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CONNECTIVITY PROPERTIES FOR COMPLEMENTS OF EXCEPTIONAL SETS

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Introduction

We consider a $\mathscr{P}$-harmonic space X with a countable base in the sense of Constantinescu—Cornea [3]. The following problem is studied here: Let X be connected and E an exceptional set of X. Under what circumstances is $X\setminus E$ not connected? It is known that this is never the case if E is polar. But this is possible if E is totally thin, as is shown by examples.

In Section 1 we derive a necessary condition for the complement of a closed totally thin set E to be nonconnected, in terms of absorbent sets and the boundary of $X\setminus E$ [Theorem 1.5]. This condition is never valid in an elliptic space. For the case where the exceptional set is assumed to consist of irregular (or even unstable) boundary points of some open set, we obtain a more precise necessary condition in Section 2 [Corollary 2.9].

1. Connectivity of the complement of a totally thin set

Let X be a Constantinescu—Cornea $\mathscr{P}$-harmonic space with a countable base. For unexplained symbols in this article we refer to [3].

Let $A\subset X$. Then

$$b(A)=\{x\in X\mid A \text{ is not thin at } x\}.$$

Let U be an open set of X. We denote by U_{reg} (respectively U_{ir}) the set of regular (respectively irregular) points of ∂U. Then by [3, Theorem 6.3.3]

$$U_{reg}=\partial U\cap b(X\setminus U),\ U_{ir}=\partial U\setminus b(X\setminus U).$$

It is known that $X\setminus E$ is connected if X is connected and E is a polar set of X [3, Proposition 6.2.5]. In what follows we shall investigate the modified situation if E is totally thin. In this case $X\setminus E$ may be nonconnected, as the following examples show.

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Example 1.1. Let $X=\mathbf{R}^2$ have the harmonic structure corresponding to the heat equation and

$$E = \{(x, t)\in X \mid t = 0\}.$$

Then $X\setminus E$ is not connected and E is totally thin [3, Exercise 6.3.8].

Example 1.2. Let X be the space defined in [3, Exercise 3.2.13] and

$$E = \{(0, 0, 0)\}.$$

Then E is a polar set of the absorbent set A of X:

$$A = \{(x, y, 0)\in X \mid 0 \leqq x^2+y^2 < 1\}.$$

Hence E is totally thin in X [3, Exercise 6.3.13], and $X\setminus E$ is not connected.

Throughout this paper, we shall use the following result of Berg [1, Théorème 1]; see also [3, Exercise 7.2.11]: *A* set which is closed and finely open is absorbent.

Lemma 1.3. *Let X be connected and E a closed set of X such that $X\setminus E=U_1\cup U_2$, with U_1, U_2 open, nonempty, and disjoint. Let $x\in(U_1)_{reg}$ and $x\in(U_2)_{reg}$. Then $x\in(X\setminus E)_{reg}$.*

Proof. Let $U'=X\setminus E=U_1\cup U_2$ and $f\in\mathscr{K}(\partial U')$ be arbitrary. Then

$$h = \begin{cases} H_f^{U_1} & \text{on} \quad U_1 \\ H_f^{U_2} & \text{on} \quad U_2 \end{cases}$$

is harmonic on U' and $\lim_{U'\ni y\to x} h(y)=f(x)$. Thus $h\leqq H_f^{U'}$, since $h\in\mathscr{U}_f^{U'}$, and $H_f^{U'}\leqq h$, since $h\in\overline{\mathscr{U}}_f^{U'}$, and therefore $\lim_{U'\ni y\to x} H_f^{U'}(y)=f(x)$. The conclusion follows.

Lemma 1.4. *With the notations of Lemma 1.3, let $x\in E$ and E be thin at x. Then either $x\in\mathscr{b}(U_1)\cap(X\setminus\mathscr{b}(U_2))$ or $x\in\mathscr{b}(U_2)\cap(X\setminus\mathscr{b}(U_1))$.*

Proof. If $x\in\partial U_1\cap U_2$, then $x\in\mathscr{b}(U_2)$. If now $x\in\mathscr{b}(U_1)$, then x is in the fine closure of U_1. Hence $x\in\overline{U}_1$ and so $U_1\cap U_2\neq\emptyset$, a contradiction. Therefore we may assume $x\in\partial U_1\cap\partial U_2$.

Since E is thin at x, x is not a regular point for $X\setminus E$. Thus by Lemma 1.3 either $x\notin(U_1)_{reg}$ or $x\notin(U_2)_{reg}$. Let $x\notin(U_1)_{reg}$. Then $X\setminus U_1$ is thin at x. We must have $x\in\mathscr{b}(U_1)$, otherwise $X=U_1\cup(X\setminus U_1)$ would be thin at x, by [3, Theorem 6.3.1] a contradiction. Since $U_2\subset X\setminus U_1$ and $X\setminus U_1$ is thin at x, we have $x\in X\setminus\mathscr{b}(U_2)$.

By a *local non-trivial absorbent set* of X we mean hereafter an absorbent set with respect to some open subspace of X, which does not reduce to a union of components of this subspace.

Theorem 1.5. *Let X be connected and E a closed totally thin set. If $X\setminus E$ is not connected, there exists a local non-trivial absorbent set whose boundary intersects E in a set containing points interior to E in its relative topology.*

Proof. There exist nonempty open sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$ and $X \setminus E = U_1 \cup U_2$. We can assume that $E = \partial\bar{U}_1 \cap \partial\bar{U}_2$. If this is not the case, denote $E' = E \cap \partial\bar{U}_1$, or, equivalently, $E' = E \cap \partial\bar{U}_2$. Then $X \setminus E'$ is not connected. If the assertion can be proved for E', it also holds for E, since $E' \subset E$.

Let

$$A_1 = \mathscr{b}(U_1) \cap E, \quad A_2 = \mathscr{b}(U_2) \cap E.$$

Then $E = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ by Lemma 1.4. The sets A_1, A_2 are G_δ-sets in the relative topology of E [3, Corollary 7.2.1]. Let $\{G_m\}_{m \in \mathbf{N}}$, $\{H_n\}_{n \in \mathbf{N}}$ be open sets in E with

$$A_1 = \bigcap\nolimits_{m \in \mathbf{N}} G_m, \quad A_2 = \bigcap\nolimits_{n \in \mathbf{N}} H_n.$$

We shall prove that either A_1 or A_2 contains interior points. Assume the contrary. Then A_1 and A_2 both are dense on E. Hence G_m and H_n are dense open sets of E for every $m, n \in \mathbf{N}$. Since E is a Baire space, by local compactness,

$$\left(\bigcap\nolimits_{m \in \mathbf{N}} G_m\right) \cap \left(\bigcap\nolimits_{n \in \mathbf{N}} H_n\right) = A_1 \cap A_2$$

is dense on E, a contradiction.

So there exists an open non-empty subset D of E, for example, with $D \subset A_1 \setminus A_2$. Then $D \subset \mathscr{b}(U_1)$ and $D \cap \mathscr{b}(U_2) = \emptyset$. Thus D is contained in the fine interior of $\bar{U}_1$. Using the result of Berg [1, Théorème 1] we see that $\bar{U}_1 \cap ((X \setminus E) \cup D)$ is an absorbent set of $(X \setminus E) \cup D$. Since $E = \partial\bar{U}_1 \cap \partial\bar{U}_2$, we have $D \subset \bar{U}_1 \cap \overline{(X \setminus \bar{U}_1)}$. Thus D is the boundary of this absorbent set. This is consequently non-trivial.

Corollary 1.6. *Let X be connected and elliptic. If E is a closed totally thin set, then $X \setminus E$ is connected.*

For an open subset U of X, the set U_{ir} is semi-polar [3, Corollary 7.2.2]. There is a better result that holds for closed subsets of ∂U containing only points of U_{ir}.

Lemma 1.7. *Let U be an open subset of X and S a closed subset of ∂U with $S \subset U_{ir}$. Then S is totally thin.*

Proof. Since $X \setminus U$ is thin at every point of U_{ir}, S is thin at every point of U_{ir}. Also, S is thin at every point of U and at every point of $X \setminus \bar{U}$. Finally, let $x \in U_{reg}$. Since S is closed in ∂U, x has a neighbourhood which does not intersect S. Hence S is thin at x. Thus S is totally thin.

Corollary 1.8. *Let U be an open set of X and B an open set of $\partial\bar{U}$ with $B \subset U_{ir}$. If there exists no nontrivial local absorbent set whose boundary intersects B, then $B = \emptyset$.*

Proof. Let $B \neq \emptyset$ and Y an open, connected set of X such that $Y \cap \partial\bar{U}$ is an open set of $\partial\bar{U}$ and $B \supset Y \cap \partial\bar{U}$. Thus $Y \cap \partial\bar{U} \subset U_{ir}$, $Y \cap \partial\bar{U}$ is closed in Y,

$$Y \setminus \partial\bar{U} = (Y \setminus \bar{U}) \cup (U \cap Y),$$

and by the assumption both sets on the right are nonempty. This implies that $Y \setminus \partial\bar{U}$ is not connected. But $Y \cap \partial\bar{U}$ is totally thin by Lemma 1.7. By Theorem 1.5 we obtain $B = \emptyset$.

2. On unstable and irregular points

In this section we shall investigate the role of irregular points more closely.

Let U be an open relatively compact set of X. Any point of ∂U which lies in the fine closure of $X\setminus\bar{U}$ is a *stable* point of U [2, p. 102], [4, Theorem 4.3]. The set of stable points of U is denoted by U_s. Then by [2, p. 102]

$$U_s = \partial U \cap \mathscr{b}(X\setminus\bar{U}).$$

Obviously $U_s \subset U_{reg}$. Let $S(U)$ denote the set of real continuous functions on the closure of an open set U which are superharmonic on U and $\mathrm{Ch}_{S(U)}\bar{U}$ denote the Choquet boundary of $S(U)$ [2, p. 87]. Then

$$U_s \subset \mathrm{Ch}_{S(U)}\bar{U} \subset U_{reg}. \tag{2.1}$$

The first inequality was established in [2, Corollary 3.5]. The second one follows from the definitions in [2].

Let S be a subset of $\partial\bar{U}$. We say that the absorbent condition holds for S if every $x\in\partial\bar{U}\setminus\bar{S}$ has an open neighbourhood V with $\partial\bar{U}\cap\bar{V}\subset\partial U\setminus\bar{S}$ such that $\bar{U}\cap V$ is a non-trivial absorbent set of V.

In a global form, this is equivalent to the statement "$\bar{U}\setminus\bar{S}$ is a non-trivial absorbent set of $X\setminus\bar{S}$". This follows from the fact that if the condition holds, then, by the sheaf property of hyperharmonic functions, the function which is 0 on $\bar{U}\setminus\bar{S}$ and ∞ on $(X\setminus\bar{S})\setminus\bar{U}$ is hyperharmonic on X. The converse is obvious.

The set S may be empty. In this case the condition signifies that U is a non-trivial absorbent set of X.

Proposition 2.1. *Either the set U_s is dense on $\partial\bar{U}$, or the absorbent condition holds for U_s.*

Proof. Assume that U_s is not dense on $\partial\bar{U}$. Let V be an open set with $\partial\bar{U}\cap V$ containing only nonstable points of U and $x\in\partial\bar{U}\cap V$. Then x is not in the fine closure of $X\setminus\bar{U}$. Since $x\in\bar{U}$, x must be a fine interior point of $\bar{U}$. This holds for every $x\in\partial\bar{U}\cap V$. Thus $V\cap\bar{U}$ is finely open in V and hence an absorbent set in V. Since $\partial(V\cap\bar{U})\neq\emptyset$, it is non-trivial.

Corollary 2.2. *Either the set $\mathrm{Ch}_{S(U)}\bar{U}$ (respectively U_{reg}) is dense on $\partial\bar{U}$, or the absorbent condition holds for $\mathrm{Ch}_{S(U)}\bar{U}$ (respectively U_{reg}).*

Proof. Obvious by (2.1).

Corollary 2.3. *If X is elliptic, the sets U_s, $\mathrm{Ch}_{S(U)}\bar{U}$ and U_{reg} are dense on $\partial\bar{U}$.*

Remark 2.4. Corollary 2.3 generalizes a result for U_{reg} in [5, Théorème 8.2]; see also [3, Exercise 3.1.16]. The corresponding result for U_s was proved in another way in [7, Corollary 3].

Proposition 2.5. *Let B be an open subset of $\partial\overline{U}$. Then $B\subset\partial\overline{U}\setminus U_s$ if and only if the absorbent condition holds for $\partial\overline{U}\setminus B$.*

Proof. By Proposition 2.1 we only have to prove that if $x\in B$, with V open such that $x\in V\cap\partial\overline{U}\subset B$ and $V\cap\overline{U}$ an absorbent set of V, then $x\notin U_s$. Since $V\cap\overline{U}$ is finely open, x is not in the fine closure of $X\setminus\overline{U}$. The conclusion follows.

An irregular point $x\in\partial U$ is *semi-regular* if for any $f\in\mathscr{C}(\partial U)$ there is a limit $\lim_{U\ni y\to x} H_f^U(y)$ [6, p. 357]. The set of semi-regular points of U is denoted by U_{sem}. It was proved in [6, Corollary 5] that U_{sem} is open in ∂U.

Corollary 2.6. *The absorbent condition holds for $\partial\overline{U}\setminus U_{sem}$. Consequently, if X is elliptic, there exist no semi-regular points on $\partial\overline{U}$.*

Proof. Since $U_{sem}\cap\partial\overline{U}$ is open and $U_{sem}\subset U_{ir}$,

$$U_{sem}\cap\partial\overline{U}\subset\partial\overline{U}\setminus U_s.$$

Thus the conclusion follows from Proposition 2.5.

The converse of Corollary 2.3 does not hold: There exist nonelliptic spaces where the set U_s is dense on $\partial\overline{U}$ for every open relatively compact U, as shown by the following example.

Example 2.7. Let X' be the harmonic space defined in [3, Exercise 3.2.11]. Then

$$X:=\{(x,y)\in X'\,|\,y\geqq 0\}$$

is an absorbent set of X' and can be regarded as a harmonic space with the corresponding hyperharmonic sheaf [3, Exercise 6.1.8]. Let

$$Y:=\{(x,y)\in X\,|\,y=0\};$$

Y is also an absorbent set of X. Let $U\subset X$ be open and relatively compact. Then $(\partial\overline{U}\cap Y)_{int}=\emptyset$ if U is seen as a subset of X. The set U_s must be dense on $\partial\overline{U}$, since the only nontrivial local and global absorbent sets of X are contained in Y.

Remark 2.8. All the results of this section only apply to those boundary points of U that are located on $\partial\overline{U}$. For example, if X is elliptic, U_{reg} is not always dense on ∂U, as can be seen by taking U such that $\partial U\cap(\overline{U})_{int}$ contains isolated polar points. For every U, we even obtain $\partial U\cap(\overline{U})_{int}\subset\partial U\setminus U_s$.

By Lemma 1.7 every closed subset of U_{ir} is totally thin. Those totally thin sets which are closed and contain only irregular points for some open set can be regarded as a special class of exceptional sets. The connection between the results of Sections 1 and 2 is made clear by the following corollary.

Corollary 2.9. *Let X be connected and E a closed set of X such that $X\setminus E=U_1\cup U_2$, with U_1, U_2 open and nonempty such that $U_1\cap U_2=\emptyset$. If $E\cap\partial\overline{U}_1$ is a totally thin set containing only irregular points of U_1, then $\overline{U}_1$ is a non-trivial ab sorbent set of X.*

Proof. Since all irregular points are unstable, the assertion follows from Proposition 2.1.

Remark 2.10. Corollary 2.9 remains valid if $E \cap \partial \bar{U}_1$ contains only unstable points of U_1. In this case $E \cap \partial \bar{U}_1$ is not necessarily totally thin.

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