corollary 1.** The inequality in Corollary 1 implies (5), and hence Theorem 2 shows that
\[ h(P) = \int_{\partial D} s^+(Q) \omega(P, dQ) \]
is a nonnegative harmonic majorant of $s$. We infer from Lemma 5 and Fubini’s theorem that
\[ \mathcal{M}h(Y_0) \]
\[ \sim \int_{\mathbb{R}^n} \int_{\partial D} \int_{\mathbb{R}^n} s^+(X, Y)(1 + |X - X'|)^{(1-n)/2} \exp(-\lambda D |X - X'|) \frac{\partial f_D}{\partial n_Y}(Y) dX d\sigma(Y) dX' \]
\[ \sim \int_{\partial D} \int_{\mathbb{R}^n} s^+(X, Y) \frac{\partial f_D}{\partial n_Y}(Y) dX d\sigma(Y) < \infty. \]

Hence $\mathcal{M}h(Y_0) < \infty$, so that the corollary follows from Theorem 3.
Note added in proof: Professor S. J. Gardiner kindly informed the author that in a paper to be published in Bull. London Math. Soc. he obtained results which imply our Theorem 3. His methods are different from ours.

References


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