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F-HARMONIC MEASURES, QUASIHYPERBOLIC DISTANCE AND MILLOUX'S PROBLEM

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1. Introduction

Let B^2 be the unit disk in the plane and C a relatively closed subset of B^2. If ω is the harmonic measure of C with respect to the open set $B^2\setminus C$, then A. Beurling's projection theorem can be used to estimate ω from below. This, in turn, leads to estimates, called Milloux's problem, for bounded analytic functions $f\colon B^2\to\mathbf{C}$ which are small on the set C.

In this paper we consider these problems for F-harmonic measures ω and quasiregular mappings which are generalizations of the harmonic measure and analytic functions, respectively, to higher dimensional euclidean spaces. If C is a relatively closed subset of a domain G in R^n without compact components, then in Section 2 we obtain lower bounds for $\omega(x)$ which only depend on the quasihyperbolic distance of x and C and on the ellipticity constant of F. The estimates are useful since both the F-harmonic measures and the quasihyperbolic distance are, in a sense, quasiconformal invariants. Although some of these bounds can be derived from the estimates due to V. G. Maz'ja [M], we use a new tool: Harnack's inequality for monotone super-F-extremals.

In Section 3 we obtain a lower bound for ω in the unit ball B^n of R^n provided that the set C meets each sphere $\partial B^n(t)$, $0\leqq t<1$. This estimate is based on a variation of the Carleman method introduced in [GLM 3]. The method gives essentially the same lower bound as derived in Section 2 where C has no compact components in B^n. These rather elementary methods produce in B^n, $n\geqq 2$, lower bounds for ω which are, except for certain numerical constants, as good as lower bounds derived from Beurling's projection theorem in B^2 for the ordinary harmonic measure. Section 4 deals with Milloux's problem for quasiregular mappings. Milloux's problem is closely related to Hadamard's three circle theorem and its local version, based on Maz'ja's estimates, proved by S. Rickman [R], see also [V].

Throughout this paper $B^n(x,r)$ denotes the open n-ball of R^n with center x and radius $r>0$. We also use the abbreviations $B^n(r)=B^n(0,r)$ and $B^n=B^n(1)$.

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2. F-harmonic measure and quasihyperbolic distance

2.1. *F-harmonic measure.* Let $F\colon R^n \times R^n \to R$ be a variational kernel satisfying the usual assumptions of measurability, convexity, differentiability and homogeneity in the conformally invariant case $F(x, h) \approx |h|^n$, see [GLM 1, p. 48]. In particular, there are constants $0 < \alpha \leqq \beta < \infty$ such that for a.e. $x \in R^n$

$$\alpha |h|^n \leqq F(x, h) \leqq \beta |h|^n \tag{2.2}$$

for all $h \in R^n$. Let G be an open set in R^n and let C be a set in $\bar{G}$ such that $C \cap G$ is closed in G. We let $\omega = \omega(C, G \setminus C; F)$ denote the F-harmonic measure C with respect to the open set $G \setminus C$. This means that for $x \in G \setminus C$

$$\omega(x) = \inf \{u(x)\colon u \in \mathcal{U}\}$$

where $\mathcal{U}$ is the upper class for ω, i.e. each $u \in \mathcal{U}$ is a non-negative super-F-extremal in $G \setminus C$ and for each $y \in \partial(G \setminus C)$

$$\varliminf_{x \to y} u(x) \geqq \chi_C(y)$$

where χ_C is the characteristic function of C. For the definition and properties of ω see [GLM 2] and [GLM 3, 2.7]. Here we are mainly interested in the case where C is a closed subset of a domain G.

For the next lemma we recall that a continuous function in an open set G is monotone if for all domains D with compact closure in G

$$\operatorname{osc}(u, D) = \operatorname{osc}(u, \partial D)$$

where $\operatorname{osc}(u; A) = \sup_A u - \inf_A u$.

2.3. Lemma. *Suppose that u is a continuous, monotone and non-negative super-F-extremal in G. Then*

$$\sup_{x \in B(r/2)} u(x) \leqq e^{c(\beta/\alpha)^{1/n}} \inf_{x \in B(r/2)} u(x) \tag{2.4}$$

whenever the ball $B(r) = B^n(x_0, r)$ lies in G. The constant c depends only on n.

Proof. The proof is similar to [GLM 1, Theorem 4.15], however, the details are different. Fix a ball $B(r) = B^n(x_0, r)$ in G. We may assume that $\bar{B}(r) \subset G$. We may also assume that $u > 0$ in $\bar{B}(r)$ since if $u(x) = 0$ at some point $x \in \bar{B}(r)$, then it easily follows from the F-comparison principle and from the continuity of u that $u \equiv 0$ in a neighborhood of $\bar{B}(r)$ and hence the inequality (2.4) would be trivial.

Next let $v = \log u$ in $B(r)$. It follows from [LM, Lemma 2.12] that for each $0 < \varrho < r$

$$\int_{B(\varrho)} |\nabla v|^n \, dm \leqq c_1(\beta/\alpha)(\log(r/\varrho))^{1-n} \tag{2.5}$$

where the constant c_1 depends only on n. To complete the proof for (2.4), note that since u is monotone, v is also monotone and [GLM 1, Lemma 2.7] yields for $\varrho=r/\sqrt{2}$

$$\omega(v, B(r/2))^n \log(2\varrho/r) \leqq \int_{r/2}^{\varrho} \omega(v, B(t))^n/t\, dt$$

$$= \int_{r/2}^{\varrho} \omega(v, \partial B(t))^n/t\, dt \leqq A \int_{B(\varrho)} |\nabla v|^n\, dm \leqq A c_1(\beta/\alpha)(\log(r/\varrho))^{1-n}$$

where (2.5) has been used in the last step and A depends only on n. This implies

$$\log \frac{\sup u}{\inf u} = \omega(v, B(r/2)) \leqq [A c_1(\beta/\alpha)]^{1/n} (\log \sqrt{2})^{-1}$$

and we obtain (2.4) with

$$c = (Ac_1)^{1/n} (\log \sqrt{2})^{-1}.$$

The Harnack's inequality of Lemma 2.3 can be used for F-harmonic measures in the following situation.

2.6. Lemma. *Suppose that C is a relatively closed subset of G without compact components. Let $\omega^*(x)=\omega(C, G\setminus C; F)(x)$ for $x\in G\setminus C$ and $\omega^*(x)=1$ for $x\in C$. Then ω^* is continuous in G and* (2.4) *holds for ω^* in each ball $B^n(x_0, r)\subset G$.*

Proof. The continuity of ω^* in G follows from [GLM 3, Remark 2.14]. Next the F-comparison principle implies that ω^* is a super-F-extremal in G and since $0<\omega^*\leqq 1$ and since each point $x\in G$ where $\omega^*(x)=1$ belongs to a continuum reaching to ∂G, ω^* is monotone in G. Thus the inequality (2.4) for ω^* follows from Lemma 2.3.

2.7. *Quasihyperbolic distance.* Suppose that G is a proper subdomain of R^n. The quasihyperbolic metric k_G in G is defined as

$$k_G(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \operatorname{dist}(x, \partial G)^{-1}\, ds \tag{2.8}$$

where the infimum is taken over the family of all rectifiable curves γ in G joining x_1 to x_2. For the properties of k_G see [GP]. If C is a subset of G, then for $x\in G$

$$k_G(x, C) = \inf_{y\in C} k_G(x, y)$$

denotes the quasihyperbolic distance of x and C. If $C=\emptyset$, we set $k_G(x, C)=\infty$.

2.9. Theorem. *Suppose that G is a proper subdomain of R^n and C is a relatively closed subset of G without compact components. Then for each $x\in G\setminus C$*

$$\omega(x) \geqq b \exp(-a k_G(x, C)) \tag{2.10}$$

where $\omega=\omega(C, G\setminus C; F)$ and $a=c(\beta/\alpha)^{1/n}$. The constant $c>0$ depends only on n and $b>0$ depends only on n and α/β.

Proof. Fix $x \in G \setminus C$ and set $\delta = k_G(x, C)$. If $\delta = \infty$, then there is nothing to prove. Suppose $0 < \delta < \infty$ and let $0 < \varepsilon < \delta/2$. Pick $y \in C$ and a rectifiable curve γ in G joining x to y with

$$\delta = k_G(x, C) > \int_\gamma \operatorname{dist}(z, \partial G)^{-1}\, ds - \varepsilon. \tag{2.11}$$

Next choose points $z_1, \ldots, z_{j+1}$ on γ and radii $r_1, \ldots, r_j$ inductively as follows. Set $z_1 = y$ and $r_1 = \operatorname{dist}(z_1, \partial G)/2$. Assume that $z_1, \ldots, z_i$ have been chosen and let γ_i denote the part of γ from z_i to x. If $\gamma_i \subset \bar{B}^n(z_i, r_i)$, then set $j = i$ and $z_{j+1} = x$. If $\gamma_i \not\subset \bar{B}(z_i, r_i)$, then let z_{i+1} be the last point where γ_i meets $\partial B^n(z_i, r_i)$ and put $r_{i+1} = \operatorname{dist}(z_{i+1}, \partial G)/2$. Since γ is a compact subset of G, the process ends after a finite number of steps.

Fix $i = 1, \ldots, j-1$. Let γ_i be the part of γ from z_i to z_{i+1}. Pick $z' \in \partial G$ such that

$$2r_i = \operatorname{dist}(z_i, \partial G) = |z_i - z'|.$$

Then for $z \in \gamma_i \cap B^n(z_i, r_i)$

$$\operatorname{dist}(z, \partial G) \leqq |z - z'| \leqq |z - z_i| + |z_i - z'| \leqq r_i + 2r_i = 3r_i$$

and thus

$$\int_{\gamma_i} \operatorname{dist}(z, \partial G)^{-1}\, ds \geqq \int_{\gamma_i \cap B^n(z_i, r_i)} \operatorname{dist}(z, \partial G)^{-1}\, ds$$
$$\geqq r_i/3r_i = 1/3.$$

Hence

$$\int_\gamma \operatorname{dist}(z, \partial G)^{-1}\, ds \geqq \sum_{i=1}^{j-1} \int_{\gamma_i} \operatorname{dist}(z, \partial G)^{-1}\, ds \geqq (j-1)/3$$

and by (2.11)

$$\delta > (j-1)/3 - \varepsilon > (j-1)/3 - \delta/2.$$

This yields

$$k_G(x, C) = \delta > (j-1)/6.$$

Next let ω^* be defined as in Lemma 2.6. We apply (2.4) to ω^* in each $B_i = B^n(z_i, r_i)$, $i = 1, \ldots, j$. Note that $B^n(z_i, 2r_i) \subset G$ and hence

$$1 = \omega^*(y) = \omega^*(z_1) \leqq \lambda \inf_{B_1} \omega^* \leqq \lambda \sup_{B_2} \omega^* \leqq \lambda^2 \inf_{B_2} \omega^*$$
$$\leqq \ldots \leqq \lambda^j \inf_{B_j} \omega^* \leqq \lambda^j \omega^*(x) \leqq \lambda^{6\delta+1} \omega(x)$$

where

$$\lambda = \exp\left(c(\beta/\alpha)^{1/n}\right)$$

and c depends only on n. This implies the required inequality (2.10) with

$$a = 6 \log \lambda = 6c(\beta/\alpha)^{1/n}, \quad b = 1/\lambda = \exp\left(-c(\beta/\alpha)^{1/n}\right).$$

2.12. Remark. The use of the Harnack inequality to estimate harmonic functions or F-extremals is well-known, see e.g. [GLM 2] and [V 2].

The next lemma, needed in Section 3, shows that Theorem 2.9 can be used in some cases to estimate ω although C is not a subset of G.

2.13. Lemma. *Let* G *be the annulus* $B^n \setminus \bar{B}^n(r_0)$, $0<r_0<1$, *and* $C=\partial B^n(r_0)$. *Then for each* $x \in G$

$$\omega(x) \geqq \begin{cases} b\big(r_0(1-|x|)\big)^a, & \text{if} \quad r_0 \leqq 1/2, \\ b\big((1-r_0)^{-1}(1-|x|)\big)^a, & \text{if} \quad r_0 > 1/2, \end{cases} \tag{2.14}$$

where $\omega=\omega(C, G; F)$ *and* a, b *are the constants of Theorem* 2.9 *depending only on* n *and* α/β.

Proof. Fix $x \in G$. Without loss of generality we may assume $x=te_n$ where e_n is the n-th unit vector of R^n. Let H denote the upper half space of R^n and write $G'=H \cap B^n$ and $C'=H \cap \bar{B}^n(r_0)$. Then C' is a relatively closed subset of G'. If ω' is the F-harmonic measure of C' with respect to G', then Carleman's principle [GLM 3, Lemma 2.8] implies $\omega' \leqq \omega$ in $G' \setminus C'$ and it remains to find the lower bounds in (2.14) for ω'.

By Theorem 2.9

$$\omega'(x) \geqq b e^{-ak(x)} \tag{2.15}$$

where $k(x)=k_{G'}(x, C')$. Let γ be the line segment from $r_0 e_n$ to x. Then

$$k(x) \leqq \int_\gamma \operatorname{dist}(z, \partial G')^{-1}\, ds = \int_{r_0}^{t} \min(1-s, s)^{-1}\, ds \tag{2.16}$$

and the last integral has the following values:

(i) $-\log\big(4r_0(1-t)\big)$ for $r_0 \leqq 1/2 < t$,

(ii) $\log\big((1-r_0)/(1-t)\big)$ for $1/2 \leqq r_0 < t$, and

(iii) $\log(t/r_0)$ for $r_0 < t \leqq 1/2$.

Since for $0<t\leqq 1/2$, $t \leqq (4(1-t))^{-1}$ and hence the upper bound of (i) for $k(x)$ can be used in the case (iii) as well. Thus (2.15) and (2.16) yield the required estimate (2.14).

2.17. Remarks. (a) Let B^{n-1} be the open unit ball of R^{n-1} and let $G=B^{n-1} \times R$ be a circular infinite cylinder in R^n. Set $C=B^{n-1} \times \{0\}$ and fix $x=(0, \dots, 0, t) \in G$ where $t>0$. Then $k_G(x, C) \leqq t$ and hence Theorem 2.9 implies

$$\omega(x) \geqq b \exp(-at).$$

On the other hand, it was shown in [HM, Theorem 3.35] that

$$\omega(x) \leqq M \exp(-a_1 t)$$

where $M<\infty$ is an absolute constant and $a_1>0$ depends only on n and α/β. Hence Theorem 2.9 is essentially the best possible.

(b) It is not possible to choose $b=1$ in Theorem 2.9. For ordinary plane harmonic measure this follows from Theorem 2.24 below.

2.18. Theorem. *Let G, C and ω be as in Theorem 2.9. Then for $x \in G \setminus C$*

$$\omega(x) \geqq 1 - ck_G(x, C)^\delta \tag{2.19}$$

where the constants $c > 0$ and $\delta \in (0, 1)$ depend only on n and α/β.

Proof. First extend ω to $\omega^*: G \to R$ as in Lemma 2.6. Then $\omega^* \in C(G) \cap \operatorname{loc} W_n^1(G)$ is a monotone super-F-extremal in G. Next we can use the proof of Lemma 4.2 in [GLM 1] to obtain

$$\int_{B^n(x_0, r)} |\nabla \omega^*|^n \, dm \leqq c_1(\beta/\alpha) \operatorname{osc}(\omega^*, B^n(x_0, \varrho))^n (\ln(\varrho/r))^{1-n} \tag{2.20}$$

where $0 < r < \varrho$ and $B^n(x_0, \varrho) \subset G$. Here c_1 depends only on n. In fact, the proof given in [GLM 1] can directly be used for sub-F-extremals via the important extremality property [GLM 1, Theorem 5.17, (ii)] of regular sub-F-extremals. Since a function u is a super-F-extremal if and only if $-u$ is a sub-F-extremal, the proof also gives (2.20).

Next the proof of Theorem 4.7 in [GLM 1] together with the inequality (2.20) and the monotonicity of ω^* yields

$$\operatorname{osc}(\omega^*, B^n(x_0, r)) \leqq e(r/\varrho)^\delta \tag{2.21}$$

where e is Neper's number and $\delta = c_2(\alpha/\beta)^{1/n} > 0$. Here c_2 depends only on n and $0 < r < \varrho$ with $B^n(x_0, \varrho) \subset G$.

To prove (2.19) we may assume that $C \neq \emptyset$. Fix $x \in G \setminus C$ and suppose first that

$$k(x) = k_G(x, C) < 1/e.$$

Pick $y \in C$ such that

$$k_G(x, y) \leqq \min(2k(x), 1/e).$$

By [GP, Lemma 2.1]

$$k_G(x, y) \geqq \log\left(1 + \frac{|x-y|}{\operatorname{dist}(y, \partial G)}\right),$$

and since $\log(1+t) \geqq t/2$ for $0 \leqq \log(1+t) \leqq 1/e$, we obtain

$$2k(x) \geqq k_G(x, y) \geqq \frac{|x-y|}{2 \operatorname{dist}(y, \partial G)}.$$

Set $\varrho = \operatorname{dist}(y, \partial G)$ and $r = |x-y|$. Then (2.21) yields

$$\omega^*(y) - \omega^*(x) \leqq \operatorname{osc}(\omega^*, B^n(y, r)) \leqq e(r/\varrho)^\delta \leqq e4^\delta k(x)^\delta$$

and since $\omega^*(y) = 1$ and $\omega^*(x) = \omega(x)$, we obtain the desired estimate (2.19) with $c = e4^\delta$.

If $k(x) \geqq 1/e$, then

$$1 - e4^\delta k(x)^\delta \leqq 1 - e4^\delta e^{-\delta} \leqq 1 - e4^\delta e^{-1} \leqq 0$$

and hence we again obtain (2.19) with the same c and δ as above.

2.22. Remark. The above proof yields

$$\omega(x) \geqq 1-e(|x-y|/\operatorname{dist}(y, \partial G))^{\delta}$$

for each $x \in G \setminus C$ and $y \in C$.

2.23. Corollary. *Let G, C and ω be as in Theorem 2.9. Then for each $x \in G \setminus C$*

$$\omega(x) \geqq \max(1-ck_G(x, C)^{\delta},\ b\exp(-ak_G(x, C)))$$

where the positive constants a, b, c and δ depend only on n and α/β.

For the ordinary plane harmonic measure A. Beurling's projection theorem gives a more precise result than (2.19).

2.24. Theorem. *Let G be a plane domain and C a relatively closed subset of G without compact components. If ω is the ordinary harmonic measure of C with respect to $G \setminus C$, then for $x \in G \setminus C$*

$$\omega(x) \geqq 1-\frac{8}{\pi} k_G(x, C)^{1/2}. \tag{2.25}$$

The exponent $1/2$ is best possible.

Proof. As in the proof of Theorem 2.18 let $x \in G \setminus C$. Let $\varepsilon > 0$ and choose $y \in C$ such that

$$k_G(x, y) < k_G(x, C)+\varepsilon.$$

Assume first that $k_G(x, y) < 1/e$. Then, see the proof for Theorem 2.18,

$$k_G(x, y) \geqq \log\left(1+\frac{|x-y|}{\operatorname{dist}(y, \partial G)}\right) \geqq \frac{1}{2}\frac{|x-y|}{\operatorname{dist}(y, \partial G)}.$$

Let $r=|x-y|$ and $\varrho=\operatorname{dist}(y, \partial G)$. For the rest of the proof we may assume $y=0$, $x=re_1$ and $\varrho=1$. Let ω' be the ordinary harmonic measure of $C \cap B^2$ with respect to $B^2 \setminus C$. By Beurling's projection theorem [B], $\omega'(x) \geqq \omega^*(x)$ where ω^* is the harmonic measure of $C^*=(-e_1, 0]$ with respect to $B^2 \setminus C^*$. On the other hand, it is well known that for each $0<t<1$

$$\omega^*(te_1) = 1-\frac{4}{\pi}\arctan\sqrt{t}. \tag{2.26}$$

By Carleman's principle, $\omega(x) \geqq \omega'(x)$ and hence the above inequalities yield

$$\omega(x) \geqq \omega'(x) \geqq \omega^*(x) = 1-\frac{4}{\pi}\arctan\sqrt{r} \geqq 1-\frac{4}{\pi}\sqrt{r}$$

$$\geqq 1-\frac{4\sqrt{2}}{\pi} k_G(x, y)^{1/2} \geqq 1-\frac{8}{\pi}(k_G(x, C)+\varepsilon)^{1/2}.$$

Letting $\varepsilon \to 0$ we obtain the desired result in the case $k_G(x, C) < 1/e$.

If $k_G(x, C) \geqq 1/e$, then $8/(\pi\sqrt{e}) > 1$ and the estimate (2.25) is trivial.

Letting $\omega=\omega^*$ where ω^* is as above we see from (2.26) that the constant $1/2$ cannot be replaced by any constant $\delta>1/2$.

2.27. Remark. Let G, C and ω be as in Theorem 2.24. It follows from the above proof that

$$\omega(x) \geqq 1-\frac{4}{\pi}(|x-y|/\operatorname{dist}(y, \partial G))^{1/2}$$

and $4/\pi$ and $1/2$ cannot be replaced by any smaller and bigger constant, respectively.

3. Estimates in a ball

The following theorem is a counterpart of an estimate due to T. Carleman [C] and A. Beurling [B] for F-harmonic measures. The proof is based on the proof of Phragmen—Lindelöf's principle in [GLM 3].

3.1. Theorem. *Suppose that C is a closed subset of the unit ball B^n with the property that the spheres $\partial B^n(t)$ meet C for all $0\leqq t<1$. Let ω be the F-harmonic measure of C with respect to $B^n\setminus C$. Then for each $x\in B^n\setminus C$*

$$\omega(x) \geqq c_1(1-|x|)^a \tag{3.2}$$

where $c_1>0$ depends only on n and α/β and a is the constant of Theorem 2.9.

Proof. Let ω' be the F-harmonic measure of ∂B^n with respect to $B^n\setminus C$. Then [GLM 3, Lemma 3.18] yields for each $x\in B^n\setminus C$

$$\omega'(x) \leqq 4\exp\left(-c(\alpha/\beta)^{1/n}\int_{|x|}^{1}\frac{dt}{t}\right) = 4|x|^{\varepsilon} \tag{3.3}$$

where $\varepsilon=c(\alpha/\beta)^{1/n}\leqq 1$ and $c>0$ depends only on n. This was proved in [GLM 3] under the additional assumption that $B^n\setminus C$ is a regular domain but since each component of $B^n\setminus C$ can be approximated from inside by regular domains and since the corresponding F-harmonic measures bound ω' from above and satisfy (3.3), the inequality (3.3) for ω' follows. Unfortunately, (3.3) does not immediately give (3.2) because the right side of (3.3) can be $\geqq 1$. To overcome this difficulty we first pick $r_0\in(0, 1/2]$ such that $4r_0^{\varepsilon}=1/2$, i.e.,

$$r_0 = (1/8)^{1/\varepsilon}. \tag{3.4}$$

Then r_0 depends only on n and α/β. Now (3.3) yields

$$\omega'(x) \leqq \tfrac{1}{2} \tag{3.5}$$

for all $x\in B^n(r_0)\setminus C$.

Next let u belong to the upper class for ω, see 2.1. Then $1-u$ is a sub-F-extremal in $B^n\setminus C$ and

$$\overline{\lim_{x\to y}}\,(1-u(x)) \leqq \chi_{\partial B^n}(y)$$

for all $y\in\partial(B^n\setminus C)$. The F-comparison principle, cf. [GLM 3, Lemma 2.3], yields

$$1-\omega' \leqq \omega$$

in $B^n\setminus C$ and hence, by (3.5), $\omega(x)\geqq 1/2$ for all $x\in B^n(r_0)\setminus C$. To complete the proof let ω^* be the F-harmonic measure of $\partial B^n(r_0)$ with respect to the annulus $G=B^n\setminus \bar{B}^n(r_0)$. By Lemma 2.13

$$\omega^*(x) \geqq b\big(r_0(1-|x|)\big)^a \tag{3.6}$$

where a and b are the constants of Theorem 2.9. Finally, since $2\omega\geqq\omega^*$ in $B^n\setminus(\bar{B}^n(r_0)\cup C)$, we obtain the desired inequality (3.2) from (3.6) with $c_1=br_0^a/2$ and the constant a is the constant of Theorem 2.9. Note that in $B^n(r_0)\setminus C$ this inequality is trivial since $\omega\geqq 1/2$ and

$$b\big(r_0(1-|x|)\big)^a \leqq br_0^a \leqq b/2 \leqq 1/2$$

there.

3.7. Remarks. (a) It follows from Beurling's projection theorem, cf. the proof for Theorem 2.24, that for the ordinary plane harmonic measure ω in the situation of Theorem 3.1

$$\omega(x) \geqq 1-\frac{4}{\pi}\arctan\sqrt{|x|} \geqq \frac{1}{\pi}(1-|x|) \tag{3.8}$$

where the last inequality follows by elementary calculus. Simple examples based on non-lipschitzian quasiconformal mappings and the invariance property [GLM 2, Theorem 5.4] of F-harmonic measures show that in the general situation of Theorem 3.1 it is not possible to choose $a=1$ in (3.2), however, see Theorem 3.13 below.

(b) T. Carleman [C] proved the estimate (3.2) for the ordinary harmonic measure ω under the assumption that C does not have compact components. He made use of the fact that in the plane the open set $B^2\setminus C$ then consists of simply connected domains. This allows the use of special conformal techniques, see [A, p. 42]. The corresponding result can be derived directly from Theorem 2.9 for general F-harmonic measures: *If C is a closed subset of B^n without compact components and if $0\in C$, then in $B^n\setminus C$*

$$\omega(x) \geqq b(1-|x|)^a \tag{3.9}$$

where ω is the F-harmonic measure of C with respect to $B^n\setminus C$ and a and b are the constants of Theorem 2.9.

To prove (3.9) fix $x\in B^n\setminus C$. We may assume $x=te_n$, $0<t<1$. First note that

$$k_{B^n}(x, C) \leqq -\log(1-|x|). \tag{3.10}$$

For this observe that since $0 \in C$,

$$k_{B^n}(x, C) \leqq k_{B^n}(x, 0) \tag{3.11}$$

and if γ is the line segment $[0, x]$, then

$$k_{B^n}(x, 0) \leqq \int_\gamma \operatorname{dist}(z, \partial B^n)^{-1}\, ds = -\log(1-|x|). \tag{3.12}$$

Next let γ: $[0, 1] \to B^n$ be any rectifiable curve (parametrized by arc length) joining x to 0 and let θ be the projection of R^n onto x_n-axis. Then $|\theta'| \leqq 1$ and $\gamma_1 = \theta \circ \gamma$ is rectifiable and joins x to 0 in B^n. Moreover, for $0 \leqq s \leqq 1$

$$\operatorname{dist}(\gamma(s), \partial B^n) \leqq \operatorname{dist}(\gamma_1(s), \partial B^n)$$

and hence

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial B^n)^{-1}\, ds &= \int_0^1 |\gamma'(s)| \operatorname{dist}(\gamma(s), \partial B^n)^{-1}\, ds \\ &\geqq \int_0^1 \frac{|\theta'(\gamma(s))|\,|\gamma'(s)|}{\operatorname{dist}(\gamma_1(s), \partial B^n)}\, ds \geqq \int_{\gamma_1} \operatorname{dist}(z, \partial B^n)^{-1}\, ds \\ &\geqq -\log(1-|x|).\end{aligned}$$

Thus $k_{B^n}(x, 0) \geqq -\log(1-|x|)$ and, by (3.12), $k_{B^n}(0, x) = -\log(1-|x|)$ and the inequality (3.10) follows from (3.11).

The inequality (3.9) now follows from (2.10) and (3.10).

If F is the n-Dirichlet kernel, i.e., $F(x, h) = |h|^n$, then it is possible to derive a lower bound for ω as good as given by Beurling's projection theorem, see (3.8).

3.13. Theorem. *Let C be as in Theorem 3.1 and $F(x, h) = |h|^n$. If ω is the F-harmonic measure of C with respect to $B^n \setminus C$, then for all $x \in B^n \setminus C$*

$$\omega(x) \geqq c_2(1-|x|) \tag{3.14}$$

where $c_2 > 0$ depends only on n.

Proof. As in the proof for Theorem 3.1 we first show that

$$\omega(x) \geqq \tfrac{1}{2} \tag{3.15}$$

for $x \in B^n(r_0) \setminus C$ where $r_0 \in (0, 1/2]$ depends only on n because $\alpha/\beta = 1$ for this special F. Next let $A = B^n \setminus (\bar{B}^n(r_0) \cup C)$ and for $x \in A$ define

$$v(x) = \log|x| / (2 \log r_0).$$

Then v is an F-extremal in A and at each point $y \in \partial A$

$$\varlimsup_{x \to y} v(x) \leqq \varliminf_{x \to y} u(x)$$

where u is any function in the upper class for ω. Thus $\omega \geqq v$ in A and by elementary calculus

$$v(x) \geqq (|x|-1)/(2 \log r_0)$$

in A and the inequality (3.14) follows in A with $c_2=-(2\log r_0)^{-1}>0$. If $x\in \bar{B}^n(r_0)\setminus C$, then (3.14) follows from (3.15) because $r_0\leqq 1/2$ and hence

$$c_2(1-|x|)\leqq (2\log 2)^{-1}(1-|x|)\leqq (4\log 2)^{-1}<\tfrac{1}{2}.$$

The proof is complete.

3.16. Remark. The author conjectures that if $F(x,h)=|h|^n$ and if $C\subset \bar{B}^n$ is such that $C\cap B^n$ is closed in B^n, then

$$\omega(x)\geqq \omega'(|x|e_1)$$

where ω is the F-harmonic measure of C in $B^n\setminus C$ and ω' is the F-harmonic measure of

$$C'=\{-te_1\colon \partial B^n(t)\cap C\neq\emptyset,\ 0\leqq t\leqq 1\}$$

in $B^n\setminus C'$, i.e. Beurling's projection theorem is true for this special F-harmonic measure. Note that Beurling's projection theorem is not needed for the estimate (3.14).

4. Milloux's problem for quasiregular mappings

Suppose that $f\colon G\to R^n$ is a K-quasiregular mapping, see [MRV], and that $|f|\leqq 1$. Let C be a relatively closed subset of G. Then there is a kernel F depending on f such that $-\log|f(x)|$ is a super-F-extremal and $\beta/\alpha\leqq K^2$. For these results see [GLM 1—2]. Let ω be the F-harmonic measure of C with respect to $G\setminus C$; set $\omega(x)=1$ for $x\in C$.

4.1. Lemma. *If $|f(x)|\leqq m$ in C, then*

$$|f(x)|\leqq m^{\omega(x)} \tag{4.2}$$

for all $x\in G$.

Proof. Cf. [GLM 2, Theorem 5.8]. Since $v(x)=-\log|f(x)|$ is a super-F-extremal in G and since for all $y\in\partial(G\setminus C)$

$$\varliminf_{x\to y} v(x)\geqq -(\log m)\chi_C(y),$$

$v\geqq-(\log m)\omega$ in $G\setminus C$. Hence (4.2) holds in $G\setminus C$ and trivially in C.

4.3. Theorem. *Suppose that f is a K-quasiregular mapping of a proper subdomain G of R^n into B^n and that $|f|\leqq m<1$ in the relatively closed subset $C\neq\emptyset$ of G without compact components. Then*

$$|f(x)|\leqq m^{g(x)}$$

where $g\colon G\to(0,1]$ is the function

$$g(x)=\max\left(1-ck_G(x,C)^{\delta},\ b\exp(-ak_G(x,C))\right)$$

and the positive constants a, b, c and δ depend only on n and K.

Proof. The proof follows from Corollary 2.23, Lemma 4.1 and from the fact that $\beta/\alpha \leqq K^2$.

As above Theorem 3.1 and Lemma 4.1 yield

4.4. Theorem. *Let f be a K-quasiregular mapping of B^n into B^n such that $|f| \leqq m < 1$ in the relatively closed subset C of B^n meeting each sphere $\partial B^n(t)$, $0 \leqq t < 1$. Then for all $x \in B^n$*

$$|f(x)| \leqq m^{c(1-|x|)^a}$$

where the positive constants a and c depend only on n and K.

References

[A] Ahlfors, L. V.: Conformal invariants. Topics in geometric function theory. - McGraw-Hill Book Company, New York, 1973.

[B] Beurling, A.: Études sur un problème de majoration. - Thèse, Uppsala, 1933.

[C] Carleman, T.: Sur les fonctions inverses des fonctions entières d'ordre fini. - Ark. Mat. Astr. Fys. 15: 10, 1921.

[GP] Gehring, F. W., and B. Palka: Quasiconformally homogeneous domains. - J. Analyse Math. 30, 1976, 172—199.

[GLM 1] Granlund, S., P. Lindqvist, and O. Martio: Conformally invariant variational integrals. - Trans. Amer. Math. Soc. 277, 1983, 43—73.

[GLM 2] Granlund, S., P. Lindqvist, and O. Martio: F-harmonic measure in space. - Ann. Acad. Sci. Fenn. A I Math. 7, 1982, 233—247.

[GLM 3] Granlund, S., P. Lindqvist, and O. Martio: Phragmén—Lindelöf's and Lindelöf's theorems. - Ark. Mat. 23, 1985, 103—128.

[HM] Heinonen, J., and O. Martio: Estimates for F-harmonic measures and Øksendal's theorem for quasiconformal mappings. - To appear.

[LM] Lindqvist, P., and O. Martio: Regularity and polar sets for supersolutions of certain degenerate elliptic equations. - To appear.

[MRV] Martio, O., S. Rickman, and J. Väisälä: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. A I Math. 448, 1969, 1—40.

[M] Maz'ja, V. G.: The continuity at a boundary point of the solutions of quasilinear elliptic equations. - Vestnik Leningrad. Univ. 25: 13, 1970, 42—55 (Russian).

[R] Rickman, S.: On the number of omitted values of entire quasiregular mappings. - J. Analyse Math. 37, 1980, 100—117.

[V 1] Vuorinen, M.: Capacity densities and angular limits of quasiregular mappings. - Trans. Amer. Math. Soc. 263, 1981, 343—354.

[V 2] Vuorinen, M.: On the Harnack constant and the boundary behavior of Harnack functions. - Ann. Acad. Sci. Fenn. A I Math. 7, 1982, 259—277.

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