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ON THE DIFFERENCE QUOTIENTS OF AN ANALYTIC FUNCTION

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Abstract. For domains $D, R \subset \mathbf{C}$ we define

$$E(D, R) = \{(f(b)-f(a))/(b-a)\colon\ f'(D) \subseteqq R,\ a, b \in D\}.$$

The set $E(D, R)$ is determined for certain classes of convex domains D (including disks) and thin convex ring domains R. As a special case we obtain sharp bounds for the global deformation effected by conformal quasi-isometric mappings with small strain in such convex domains.

Introduction

If f is a real-valued differentiable function on an open interval $J \subseteqq \mathbf{R}$ and $f'(J) \subseteqq E$, then by the mean value theorem $f[b, a] = (f(b)-f(a))/(b-a) \in E$ for any pair a, b of points of J. As is well known, this is not the case for analytic functions. Indeed, if $R \subseteqq \mathbf{C}$ is any nonconvex domain, then there exist functions f analytic on the unit disk Δ for which $f'(\Delta) \subseteqq R$, but for which $f[b, a]$ is not in R for some b, a in Δ. For any such nonconvex region R it is therefore reasonable to ask what values the difference quotients $f[b, a]$ of an analytic function f on Δ can take when the values of $f'(z)$ are constrained to lie in R. More generally, one can consider

$$E(D, R) = \{f[b, a] : f'(D) \subseteqq R,\ a, b \in D\},$$

for any two regions D and R. In this paper we determine $E(D, R)$ for certain classes of convex D and thin ring-shaped R, but before explaining in detail exactly which regions we consider we motivate what is to follow with some heuristic considerations, and in the course of doing so set down some of the notation to be used in subsequent sections of this paper. The author hopes that the index of notation to be found at the very end will be of help to the reader.

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As will be seen later on, for the regions R we deal with, there is no essential difference between what happens in the general case of bounded convex D with smooth boundary of positive curvature and the particular case of $D=\Delta$, so that in this initial discussion we restrict ourselves to the latter; for convenience we abbreviate $E(\Delta, R)$ by $E(R)$. In addition, the reader might find this introduction easier to follow if he keeps in mind the special and perhaps most significant case in which

$$R = A_M = \{z\colon 1 < |z| < M\}, \quad M > 1.$$

It is easy to see that $E(R) \subseteq \operatorname{conv}(R)$, so that the interesting points of $\partial E(R)$ are those not on $\partial \operatorname{conv}(R)$. For example, when $R=A_M$, and M is not too large, one expects $\partial E(A_M)$ to consist of the circle $|z|=M$ and an interior circle of radius smaller than 1. (For M sufficiently large $E(A_M)$ is the entire disk $|z|<M$; in fact this is the case for any M greater than the so-called John constant, the numerical value of which is not known.) Since for nonconstant f', $f[b, a]$ is a nonconstant analytic function in each variable, the extreme behavior of this expression is to be expected for $a, b \in \partial\Delta$. However, it appears that for the regions R that we consider extreme behavior is in addition only manifested as $|b-a| \to 0$, so that one is naturally led to a corresponding problem for functions in the upper half-plane H. Since the set of values $\{f[B, A]\colon f'(H) \subseteq R\}$ does not depend on which points $A \neq B$ on ∂H are chosen we are motivated to study the functional

$$\text{(0-1)} \qquad \tfrac{1}{2}\int_{-1}^{1} h(z)\,dz, \quad \text{for} \quad h(H) \subseteq R.$$

Function-theoretic intuition suggests that the extreme values of this functional are attained when h is a one-to-one mapping of H onto the universal covering surface $\tilde{R}$ of R, even though purely geometric intuition might suggest otherwise, as we point out below. Let $S=\{z\colon 0<\operatorname{Re} z<1\}$, and let ξ be the one-to-one mapping of H onto S under which the points $-1, 0$ and 1 of ∂H correspond to $+\infty i$, 0 and $-\infty i$ of ∂S, respectively; ξ is given analytically by

$$\xi(z) = \frac{i}{\pi} \ln \frac{z-1}{z+1} + 1,$$

where we understand the branch of $\ln (z-1)/(z+1)$ to be the one which is single-valued outside $[-1, 1]$ and tends to 0 as z tends to infinity. Let $g=g_R$ be any fixed one-to-one mapping of S onto $\tilde{R}$. Then all the one-to-one mappings of H onto $\tilde{R}$ are given by $g(\xi(pz+q)+i\alpha)$, where $p>0$ and $q, \alpha \in \mathbf{R}$. After a simple change of variable we see that the values of the functional in (0-1) as h ranges over the family of one-to-one mappings of H onto $\tilde{R}$ are given by

$$c(A, B, \alpha) = \frac{1}{B-A} \int_A^B g(\xi(z)+i\alpha)\,dz \quad A < B, \ \alpha \in \mathbf{R}.$$

Naturally, only certain values of A, B and α can be expected to yield points on the boundary of $E(R)$. Assuming that $\partial E(R)$ is a smooth curve and that $c_0= c(A_0, B_0, \alpha_0) \in \partial E(R)$, the three partial derivatives of $c(A, B, \alpha)$ at (A_0, B_0, α_0) must all have the direction of the tangent to $\partial E(R)$ at c_0. One is thus led to seek functions $A=A(\alpha)$, $B=B(\alpha)$ defined implicitly by the condition that the three expressions

$$\frac{1}{B-A}\int_A^B g(\xi(z)+i\alpha)\,dz-g(\xi(A)+i\alpha),\quad -\frac{1}{B-A}\int_A^B g(\xi(z)+i\alpha)\,dz+g(\xi(B)+i\alpha)$$

and

$$i\int_A^B g'(\xi(z)+i\alpha)\,dz$$

all be real multiples of each other. Simple manipulations show that this condition is equivalent to

$$\text{(0-2)}\qquad \operatorname{Im}\left\{\frac{\displaystyle\int_A^B \frac{g'(\xi(z)+i\alpha)}{z-1}\,dz}{\displaystyle\int_A^B g'(\xi(z)+i\alpha)\,dz}\right\}=\operatorname{Im}\left\{\frac{\displaystyle\int_A^B \frac{g'(\xi(z)+i\alpha)}{z+1}\,dz}{\displaystyle\int_A^B g'(\xi(z)+i\alpha)\,dz}\right\}=0,$$

provided, of course, that $\int_A^B g'(\xi(z)+i\alpha)dz\neq 0$. I believe that in a great number of cases the part of $\partial E(R)$ not on $\partial R\cap\partial\operatorname{conv}(R)$ consists of a union of curves parametrized by $Z(\alpha)=c(A(\alpha), B(\alpha), \alpha)$, where $A(\alpha)$ and $B(\alpha)$ are defined implicitly by (0-2), and I expect that if complex analysts become interested in this question, a very general theorem to this effect will ultimately be established. In this paper this conjecture is verified for a certain class of thin ring domains R which we now describe.

Let R_1 be a ring domain bounded on the inside and outside by C_0 and C_1, respectively. As is well known, for some $M>1$ there is a one-to-one conformal mapping F of A_M onto R_1 such that the circles $|z|=1$ and $|z|=M$ correspond to C_0 and C_1, respectively. We shall assume that

C_0 is a convex curve which has strictly positive curvature and F''' is bounded in A_m for some $m>1$.

The function $G(z)=F(e^{(\ln M)z})$ maps S onto R_1; more precisely, it maps S onto the universal covering surface of R_1. Also, G is periodic with period $2\pi i/\ln M$. The function $g(z)=g_\varepsilon(z)=G(\varepsilon z)$ consequently maps S onto the universal covering surface of the ring domain R_ε bounded by C_0 and the curve C_ε parametrized by $G(\varepsilon+it)$, $0\leqq t\leqq 2\pi/\ln M$. We shall prove that

0-3. *For any R_1 as above there exists $\varepsilon_0=\varepsilon_0(R_1)>0$ such that if $0<\varepsilon<\varepsilon_0(R_1)$, then there are functions $A=A(\alpha)<-1$ and $B=B(\alpha)>1$ such that*

$$\operatorname{Im}\left\{\frac{\displaystyle\int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{z-1}\,dz}{\displaystyle\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz}\right\}=\operatorname{Im}\left\{\frac{\displaystyle\int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{z+1}\,dz}{\displaystyle\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz}\right\}=0,$$

and which furthermore have the property that

0-4. *If R_1 is as above and D is a bounded convex domain such that ∂D has strictly positive curvature and such that the third derivative of any one-to-one mapping* *of Δ onto D is bounded, then there exists* $\varepsilon_1(D, R_1)>0$ *such that for* $0<\varepsilon<\varepsilon_1(D, R_1)$, $E(D, R_\varepsilon)$ *is bounded on the outside by* C_ε *and on the inside by the curve* C_ε^* *parametrized by*

$$Z(\alpha) = \frac{1}{B(\alpha)-A(\alpha)}\int_{A(\alpha)}^{B(\alpha)} G\bigl(\varepsilon\xi(z)+i\alpha\bigr)\,dz, \quad 0 \leqq \alpha \leqq 2\pi/\ln M.$$

We point out that although the equations of (0-3) alone do not uniquely determine A and B, in Section 1 we give intervals in which our functions $A(\alpha)$ and $B(\alpha)$ are defined in a unique fashion by these equations, so that the formula for $Z(\alpha)$ in (0-4) is quite explicit.

When (0-4) is specialized to the case in which $R_1=A_e$ one has the following, after the symmetry of A_e is taken into account:

Let D be as in (0-4). *Then for all sufficiently small* $\varepsilon>0$, $1\leqq|f'(z)|\leqq e^\varepsilon$ *in* D *implies that*

$$e^\varepsilon \geqq |f[b,a]| \geqq \frac{1}{2B}\int_{-B}^{B} e^{\varepsilon\xi(z)}\,dz \quad \textit{for all} \quad a, b\in D,$$

where $B>1$ *is the largest number for which*

$$\operatorname{Im}\left\{\int_{-B}^{B}\frac{e^{\varepsilon\xi(z)}}{z-1}\,dz\right\} = 0.$$

Moreover, these bounds are sharp.

Thus this special case of (0-4) yields sharp bounds for the global deformation effected by conformal quasi-isometric mappings with small strain in any convex domain of the type described. (The reader is referred to [3] for a discussion of quasi-isometric mappings.) It is of interest that the extremal mapping for the lower bound, $f(z)=\int_0^z e^{\varepsilon\xi(\zeta)}\,d\zeta$, maps H onto a domain whose boundary contains two spirals, the appearence of which might run counter to geometric intuition since they seem to imply a wasteful use of the limited stretching imposed by the condition $1\leqq|f'(z)|\leqq e^\varepsilon$.

In the hope of making the subsequent sections easier to follow, we now explain the basic idea of how (0-4) is to be proved. First of all, it is clear that for sufficiently small ε the curve C_ε is convex so that for such ε it forms part of the boundary of $E(R_\varepsilon)$. The difficulty lies in showing that the rest of the boundary is made up of C_ε^*. We begin in Section 1 by identifying intervals for $A(\alpha)$ and $B(\alpha)$ in which they are well defined by the equations in (0-3) and in the course of doing so we show that $-1-A(\alpha)$ and $B(\alpha)-1$ are of the form $\exp\bigl(-\pi\sqrt{2/\varepsilon r(\alpha)}(1+O(\sqrt{\varepsilon}))\bigr)$, where $r(\alpha)=\operatorname{Re}\{G''(i\alpha)/G'(i\alpha)\}$. Among other things, this allows us to conclude that, as one would expect, the curve C_ε^* is convex. The core of the proof of (0-4) consists of showing that

$$\frac{1}{B-A}\int_A^B G\bigl(\varepsilon\eta(z)+i\alpha\bigr)\,dz, \quad \bigl(A = A(\alpha), B = B(\alpha)\bigr)$$

lies outside of C_ε^* for any function η on H which satisfies $0 \leqq \mathrm{Re}\,\{\eta(z)\} \leqq 1$ and $\mathrm{Im}\,\{\eta(i)\}=0$. Since, as is shown in Section 1, the direction of the outward pointing normal to C_ε^* at $Z(\alpha)$ is that of $\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz$, we can do this by showing that

$$\text{(0-5)}\qquad \mathbf{G}(\eta,\alpha) = \mathrm{Re}\left\{\frac{\int_A^B G(\varepsilon\eta(z)+i\alpha)\,dz-(B-A)Z(\alpha)}{\int_A^B G'(\varepsilon\eta(z)+i\alpha)\,dz}\right\} > 0,$$

unless $\eta=\xi$, in which case the value of this expression is obviously 0. To prove (0-5), we write η as $\xi+\delta$, so that by the definition of ξ, δ is given by the Poisson integral formula,

$$\delta(z) = \mathrm{PI}\,(u) = \frac{i}{\pi}\int_{-\infty}^{\infty} P(z,t)u(t)\,dt,$$

for some u satisfying

$0 \leqq u(t) \leqq 1$ a.e. on $(-1,1)$ and $-1 \leqq u(t) \leqq 0$ a.e. on $(-\infty,-1)\cup(1,\infty)$, where

$$P(z,t) = \frac{1+zt}{(z-t)(1+t^2)} = \frac{1}{z-t}+\frac{t}{1+t^2}.$$

It follows from our assumption about F that $G'''(z)$ is bounded in $\{z\colon 0<\mathrm{Re}\,z<\varrho\}$ for any $\varrho<1$, so that

$$G(\varepsilon\eta(z)+i\alpha) = G(\varepsilon\xi(z)+i\alpha)+G'(\varepsilon\xi(z)+i\alpha)\varepsilon\delta(z)$$
$$+\tfrac{1}{2}G''(\varepsilon\xi(z)+i\alpha)\varepsilon^2\delta^2(z)+\varepsilon^3 O(\delta^3(z)),$$

where the constant implied by the big-O depends only on R_1. Hence the left-hand side of (0-5) is given by

$$\text{(0-6)}\qquad \mathbf{G}(\eta,\alpha) = \varepsilon\int_{-\infty}^{\infty} V(t,\alpha)u(t)\,dt$$
$$+\frac{\varepsilon^2}{2}\,\mathrm{Re}\left\{\frac{\int_A^B G''(\varepsilon\xi(z)+i\alpha)\delta^2(z)\,dz}{\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz}\right\}+\varepsilon^3\int_A^B O(\delta^3(z))\,dz,$$

where

$$V(t,\alpha) = \mathrm{Re}\left\{\frac{i}{\pi}\frac{\int_A^B G'(\varepsilon\xi(z)+i\alpha)P(z,t)\,dz}{\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz}\right\} = \mathrm{Re}\left\{\frac{\frac{i}{\pi}\int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{z-t}\,dz}{\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz}\right\}.$$

(The denominator has not been written explicitly in the ε^3 term since, as follows from the results of Section 1, it is bounded away from 0, and consequently may be implicitly included in the integrand $O(\delta^3(z))$.) It is relatively easy to show that $V(t,\alpha)$ has the necessary sign (positive on $(-1,1)$ and negative outside $[-1,1]$), but since it vanishes at ± 1 and moreover is of order ε outside of any neighborhood of $[A,B]$, the proof of (0-4) based on (0-6) is not straightforward and requires a fairly careful analysis of the properties of $V(t, \alpha)$, to be given in Section 2, and considerable manipulation of the integral in the ε^2 term. Fortunately, the ε^3 term can be dispensed with relative ease. In Section 3 we derive from (0-6) a lower bound for $\mathbf{G}(\eta, \alpha)$ which is strictly positive when u is not 0 a.e. on $\mathbf{R}$. Finally, in Section 4 we use this lower bound for functions in the half-plane H to prove (0-4).

The method used in Section 3 to establish the lower bound for $\mathbf{G}(\eta, \alpha)$ represents a refinement of that used in [1, Section 2] to obtain a lower bound for a similar but simpler functional. The arguments presented at the end of that paper can easily be modified to show how the determination of $\partial E(D, R_\varepsilon)$ yields sharp first order univalence criteria for functions in D.

In what is to follow we use the big-O and big-Ω notation in the following sense. If X and Y are two expressions, then the statement $X=O(Y)$ $(X=\Omega(Y))$ shall mean that there are positive constants ε_1 and T which depend only on R_1 and D such that $|X|\leqq T|Y|$ $(X\geqq TY)$ for all $\varepsilon\in(0, \varepsilon_1)$. (Actually, in Sections 1, 2 and 3 these constants only depend on R_1.) Thus, for example, by our assumption about R_1, $G'(\varepsilon\xi(z)+i\alpha)$, $G''(\varepsilon\xi(z)+i\alpha)$ and $G'''(\varepsilon\xi(z)+i\alpha)$ are all $O(1)$ and $|G'(\varepsilon\xi(z)+i\alpha)|$ is $\Omega(1)$. The reader should take note of these facts since they are used many times in the sequel without further comment.

The reader is advised that throughout x and t are used to denote real variables, so that in particular integration with respect to either of these variables is to be understood as being performed over the relevant subset of $\mathbf{R}$.

Existence of and estimates for $A(\alpha)$ and $B(\alpha)$

First of all, we observe that the assumption that the curvature of C_0 be strictly positive simply means that

$$r(\alpha) = \operatorname{Re}\left\{\frac{G''(i\alpha)}{G'(i\alpha)}\right\} \geqq r_0 > 0, \quad \alpha\in\mathbf{R}. \tag{1-1}$$

To facilitate the analysis we employ the following notation:

$$D(A, B, \alpha) = \int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz,$$

$$N^-(A, B, \alpha) = \int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{z-1}\,dz,$$

$$N^+(A, B, \alpha) = \int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{z+1}\,dz,$$

In order to establish the existence of $A(\alpha)$ and $B(\alpha)$ which satisfy the equations of (0-3) and for which (0-4) is true we need to analyze D, N^-/D and N^+/D in the vicinity of where it turns out $A(\alpha)$ and $B(\alpha)$ must lie. Let $A=-1-e^{-a/\sqrt{\varepsilon}}$ and $B=1+e^{-b/\sqrt{\varepsilon}}$, where $\pi/2\sqrt{r_1}\leqq a, b\leqq 2\pi/\sqrt{r_0}$, with $r_1=\max\{r(\alpha)\colon \alpha\in\mathbf{R}\}$ and r_0 as in (1-1). Since $\operatorname{Re}\{\xi(x)\}$ is 0 on $(-1,1)$ and is 1 on $\mathbf{R}\setminus[-1,1]$, and since $\operatorname{Im}\{\xi(x)\}=(1/\pi)\ln(|x-1|/|x+1|)$ is an odd function, we have that

(1-2) $$\int_A^B \xi(x)\,dx = O\left(\int_0^{e^{-\pi/2\sqrt{\varepsilon r_1}}} \ln x\,dx\right)$$

$$= O\left(\frac{e^{-\pi/2\sqrt{\varepsilon r_1}}}{\sqrt{\varepsilon}}\right) = O(\varepsilon^n), \quad \text{for any} \quad n>0.$$

Since $B-A=2+O(\varepsilon^2)$, we have

(1-3) $$D(A,B,\alpha) = \int_A^B G'(i\alpha)+G''(i\alpha)\varepsilon\xi(x)+O(\varepsilon^2\xi^2(x))\,dx$$

$$= 2G'(i\alpha)+G''(i\alpha)\varepsilon\int_A^B \xi(x)\,dx+O(\varepsilon^2) = 2G'(i\alpha)(1+O(\varepsilon^2)).$$

Throughout the paper we shall repeatedly have need to integrate over the contour Ξ defined by

(1-4) $$\Xi = [-2,A]\cup\{z\colon |z|=2,\ \operatorname{Im} z\geqq 0\}\cup[B,2], \quad \text{oriented from } A \text{ to } B.$$

Now,

$$2\operatorname{Im}\{N^-(A,B,\alpha)/D(A,B,\alpha)\}$$

$$= \operatorname{Im}\left\{\frac{(1+O(\varepsilon^2))}{G'(i\alpha)}\int_\Xi \frac{G'(i\alpha)+G''(i\alpha)\varepsilon\xi(z)+O(\varepsilon^2\xi^2(z))}{z-1}\,dz\right\}$$

$$= \operatorname{Im}\left\{\int_\Xi \frac{1}{z-1}+\varepsilon\frac{G''(i\alpha)}{G'(i\alpha)}\frac{\xi(z)}{z-1}+O\left(\frac{\varepsilon^2+\varepsilon^3|\xi(z)|+\varepsilon^2|\xi^2(z)|}{|z-1|}\right)dz\right\}.$$

If we take into account that $\xi(z)$ and $(z-1)^{-1}$ are bounded on the semicircular portion of Ξ and that $\xi(x)=O(1/\sqrt{\varepsilon})$ on $\mathbf{R}\setminus[A,B]$, we have that the O-term of this integral contributes $O(\varepsilon\ln(B-1))=O(\sqrt{\varepsilon})$. Since $\operatorname{Re}\{\xi(x)\}=1$ on $\mathbf{R}\setminus[A,B]$, we see that $\varepsilon\operatorname{Re}\left\{\int_\Xi \xi(z)/(z-1)\,dz\right\}=O(\sqrt{\varepsilon})$ also. Using these observations together with the fact that $\operatorname{Im}\left\{\int_\Xi (z-1)^{-1}dz\right\}=-\pi$, we conclude that

(1-5) $$2\operatorname{Im}\{N^-(A,B,\alpha)/D(A,B,\alpha)\}$$

$$= -\pi-\operatorname{Im}\left\{\varepsilon\frac{G''(i\alpha)}{G'(i\alpha)}\frac{i}{\pi}\int_{[-2,A)\cup(B,2]}\frac{\ln\dfrac{|x-1|}{|x+1|}}{x-1}\,dx\right\}+O(\sqrt{\varepsilon})$$

$$= -\pi-\frac{\varepsilon r(\alpha)}{\pi}\int_B^2\frac{\ln(x-1)}{x-1}\,dx+O(\sqrt{\varepsilon}) = -\pi+\frac{r(\alpha)}{2\pi}b^2+O(\sqrt{\varepsilon}).$$

In the same fashion one shows that

(1-6) $$2\operatorname{Im}\{N^+(A,B,\alpha)/D(A,B,\alpha)\} = -\pi+\frac{r(\alpha)a^2}{2\pi}+O(\sqrt{\varepsilon}).$$

In addition, we have

$$\frac{\partial}{\partial b} \operatorname{Im} \{N^-(A, B, \alpha)/D(A, B, \alpha)\}$$

$$= \operatorname{Im} \left\{ \frac{(B-1) N^-(A, B, \alpha) G'(\varepsilon\xi(B) + i\alpha)}{\sqrt{\varepsilon} D^2(A, B, \alpha)} - \frac{G'(\varepsilon\xi(B) + i\alpha)}{\sqrt{\varepsilon} D(A, B, \alpha)} \right\}.$$

Since $N^-(A, B, \alpha) = O(\ln(B-1)) = O(1/\sqrt{\varepsilon})$, from (1-3) it follows that this last expression is

$$\operatorname{Im} \left\{ -\frac{G'(\varepsilon\xi(B) + i\alpha)}{2G'(i\alpha)\sqrt{\varepsilon}} \right\} + O(\sqrt{\varepsilon}).$$

But

$$\frac{G'(\varepsilon\xi(B) + i\alpha)}{\sqrt{\varepsilon}\, G'(i\alpha)} = \frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} \frac{G''(i\alpha)}{G'(i\alpha)} \xi(B) + O(\varepsilon^{3/2} \xi^2(B)).$$

Since $\xi(B) = O(1/\sqrt{\varepsilon})$, and $\operatorname{Re}\{\xi(B)\} = 1$, we have

$$\frac{\partial}{\partial b} \operatorname{Im} \{N^-(A, B, \alpha)/D(A, B, \alpha)\}$$

$$= -\sqrt{\varepsilon} \frac{r(\alpha)}{2\pi} \ln(B-1) + O(\sqrt{\varepsilon}) = \frac{r(\alpha)}{2\pi} b + O(\sqrt{\varepsilon}).$$

Very similar arguments show that

$$\frac{\partial}{\partial a} \operatorname{Im} \{N^+(A, B, \alpha)/D(A, B, \alpha)\} = \frac{r(\alpha)}{2\pi} a + O(\sqrt{\varepsilon}),$$

$$\frac{\partial}{\partial b} \operatorname{Im} \{N^+(A, B, \alpha)/D(A, B, \alpha)\} = O(\varepsilon^2),$$

$$\frac{\partial}{\partial a} \operatorname{Im} \{N^-(A, B, \alpha)/D(A, B, \alpha)\} = O(\varepsilon^2).$$

By the formulas (1-5) and (1-6) and these estimates of the partial derivatives of $\operatorname{Im}\{N^-/D\}$ and $\operatorname{Im}\{N^+/D\}$ with respect to a and b, it follows that the image of the square $\pi/2\sqrt{r_1} \leqq a, b \leqq 2\pi/\sqrt{r_0}$ under the mapping

$$F(a, b) = (\operatorname{Im}\{N^-(A, B, \alpha)/D(A, B, \alpha)\}, \operatorname{Im}\{N^+(A, B, \alpha)/D(A, B, \alpha)\})$$

covers (0, 0) exactly once, so that the equations in (0-3) have exactly one solution there; that is,

1-7. *For sufficiently small $\varepsilon > 0$ there exist solutions $A(\alpha)$ and $B(\alpha)$ of the equations of (0-3). Furthermore, $A(\alpha)$ and $B(\alpha)$ are continuous and periodic* $\left(\textit{of period } \dfrac{2\pi}{\ln M}\right)$ *and $-1-A(\alpha)$ and $B(\alpha)-1$ are both of the form*

$$\exp\left(-\pi \sqrt{\frac{2}{\varepsilon r(\alpha)}}\,(1 + O(\sqrt{\varepsilon}))\right).$$

In what follows we shall frequently abbreviate $A(\alpha)$ and $B(\alpha)$ simply by A and B, respectively. This should cause no confusion.

A routine but tedious calculation using implicit differentiation on the defining relations for $A(\alpha)$ and $B(\alpha)$ given in (0-3), together with the differentiability properties of G shows that $A'(\alpha)$, $B'(\alpha)$, $A''(\alpha)$ and $B''(\alpha)$ are all $O(\exp(-\pi/2\sqrt{\varepsilon r_1}))$. From this it follows that the parametrization $Z(\alpha)$ for C_ε^* of (0-4) satisfies

(1-8)
$$Z'(\alpha) = \frac{A'-B'}{(B-A)^2}\int_A^B G(\varepsilon\xi(z)+i\alpha)\,dz$$
$$+\frac{1}{B-A}\left(B'G(\varepsilon\xi(B)+i\alpha)-A'G(\varepsilon\xi(A)+i\alpha)+i\int_A^B G'(\varepsilon\xi(z)+i\alpha)\,dz\right)$$
$$= iG'(i\alpha)(1+O(\varepsilon)),$$

where we have used the fact that $G'(\varepsilon\xi(z)+i\alpha)=G'(i\alpha)+O(\varepsilon\xi(z))$. In the same way one sees that $Z''(\alpha)=-G''(i\alpha)+O(\varepsilon)$. Thus $\operatorname{Im}\{Z''(\alpha)/Z'(\alpha)\}=r(\alpha)(1+O(\varepsilon))$, so that

1-9. *C_ε^* is convex for $\varepsilon>0$ sufficiently small.*

Finally, we estimate $Z(\alpha)$ itself. We have, by (1-2)

$$Z(\alpha) = \frac{1}{B-A}\int_A^B G(\varepsilon\xi(z)+i\alpha)\,dz$$
$$= \frac{1}{B-A}\int_A^B \left(G(i\alpha)+G'(i\alpha)\varepsilon\xi(x)+\frac{G''(i\alpha)}{2}\varepsilon^2\xi^2(x)\right)dx+O(\varepsilon^2)$$
$$= G(i\alpha)-\frac{\varepsilon^2 G''(i\alpha)}{2\pi^2(B-A)}\int_{-1}^1 \ln^2\frac{|x-1|}{|x+1|}\,dx+O\left(\varepsilon^3+\int_{[A,-1)\cup(1,B]}\varepsilon^2\ln^2\frac{|x-1|}{|x+1|}\,dx\right),$$

so that

(1-10)
$$Z(\alpha) = G(i\alpha)-\varepsilon^2 G''(i\alpha)L+O(\varepsilon^3),$$

where

$$L = \frac{1}{4\pi^2}\int_{-1}^1 \ln^2\frac{|x-1|}{|x+1|}\,dx.$$

We end this section by showing that

1-11. *The direction of $D(A(\alpha), B(\alpha), \alpha)$ is that of the outward pointing normal to C_ε^* at $Z(\alpha)$.*

To show this we must prove that $iD(A, B, \alpha)$ is the direction of the tangent to C_ε^* (in the positive sense) at $Z(\alpha)$. From the formula (1-8) for $Z'(\alpha)$ this will follow if we can show that

$$\int_A^B G(\varepsilon\xi(z)+i\alpha)dz-(B-A)G(\varepsilon\xi(A)+i\alpha)$$

(1-12) and

$$\int_A^B G(\varepsilon\xi(z)+i\alpha)\,dz-(B-A)G(\varepsilon\xi(B)+i\alpha)$$

are real multiples of $iD(A, B, \alpha)$. However, since $iN^+(A, B, \alpha)$ and $iN^-(A, B, \alpha)$ are real multiples of this number, so is

$$\frac{\varepsilon i}{\pi}(N^+(A, B, \alpha)+N^-(A, B, \alpha)) = \int_A^B G'(\varepsilon\xi(z)+i\alpha)\varepsilon z\xi'(z)\,dz$$

$$= BG(\xi(B)+i\alpha)-AG(\xi(A)+i\alpha)-\int_A^B G(\varepsilon\xi(z)+i\alpha)\,dz.$$

Similarly,

$$\frac{\varepsilon i}{\pi}(N^-(A, B, \alpha)-N^+(A, B, \alpha))$$

$$=\int_A^B G'(\varepsilon\xi(z)+i\alpha)\varepsilon\xi'(z)\,dz = G(\varepsilon\xi(B)+i\alpha)-G(\varepsilon\xi(A)+i\alpha)$$

is a real multiple of $iD(A, B, \alpha)$. Since this immediately implies the same for the two expressions in (1-12), (1-11) is proved.

2. Properties of $V(t, \alpha)$

We begin by looking at $\partial V(t, \alpha)/\partial t$. For $t\neq A, B$ we have

$$\frac{\partial V(t,\alpha)}{\partial t} = \operatorname{Re}\left\{\frac{i}{\pi D(A, B, \alpha)}\int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{(z-t)^2}\,dz\right\}$$

$$= \operatorname{Re}\left\{\frac{i}{2\pi}\frac{(1+O(\varepsilon^2))}{G'(i\alpha)}\int_A^B \frac{G'(\varepsilon\xi(z)+i\alpha)}{(z-t)^2}\,dz\right\},$$

by (1-3). For $A<t<B$ we can perform the integration over the contour Ξ of (1-4), so that

$$\frac{\partial V(t,\alpha)}{\partial t} = \operatorname{Re}\left\{\frac{i(1+O(\varepsilon^2))}{2\pi}\int_\Xi \frac{1+\varepsilon\dfrac{G''(i\alpha)}{G'(i\alpha)}\xi(z)+O(\varepsilon^2\xi^2)}{(z-t)^2}\,dz\right\}. \tag{2-1}$$

Since $\int_A^B (z-t)^{-2}dz$ is real, and $\xi(z)$ and $(z-t)^{-1}$ are bounded on the semicircular part of Ξ, and since by (1-7), $\xi(x)=O(1/\sqrt{\varepsilon})$ for $x\in\mathbf{R}\setminus[A, B]$, we have

$$\frac{\partial V(t,\alpha)}{\partial t} = \operatorname{Re}\left\{\frac{-i}{2\pi}\int_{[-2,A)\cup(B,2]} \varepsilon\frac{G''(i\alpha)}{G'(i\alpha)}\frac{\xi(x)}{(x-t)^2}+\frac{O(\varepsilon)}{(x-t)^2}\,dx\right\}+O(\varepsilon)$$

$$= \frac{\varepsilon r(\alpha)}{2\pi^2}\int_{[-2,A)\cup(B,2]}\frac{\ln\dfrac{|x-1|}{|x-1|}}{(x-t)^2}\,dx+O\left(\int_{[-2,A)\cup(B,2)}\frac{\varepsilon}{(x-t)^2}\,dx\right)+O(\varepsilon).$$

From this it follows that

2-2. *There exists a constant* $K=K(R_1)$, $0<K<1$, *such that for* $\varepsilon>0$ *sufficiently small and all* α, $\partial V(t,\alpha)/\partial t$ *is negative on* $[K, B)$ *and positive on* $(A, -K]$.

Furthermore, we have that

2-3. *For sufficiently small* $\varepsilon>0$ *and all* α, $-\partial V(t,\alpha)/\partial t=\Omega(\sqrt{\varepsilon}/(B-1))$ *for* $|t-1|\leqq(B-1)/2$ *and* $\partial V(t,\alpha)/\partial t=\Omega(-\sqrt{\varepsilon}/(A+1))$ *for* $|t+1|=-(A+1)/2$.

To see that (2-3) is true, let $|t-1|\leqq(B-1)/2$. Then

$$\int_{[-2,A)\cup(B,2]}\frac{dx}{(x-t)^2}=O\left(\frac{1}{B-1}\right),$$

and

$$-\frac{\varepsilon r(\alpha)}{2\pi^2}\int_{[-2,A)\cup(B,2]}\frac{\ln\dfrac{|x-1|}{|x+1|}}{(x-t)^2}dx=-\frac{\varepsilon r(\alpha)}{2\pi^2}\int_B^2\frac{\ln(x-1)}{(x-t)^2}dx+O\left(\frac{\varepsilon}{B-1}\right).$$

For $|t-1|\leqq(B-1)/2$ and $x\in(B,2]$, $|x-t|\leqq 3|x-1|/2$, so that

$$-\frac{\varepsilon r(\alpha)}{\pi^2}\int_B^2\frac{\ln(x-1)}{(x-t)^2}dx\geqq-\frac{4\varepsilon r(\alpha)}{9\pi^2}\int_B^2\frac{\ln(x-1)}{(x-1)^2}dx=\Omega\left(\frac{\sqrt{\varepsilon}}{B-1}\right),$$

which proves the first lower bound of (2-3). The lower bound for the interval around $t=-1$ is established in exactly the same fashion.

We now examine $V(t,\alpha)$ itself. For $t\in(A,B)$ we proceed as we did in dealing with $\partial V(t,\alpha)/\partial t$ and obtain

$$\text{(2-4)}\qquad V(t,\alpha)=\operatorname{Re}\left\{\frac{i(1+O(\varepsilon^2))}{2\pi G'(i\alpha)}\int_A^B\frac{G'(\varepsilon\xi(z)+i\alpha)}{z-t}dz\right\}$$

$$=\operatorname{Re}\left\{\frac{i(1+O(\varepsilon^2))}{2\pi}\int_\Xi\left(\frac{1}{z-t}+\varepsilon\frac{G''(i\alpha)}{G'(i\alpha)}\frac{\xi(z)}{z-t}+O\left(\frac{\varepsilon^2\xi^2(z)}{z-t}\right)\right)dz\right\}$$

$$=\frac{1}{2}+\frac{\varepsilon r(\alpha)}{2\pi^2}\int_{[-2,A)\cup(B,2]}\frac{\ln\dfrac{|x-1|}{|x+1|}}{x-t}dx+O\left(\int_{[-2,A)\cup(B,2)}\frac{\varepsilon}{|x-t|}dx\right)+O(\varepsilon),$$

where we have used the fact that $\operatorname{Im}\left\{\int_A^B(z-t)^{-1}dz\right\}=-\pi$ for t in (A,B). Thus,

2-5. *If* $M\in(0,1)$ *is any constant, then for all sufficiently small* $\varepsilon>0$ *and all* α, $V(t,\alpha)=\Omega(1)$ *on* $[-M,M]$.

Since $V(\pm1,\alpha)=0$ by equations (0-3), facts (2-2) and (2-5) imply that $V(t,\alpha)$ has the correct sign in (A,B), that is,

2-6. *For* $\varepsilon>0$ *sufficiently small and all* α, $V(t,\alpha)>0$ *on* $(-1,1)$ *and* $V(t,\alpha)<0$ *on* $(A,B)\diagdown[-1,1]$.

To finish the analysis of $V(t,\alpha)$ for $t\in(A,B)$ we note that from (2-3) and the fact that $V(\pm1,\alpha)=0$ it follows that $|V(t,\alpha)|=\Omega(\sqrt{\varepsilon})$ for $|t-1|=(B-1)/2$ and for

$|t+1| = -(A+1)/2$, so that we conclude from (2-2) and (2-5) that

$$|V(t, \alpha)| = \Omega(\sqrt{\varepsilon}) \tag{2-7}$$

for t in $\left\{t \in (A, B)\colon |t-1| \geqq \frac{B-1}{2} \text{ and } |t+1| \geqq -\frac{A+1}{2}\right\}$.

Lastly, we analyze the behavior of $V(t, \alpha)$ for $t \in \mathbf{R} \setminus [A, B]$. Since for $t \in \mathbf{R} \setminus [A, B]$, $\int_A^B (z-t)^{-1} dz$ is real, from the second expression for $V(t, \alpha)$ in (2-4) it follows that for such t we have

$$V(t, \alpha) = \operatorname{Re}\left\{\frac{i}{2\pi}\int_A^B \left(\varepsilon \frac{G''(i\alpha)}{G'(i\alpha)}\frac{\xi(z)}{z-t} + \frac{O(\varepsilon^2)}{z-t} + O(\varepsilon^2)\frac{\xi(z)}{z-t} + \frac{O(\varepsilon^2 \xi^2(z))}{z-t}\right) dz\right\}.$$

On the contour $[A, A+i] \cup [A+i, B+i] \cup [B+i, B]$, $\xi(z) = O(1/\sqrt{\varepsilon})$, so that, if we integrate over this contour, the contribution of the last three terms of the integrand is $O(\varepsilon(|\ln|t-A|| + |\ln|t-B||))$. Upon taking into account the boundedness of $\operatorname{Re}\{\xi(z)\}$ we have that

$$V(t, \alpha) = -\frac{\varepsilon r(\alpha)}{2\pi^2}\int_A^B \frac{\ln \dfrac{|x-1|}{|x+1|}}{x-t} dx + O(\varepsilon(|\ln|t-A|| + |\ln|t-B||)). \tag{2-8}$$

Let

$$\varkappa(t) = \int_{-1}^1 \frac{\ln\dfrac{|x-1|}{|x+1|}}{x-t} dx = 2\int_0^1 \frac{x \ln\dfrac{|x-1|}{|x+1|}}{x^2-t^2} dx. \tag{2-9}$$

We shall show that

2-10. *For all sufficiently small* $\varepsilon > 0$ *and all* α, $V(t, \alpha) < -9\varepsilon r(\alpha)\varkappa(t)/20\pi^2$, *for* $t \in \mathbf{R} \setminus [A, B]$.

First of all, it follows from (2-8) that there exists an $s_0 > 1$ such that for sufficiently small $\varepsilon > 0$ the inequality of (2-10) holds for $t \in [-s_0, s_0] \setminus [A, B]$. Now, for $|t| \geqq s_0$ we have

$$V(t, \alpha) = \operatorname{Re}\left\{\frac{\varepsilon i}{2\pi}\int_{-1}^1 \frac{G''(i\alpha)}{G'(i\alpha)}\frac{\xi(x)}{x-t} dx\right\} + O\left(\int_{[A,-1)\cup(1,B]} \varepsilon|\xi(x)|\, dx + \varepsilon^2\right)$$

$$= -\frac{\varepsilon r(\alpha)}{2\pi^2}\varkappa(t) + O(\varepsilon^2),$$

since $\xi(x)$ is pure imaginary on $[-1, 1]$. Because $\varkappa(t) > 0$ for $|t| > 1$, it follows that (2-10) is true for $s_0 \leqq |t| \leqq s_1$, for any fixed s_1. Finally, we note that for $|t| \geqq s_0$

$$V(t, \alpha) = \operatorname{Re}\left\{\frac{-i}{\pi t D(A, B, \alpha)} \int_A^B G'(\varepsilon\xi(z)+i\alpha)\left(\sum_{n=0}^{\infty} \frac{z^n}{t^n}\right) dz\right\}$$

$$= \operatorname{Re}\left\{\frac{-i}{\pi D(A, B, \alpha)} \int_A^B G'(\varepsilon\xi(z)+i\alpha) z\, dz\right\} t^{-2}$$

$$+ \sum_{n=3}^{\infty} \operatorname{Re}\left\{\frac{i}{2\pi} \int_A^B (1+O(\varepsilon\xi(x)))(1+O(\varepsilon^2)) x^n\, dx\right\} t^{-n},$$

where we have taken into account the definition of $D(A, B, \alpha)$ given at the beginning of Section 1 and (1-3). However,

$$\operatorname{Re}\left\{\frac{i}{\pi} \int_A^B (1+O(\varepsilon\xi(x)))(1+O(\varepsilon^2)) x^n\, dx\right\}$$

$$= O\left(\varepsilon \int_A^B |\xi(x)|\, x^n\, dx\right) = O\left(\left(\frac{s_0+1}{2}\right)^n \varepsilon\right),$$

since $|A|, |B| \leqq (s_0+1)/2$, for ε sufficiently small. Also,

$$\operatorname{Re}\left\{\frac{-i}{\pi D(A, B, \alpha)} \int_A^B G'(\varepsilon\xi(z)+i\alpha) z\, dz\right\} = \frac{\varepsilon r(\alpha)}{2\pi^2} \int_{-1}^{1} x \ln \frac{|x-1|}{|x+1|}\, dx + O(\varepsilon^2).$$

Thus, for $|t| \geqq s_0$

$$V(t, \alpha) = \left(\frac{\varepsilon r(\alpha)}{2\pi^2} \int_{-1}^{1} x \ln \frac{|x-1|}{|x+1|}\, dx + O(\varepsilon^2)\right) t^{-2} + O(\varepsilon t^{-3}).$$

Since

$$\varkappa(t) = -\left(\int_{-1}^{1} x \ln \frac{|x-1|}{|x+1|}\, dx\right) t^{-2} + O(t^{-3}),$$

by appropriately choosing s_1 we see that the desired bound holds for $|t| \geqq s_1$ and all sufficiently small ε. This finishes the proof of (2-10). It is to be noted that (2-10) implies that $V(t, \alpha)$ has the correct sign (negative) outside of $[A, B]$.

3. A lower bound for $\mathbf{G}(\eta, \alpha)$

This section is devoted to deriving a positive lower bound for the expression $\mathbf{G}(\eta, \alpha)$ defined in (0-5) based on (0-6), as outlined in the introduction. To accomplish this end it is necessary initially to break u up into three parts $u = u_1 + u_2 + u_3$, where $u_i = u\chi_i$, $1 \leqq i \leqq 3$, χ_1, χ_2, and χ_3 being the characteristic functions of the sets

$$X_1 = \left\{t \in (A, B)\colon\ |t-1| \leqq \frac{B-1}{2} \text{ and } |t+1| \leqq -\frac{A+1}{2}\right\},$$

$$X_2 = (A, B) \setminus X_1,$$

$$X_3 = \mathbf{R} \setminus [A, B],$$

respectively. We shall denote $\mathrm{PI}(u_i)$ by δ_i. For a bounded measurable function u on $\mathbf{R}$ we define

$$\|u\| = \int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2}\,dt.$$

We shall show that

$$\mathbf{G}(\eta, \alpha) = \Omega\big(\varepsilon^{3/2}(\gamma(\varepsilon)\|u_1\|^2 + \|u_2\|) + \varepsilon^2\|u_3\|\big), \tag{3-1}$$

where we are using the abbreviation

$$\gamma(\varepsilon) = \gamma(\varepsilon, \alpha) = \exp\left(\pi \sqrt{\frac{2}{\varepsilon r(\alpha)}}\right).$$

We begin by decomposing the right hand side of (0-6) into five parts as follows:

$$\mathbf{G}(\eta, \alpha) = \varepsilon(\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3) + \varepsilon^2(\mathbf{G}_4 + \mathbf{G}_5),$$

where the $\mathbf{G}_i = \mathbf{G}_i(\eta, \alpha)$ are given by

$$\mathbf{G}_1 = \int_{X_1} V(t, \alpha) u_1(t)\,dt,$$

$$\mathbf{G}_2 = \int_{X_2} V(t, \alpha) u_2(t)\,dt,$$

$$\mathbf{G}_3 = \int_{X_3} V(t, \alpha) u_3(t)\,dt + \frac{\varepsilon}{2}\operatorname{Re}\left\{\int_A^B \frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A, B, \alpha)}\delta_3^2(z)\,dz\right\},$$

$$\mathbf{G}_4 = \operatorname{Re}\left\{\int_A^B \frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A, B, \alpha)}(\delta_1(z)+\delta_2(z))\,\delta_3(z)\,dz\right\},$$

$$\mathbf{G}_5 = \operatorname{Re}\left\{\int_A^B \left(\frac{G''(\varepsilon\xi(z)+i\alpha)(\delta_1(z)+\delta_2(z))^2}{2D(A, B, \alpha)} + \varepsilon O(\delta^3(z))\right) dz\right\}.$$

First of all, we note that by (1-7), (2-3) and (2-6) it follows that

$$\mathbf{G}_1 = \Omega(\sqrt{\varepsilon}\,\gamma(\varepsilon)\|u_1\|^2), \tag{3-2}$$

and that by (2-6) and (2-7) it follows that

$$\mathbf{G}_2 = \Omega(\sqrt{\varepsilon}\|u_2\|). \tag{3-3}$$

Next we establish a few estimates that we will need for the analysis of the other terms. In the first place, it follows from the translation to H of the well known theorem of Riesz on the p-norm of the conjugate of a harmonic function in Δ that

3-4. *If* $|u(t)| \leqq 1$ a.e. *on* $\mathbf{R}$ *and* $\delta = \mathrm{PI}(u)$, *then* $\int_{-2}^{2} |\delta(x)|^n dx = O(\|u^n\|) = O(\|u\|)$ *for each* $n \geqq 2$.

We observe that

3-5. *If* $|u(t)| \leqq 1$ a.e. *on* $\mathbf{R}$ *and* $\delta = \mathrm{PI}(u)$, *then* $\int_{\Xi} |\delta(z)|^n |dz| = O(\|u\|)$ *for each* $n \geqq 2$.

To see this, we note that $\int_{[-2,A)\cup(B,2]} |\delta(x)|^n\,dx = O(\|u\|)$ by (3-4). That the part of the integral corresponding to the semicircular portion is also $O(\|u\|)$, follows from the translation to H of the fact that for a function $f\in H^1(\Delta)$, $\int_{-1}^{1} |f(x)|\,dx \leqq (1/2)\int_0^{2\pi} |f(e^{i\theta})|\,d\theta$ (see [2, page 122]).

We also have

$$\text{(3-6)}\qquad \int_{[-2,A)\cup(B,2]} |\delta_1(x)|^n\,dx = O(\gamma(\varepsilon)\|u_1\|^2) \quad \text{for each} \quad n \geqq 2.$$

To justify this we observe that for $x\in(B,2]$

$$|\delta_1(x)| = \left|\frac{1}{\pi}\int_{X_1} \frac{1+xt}{x-t}\,\frac{u_1(t)}{1+t^2}\,dt\right| = O\left(\int_{X_1} \frac{1}{x-t}\,\frac{|u_1(t)|}{1+t^2}\,dt\right) = O\left(\frac{\|u_1\|}{x-1}\right)$$

so that

$$\int_B^2 |\delta_1(x)|^n\,dx = O\left(\|u_1\|^n \int_B^2 \frac{dx}{(x-1)^n}\right) = O\left(\frac{\|u_1\|^n}{(B-1)^{n-1}}\right) = O(\gamma^{n-1}(\varepsilon)\,\|u_1\|^n).$$

In the same manner one sees that $\int_{-2}^{A} |\delta_1(x)|^n dx = O(\gamma^{n-1}(\varepsilon)\|u_1\|^n)$. Since $\|u_1\| \leqq B-A-2 = O(1/\gamma(\varepsilon))$, we have (3-6).

We can now derive an upper bound for the term $\mathbf{G}_5$. If we perform the integration over the contour Ξ, we have

$$\begin{aligned}|\mathbf{G}_5| &= O\left(\int_\Xi |\delta_1(z)|^2 + |\delta_2(z)|^2 + \varepsilon\left(|\delta_1(z)|^3 + |\delta_2(z)|^3 + |\delta_3(z)|^3\right)|dz|\right) \\ &= O\left(\|u_2\| + \varepsilon\,\|u_3\| + \int_\Xi |\delta_1(z)|^2 + \varepsilon\,|\delta_1(z)|^3\,|dz|\right),\end{aligned}$$

by (3-5). However, $\delta_1(z) = O(\|u_1\|)$ on the semicircular part of Ξ by the Poisson integral formula. We therefore conclude from (3-6) that

$$\text{(3-7)}\qquad \mathbf{G}_5 = O(\gamma(\varepsilon)\,\|u_1\|^2 + \|u_2\| + \varepsilon\,\|u_3\|).$$

Now we turn to $\mathbf{G}_3$ which is by far the most troublesome of the five terms. Since $\varepsilon\xi(z) = O(\sqrt{\varepsilon})$ on Ξ, we have by (1-3) that

$$\begin{aligned}\mathrm{Re}\left\{\int_\Xi \frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A,B,\alpha)}\,\delta_3^2(z)\,dz\right\} &= \mathrm{Re}\left\{\frac{(1+O(\varepsilon^2))}{2G'(i\alpha)}\int_\Xi (G''(i\alpha) + O(\sqrt{\varepsilon}))\delta_3^2(z)\,dz\right\} \\ &= \mathrm{Re}\left\{\frac{G''(i\alpha)}{2G'(i\alpha)}\int_A^B \delta_3^2(x)\,dx\right\} + O\left(\sqrt{\varepsilon}\int_\Xi |\delta_3(z)|^2\,|dz|\right).\end{aligned}$$

Using the fact that $\delta_3(x)$ is pure imaginary on (A,B) and the case $n=2$ of the bound (3-5), we then conclude that

$$\text{(3-8)}\qquad \frac{\varepsilon}{2}\,\mathrm{Re}\left\{\int_A^B \frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A,B,\alpha)}\,\delta_3^2(z)\,dz\right\} = \frac{\varepsilon r(\alpha)}{4}\int_A^B \delta_3^2(x)\,dx + O(\varepsilon^{3/2}\,\|u_3\|).$$

We examine the integral appearing on the right hand side of this last formula and begin with the part corresponding to $[-1,1]$. To do this we express u_3 as $\sigma+\beta$, where σ and β have support on $\mathbf{R}\setminus[A, B]$ and are even and odd, respectively. Since $-1 \leqq u_3(x) \leqq 0$ a.e. on $\mathbf{R}\setminus[A, B]$, we have

$$-1 \leqq \sigma(x) \leqq 0 \quad \text{and} \quad |\beta(x)| \leqq \min\{|\sigma(x)|, 1/2\}. \tag{3-9}$$

From the parity of σ and β and the fact that $A<-1$ and $B>1$ it follows that outside of $\mathbf{R}\setminus[-1, 1]$, the functions $\delta_\sigma = \mathrm{PI}(\sigma)$ and $\delta_\beta = \mathrm{PI}(\beta)$ are given by

$$\delta_\sigma(z) = \frac{2zi}{\pi}\int_1^\infty \frac{\sigma(t)}{z^2-t^2}\,dt \quad \text{and} \quad \delta_\beta(z) = \frac{2i}{\pi}\int_1^\infty \frac{t(z^2+1)\beta(t)}{(z^2-t^2)(1+t^2)}\,dt.$$

Since δ_σ and δ_β are therefore odd and even, respectively, we see that

$$\int_{-1}^1 \delta_3^2(x)\,dx = \int_{-1}^1 \delta_\sigma^2(x) + \delta_\beta^2(x)\,dx.$$

For $|x|<1$ and $t>1$ the expressions $1/(x^2-t^2)$ and $t(x^2+1)/((x^2-t^2)(1+t^2))$ are both negative, so that for $|x|<1$ we have by (3-9) that

$$|\delta_\sigma(x)| \leqq \frac{2}{\pi}\left|x\int_1^\infty \frac{1}{t^2-x^2}\,dt\right| = \frac{1}{\pi}\left|\ln\frac{1-x}{1+x}\right|$$

and

$$|\delta_\beta(x)| \leqq \frac{1}{\pi}\left|\int_1^\infty \frac{t(x^2+1)}{(t^2-x^2)(1+t^2)}\,dt\right| = \frac{1}{2\pi}\ln\frac{2}{1-x^2}.$$

From this we conclude that

$$\left|\int_{-1}^1 \delta_3^2(x)\,dx\right| \leqq \frac{2}{\pi^2}\int_1^\infty \varkappa(t)|\sigma(t)|\,dt + \frac{1}{\pi^2}\int_1^\infty\left(\int_{-1}^1 \frac{\dfrac{t(x^2+1)}{t^2+1}\ln\dfrac{2}{1-x^2}}{t^2-x^2}\,dx\right)|\sigma(t)|\,dt,$$

where (3-9) has been used again and $\varkappa$ is the function defined in (2-9). Since $u_3(x) \leqq 0$ a.e., it follows from (2-10) that

$$\int_{X_3} V(t, \alpha)u_3(t)\,dt \geqq \frac{9\varepsilon r(\alpha)}{20\pi^2}\left|\int_{X_3} \varkappa(t)u_3(t)\,dt\right|$$

$$= \frac{9\varepsilon r(\alpha)}{20\pi^2}\left|\int_{X_3} \varkappa(t)(\sigma(t)+\beta(t))\,dt\right| = \frac{9\varepsilon r(\alpha)}{10\pi^2}\int_1^\infty \varkappa(t)|\sigma(t)|\,dt,$$

the last equality following from the evenness of $\varkappa$ and σ and the oddness of β. Hence we have that

$$\int_{X_3} V(t, \alpha)u_3(t)\,dt + \frac{\varepsilon r(\alpha)}{4}\int_{-1}^1 \delta_3^2(x)\,dx \tag{3-10}$$

$$\geqq \frac{\varepsilon r(\alpha)}{\pi^2}\int_1^\infty\left(\frac{2}{5}\varkappa(t) - \frac{t}{4(t^2+1)}\int_{-1}^1 \frac{(x^2+1)\ln\dfrac{2}{1-x^2}}{t^2-x^2}\,dx\right)|\sigma(t)|\,dt.$$

In [1, Appendix] it is shown that

$$\varkappa(t) > \frac{t}{t^2+1}\int_{-1}^{1}\frac{(x^2+1)\ln\dfrac{2}{1-x^2}}{t^2-x^2}\,dx,$$

so that since $\sigma(t)=(u_3(t)+u_3(-t))/2$ we have from (3-8) and (3-10) that

$$\text{(3-11)}\qquad \mathbf{G}_3 \geqq \frac{3\varepsilon r(\alpha)}{40\pi^2}\int_{|t|\geqq 1}\varkappa(t)|u_3(t)|\,dt+O\left(\int_{[A,-1]\cup[1,B]}\varepsilon|\delta_3(x)|^2\,dx+\varepsilon^{3/2}\|u_3\|\right).$$

Now we examine the integral in the O-term of this lower bound. Let $s\in(1,3/2)$ be fixed for the moment, and let $v_2=u_3\chi_{\{t:|t|\geqq s\}}$, $v_1=u_3-v_2$, and $\lambda_i=\mathrm{PI}(v_i)$, $i=1,2$. It follows from the Poisson integral formula that $\lambda_2(x)=O(\|v_2\|/(s-1))$ on $[A,B]$, so that

$$\int_{[A,-1]\cup[1,B]}|\lambda_2(x)|^2\,dx = O\left(\frac{1}{\gamma(\varepsilon)}\frac{\|v_2\|^2}{(s-1)^2}\right).$$

Also, by (3-4)

$$\int_{[A,-1]\cup[1,B]}|\lambda_1(x)|^2\,dx = O(\|v_1\|).$$

Thus we have from (3-11) that

$$\text{(3-12)}\qquad \mathbf{G}_3 \geqq \frac{3\varepsilon r(\alpha)}{40\pi^2}\int_{|t|\geqq 1}\varkappa(t)|u_3(t)|\,dt+O\left(\varepsilon\|v_1\|+\frac{\varepsilon}{\gamma(\varepsilon)}\frac{\|v_2\|^2}{(s-1)^2}+\varepsilon^{3/2}\|u_3\|\right).$$

Lastly, we come to $\mathbf{G}_4$. Here again we work with δ_3 expressed as $\lambda_1+\lambda_2$ as above. We have

$$\left|\int_A^B\frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A,B,\alpha)}(\delta_1(z)+\delta_2(z))\lambda_1(z)\,dz\right| = O\left(\int_{\Xi}(|\delta_1(z)|+|\delta_2(z)|)|\lambda_1(z)|\,|dz|\right)$$

$$= O\left(\int_{\Xi}|\delta_1(z)|^2+|\delta_2(z)|^2+|\lambda_1(z)|^2\,|dz|\right).$$

Since we are assuming that $s<3/2$, it follows from the Poisson integral formula that the part of this last integral corresponding to the semicircular portion of Ξ is $O(\|u_1\|^2+\|u_2\|^2+\|v_1\|^2)$. Also, by (3-4) and (3-6) it follows that the rest of the integral is $O(\gamma(\varepsilon)\|u_1\|^2+\|u_2\|+\|v_1\|)$, so that

$$\left|\int_A^B\frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A,B,\alpha)}(\delta_1(z)+\delta_2(z))\lambda_1(z)\,dz\right| = O(\gamma(\varepsilon)\|u_1\|^2+\|u_2\|+\|v_1\|).$$

Since, as observed above, $\lambda_2(x)=O(\|v_2\|/(s-1))$ for $x\in[A,B]$, we have that

$$\left|\int_A^B\frac{G''(\varepsilon\xi(z)+i\alpha)}{D(A,B,\alpha)}(\delta_1(z)+\delta_2(z))\lambda_2(z)\,dz\right| = O\left(\frac{\|v_2\|}{(s-1)}\int_A^B|\delta_1(x)+\delta_2(x)|\,dx\right)$$

$$= O\left(\frac{\|v_2\|\sqrt{\|u_1+u_2\|}}{s-1}\right),$$

by (3-4) and the Schwarz inequality. Thus

$$\mathbf{G}_4 = \left(\gamma(\varepsilon)\|u_1\|^2 + \|u_2\| + \|v_1\| + \frac{\|v_2\| \sqrt{\|u_1+u_2\|}}{s-1}\right).$$

From this together with (3-2), (3-3), (3-7) and (3-12) we have

$$\mathbf{G}(\eta, \alpha) \geqq \Omega\big(\varepsilon^{3/2}(\gamma(\varepsilon)\|u_1\|^2 + \|u_2\|)\big) + \frac{3\varepsilon^2 r(\alpha)}{40\pi^2} \int_{|t|\geqq 1} \varkappa(t)\, |u_3(t)|\, dt$$

$$+ O\left(\frac{\varepsilon^2 \|v_2\|^2}{\gamma(\varepsilon)(s-1)^2} + \varepsilon^2 \|v_1\| + \varepsilon^{5/2}(\|v_1\| + \|v_2\|)\right)$$

$$+ O\left(\varepsilon^2 \gamma(\varepsilon)\|u_1\|^2 + \varepsilon^2 \|u_2\| + \varepsilon^2 \|v_1\| + \frac{\varepsilon^2 \|v_2\| \sqrt{\|u_1+u_2\|}}{s-1}\right).$$

It is easy to see from the formula for $\varkappa$ in (2-9) that $T(s) = \inf\{\varkappa(t)/(1+t^2)\colon 1 < t \leqq s\} \to \infty$ as $s \to 1$. It is also easy to see that $\varkappa(t) = \Omega((1+t^2)^{-1})$ for $|t| \geqq 1$. Thus we have

$$\mathbf{G}(\eta, \alpha) \geqq \Omega\big(\varepsilon^{3/2}(\gamma(\varepsilon)\|u_1\|^2 + \|u_2\|)\big) + \varepsilon^2\left(\frac{3r_0}{40\pi^2} T(s) + O(1)\right) \|v_1\|$$

$$+ \left(\Omega(\varepsilon^2) + O(\varepsilon^{5/2}) + O\left(\frac{\varepsilon^2}{\gamma(\varepsilon)(s-1)^2}\right) + O\left(\frac{\varepsilon^2 \sqrt{\|u_1+u_2\|}}{s-1}\right)\right) \|v_2\|.$$

There exists an s such that the coefficient $(3r_0/20\pi^2)T(s) + O(1)$ of $\|v_1\|$ is positive. Then for this s, $\Omega(\varepsilon^2) + O(\varepsilon^{5/2}) + O(\varepsilon^2/(\gamma(\varepsilon)(s-1)^2))$ is $\Omega(\varepsilon^2)$. There exists a $p > 0$ such that if $\|u_1+u_2\| < p$, this last $\Omega(\varepsilon^2)$ together with the $O(\varepsilon^2 \sqrt{\|u_1+u_2\|}/(s-1))$ is again $\Omega(\varepsilon^2)$. Finally, if $\|u_1+u_2\| \geqq p$, then for sufficiently small ε, $\|u_2\| > p/2$ and $\mathbf{G}(\eta, \alpha) = \Omega(\varepsilon^{3/2})$. This establishes the desired lower bound (3-1).

We end this section by observing that as a consequence of (0-6) and (2-5) it follows that

3-14. *If* $\varphi > 0$ *is any constant, then* $\mathbf{G}(\eta, \alpha) = \Omega(\varepsilon)$ *for* $\|u_2\| \geqq \varphi$.

4. Proof of (0-4)

Let D and R_1 be as in the statement of (0-4). For sufficiently small $\varepsilon > 0$ the curve C_ε which forms the outer boundary of R_ε is convex, so that $E(D, R_\varepsilon)$ is contained inside C_ε. It is very easy to see from the definition of C_ε^* that $E(D, R_\varepsilon)$ contains the ring lying between C_ε and C_ε^*, so that in order to prove (0-4) we only have to show that $E(D, R_\varepsilon)$ lies outside of C_ε^*. Without loss of generality we may assume that $0 \in D$. If for a given $\varepsilon > 0$ there is a function f on D for which $f'(D) \subseteqq R_\varepsilon$ but for which $f[b, a]$ lies inside C_ε^* for some $a', b' \in D$, then

$$1 > s_0 = \inf\{s\colon f[b, a] \text{ is inside } C_\varepsilon^* \text{ for all } a, b \in sD\} > 0.$$

It is easy to see that there then exist $a, b \in \partial(s_0 D)$ such that $f[b, a] \in C_\varepsilon^*$. Thus, all we need to do to prove (0-4) is to show that

4-1. *There exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *and* $f'(D) \subseteqq R_\varepsilon$, *then* $f[b, a]$ *lies outside of* C_ε^* *for all* $a, b \in \partial D$.

We begin by showing that

4-2. *If* $\varrho(\varepsilon) = \sup\{|b-a|:\ a, b \in \partial D,\ f'(D) \subseteqq R_\varepsilon,$ *and* $f[b, a]$ *is inside* $C_\varepsilon^*\}$, *then* $\varrho(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

To see this, say $f'(D) \subseteqq R_\varepsilon$. Then $f'(z) = G(\varepsilon\omega(z) + i\alpha)$, where $0 \leqq \operatorname{Re}\{\omega(z)\} \leqq 1$ in D, $\operatorname{Im}\{\omega(0)\} = 0$ and $\alpha \in \mathbf{R}$. Let q map Δ one-to-one onto D with $q(0) = 0$. We have

$$\frac{1}{b-a}\int_a^b \omega(z)\,dz = \frac{1}{|b-a|}\int_{[a,b]} \omega(z(s))\,ds = \frac{1}{|b-a|}\int_\Phi \omega(q(z(s)))\,|q'(z(s))|\,ds,$$

where Φ is the curve $q^{-1}([a, b])$, and where s denotes arc length. Since by our assumption about D, $\inf\{|q'(z)|:\ z \in \Delta\} > 0$, there exists $\mu(T) > 0$ such that for all pairs of points $a, b \in \partial D$ with $|a-b| \geqq T$, the real part of this last integral is bounded below by

$$\mu(T)\int_0^{2\pi} \operatorname{Re}\{\omega(q(e^{i\theta}))\}\,d\theta.$$

Thus, for all such a, b

$$\operatorname{Re}\left\{\frac{1}{b-a}\int_a^b \omega(z)\,dz\right\} \geqq \nu(T)\int_{\partial D} \operatorname{Re}\{\omega(z)\}\,ds, \tag{4-3}$$

where $\nu(T) > 0$ is again a constant which depends only on T. (It is to be noted that for a given D such a $\nu(T)$ can be calculated explicitly.) We have

$$f[b, a] = \frac{1}{b-a}\int_a^b G(\varepsilon\omega(z) + i\alpha)\,dz \tag{4-4}$$

$$= \frac{1}{b-a}\int_a^b \big(G(i\alpha) + G'(i\alpha)\varepsilon\omega(z) + O(\varepsilon^2\omega^2(z))\big)\,dz$$

$$= G(i\alpha) + \frac{\varepsilon G'(i\alpha)}{b-a}\int_a^b \omega(z)\,dz + \frac{1}{b-a}\int_a^b O(\varepsilon^2\omega^2(z))\,dz.$$

If we take either of the arcs of ∂D joining a and b as the path of integration, it easily follows from the case $p=2$ of Riesz's theorem on the p-norm of the conjugate of a harmonic function in Δ and the fact that $0 \leqq \operatorname{Re}\{\omega(z)\} \leqq 1$, that

$$\left|\frac{1}{b-a}\int_a^b O(\varepsilon^2\omega^2(z))\,dz\right| = O\left(\frac{\varepsilon^2}{|b-a|}\int_{\partial D} \operatorname{Re}\{\omega\}\,ds\right).$$

From (4-3) and (4-4) it therefore follows that for $|b-a| \geqq T$

$$\operatorname{Re}\left\{\frac{f[b, a]-G(i\alpha)}{\varepsilon G'(i\alpha)}\right\} = \left(v(T) - O\left(\frac{\varepsilon}{T}\right)\right) \int_{\partial D} \operatorname{Re}\{\omega\}\, ds \geqq 0,$$

for ε sufficiently small. Since the direction of $G'(i\alpha)$ is that of the outward pointing normal to C_0 at $G(i\alpha)$, there is an $\varepsilon_1(T)>0$ such that for $\varepsilon<\varepsilon_1(T)$, $a, b \in \partial D$ and $|b-a| \geqq T$, the point $f[b, a]$ lies outside of C_0 and therefore outside of C^*_ε. This proves (4-2).

We now begin the proof of (4-1) itself. Let $w(z)$ denote the function $(1+iz)/(1-iz)$ which maps H one-to-one onto Δ. Let $q(z)$ be any fixed one-to-one mapping of Δ onto D. By (4-2) we only need to show (4-1) under the additional assumption that $|b-a| \leqq \varrho(\varepsilon)$. Let $f'(D) \subseteqq R_\varepsilon$ and $a, b \in \partial D$ with $|b-a| \leqq \varrho(\varepsilon)$. By interchanging a and b if necessary, we can assume that in going from a to b along ∂D in the positive sense, the shorter of the two arcs of ∂D connecting these two points is traversed. For ε sufficiently small and each $\alpha \in \mathbf{R}$ there exists a unique $\tau \in (0, 1)$ such that $|w(\tau A(\alpha)) - w(\tau B(\alpha))| = |q^{-1}(a) - q^{-1}(b)|$, and consequently there exists a unique $e^{i\psi}$ such that $q(e^{i\psi} w(\tau A(\alpha))) = a$ and $q(e^{i\psi} w(\tau B(\alpha))) = b$. Furthermore, τ and ψ are continuous periodic functions of α by (1-7), and $\tau \to 0$ uniformly as $\varepsilon \to 0$. Since $G(\varepsilon z)$ maps the strip S onto the universal covering surface of R_ε, $k(\alpha) = \operatorname{Im}\{G^{-1}(f'(q(e^{i\psi} w(\tau i))))\}$ is a continuous periodic function. By the intermediate value theorem there is an α for which $k(\alpha)=\alpha$; that is,

4-5. $q(e^{i\psi} w(\tau A(\alpha))) = a$, $q(e^{i\psi} w(\tau B(\alpha))) = b$ *and* $f'(q(e^{i\psi} w(\tau z))) = G(\varepsilon h(z) + i\alpha)$, *where* $h(z) = \operatorname{PI}(v)$ *with* $0 \leqq v(t) \leqq 1$ *a.e. on* ∂H.

We have $\tau w'(\tau z) = 2i\tau/(1-\tau i z)^2$, so that by the assumption about D we have for z bounded that

$$\tau e^{i\psi} q'(e^{i\psi} w(\tau z)) w'(\tau z) = 2i\tau e^{i\psi} q'(e^{i\psi})(1 + 2i\tau\beta z + O(\tau^2)), \tag{4-6}$$

where $\beta = 1 + e^{i\psi} \dfrac{q''(e^{i\psi})}{q'(e^{i\psi})}$.

Since by hypothesis q''' is bounded on Δ, we have

$$b = q(e^{i\psi} w(\tau B)) = q(e^{i\psi}) + 2ie^{i\psi} Bq'(e^{i\psi})\tau - 2(e^{i\psi} q'(e^{i\psi}) + e^{2i\psi} q''(e^{i\psi}))B^2\tau^2 + O(\tau^3),$$

and a is given by the analogous expression in which B is replaced by A. Thus it follows that

$$\frac{2i\tau e^{i\psi} q'(e^{i\psi})}{b-a} = \frac{1 + O(\tau(B^2 - A^2) + \tau^2)}{B-A} = \frac{1 + O(\tau\varepsilon^2 + \tau^2)}{B-A}, \tag{4-7}$$

where we have used the fact that $B^2 - A^2 = O(\varepsilon^2)$.

It follows from (1-11) that in order to prove (4-1) we have only to show that

$$I = \mathrm{Re}\left\{\frac{1}{(b-a)D(A, B, \alpha)}\left(\int_a^b f'(z) - Z(\alpha)\,dz\right)\right\} > 0.$$

We have, on the one hand,

(4-8) $$I = \mathrm{Re}\left\{\frac{1}{(b-a)D(A, B, \alpha)}\int_A^B (G(\varepsilon h(z)+i\alpha)-Z(\alpha))q'(e^{i\psi}w(\tau z))e^{i\psi}\tau w'(z)\,dz\right\}$$

$$= \mathrm{Re}\left\{\frac{(1+O(\varepsilon^2))}{2G'(i\alpha)(b-a)}\right.$$

$$\left.\times\int_A^B (G'(i\alpha)\varepsilon h(z)+\varepsilon^2 G''(i\alpha)L+O(\varepsilon^2 h^2(z))+O(\varepsilon^3))q'(e^{i\psi}w(\tau z))e^{i\psi}\tau w'(\tau z)\,dz\right\}$$

$$= \mathrm{Re}\left\{\frac{\varepsilon}{2(b-a)}\int_a^b h(\tau^{-1}w^{-1}(e^{-i\psi}q^{-1}(\zeta)))\,d\zeta\right\}$$

$$+\frac{\varepsilon^2 Lr(\alpha)}{2}+O\left(\frac{\tau}{|b-a|}\int_A^B \varepsilon^3|h(x)|+\varepsilon^3+\varepsilon^2|h(x)|^2\,dx\right),$$

where (1-3) and (1-10) have been used. Since (4-7) implies that $\tau/|b-a|=O(1)$ and $\mathrm{Re}\,\{h(\tau^{-1}w^{-1}(e^{-i\psi}q^{-1}(\zeta)))\}\geqq 0$, we have by (3-4) that

$$I \geqq \frac{\varepsilon^2 Lr(\alpha)}{2}+O(\varepsilon^2\|v\|+\varepsilon^3).$$

By (1-1) we have that $r(\alpha)=\Omega(1)$, so that

(4-9) $$I = \Omega(\varepsilon^2)+O(\varepsilon^2\|v\|+\varepsilon^3).$$

We shall use this lower bound when $\|v\|$ is small. For other v we need a different lower bound which we now proceed to derive.

From (4-6) and (4-7) it follows that

(4-10) $$I = \frac{\mathrm{G}(h, \alpha)}{B-A}+\mathrm{Re}\left\{\frac{2i\tau\beta}{(B-A)D(A, B, \alpha)}\int_A^B (G(\varepsilon h(z)+i\alpha)-Z(\alpha))z\,dz\right\}$$
$$+O\left((\tau\varepsilon^2+\tau^2)\int_A^B |G(\varepsilon h(z)+i\alpha)-Z(\alpha)|\,|dz|\right).$$

By (1-3), (1-7) and (1-10),

$$\mathrm{Re}\left\{\frac{2i\tau\beta}{(B-A)D(A, B, \alpha)}\int_A^B (G(\varepsilon h(z)+i\alpha)-Z(\alpha))z\,dz\right\}$$

$$= \mathrm{Re}\left\{\frac{i\tau\beta(1+O(\varepsilon^2))}{2G'(i\alpha)}\int_{-1}^1 (G(\varepsilon h(z)+i\alpha)-Z(\alpha))z\,dz\right\}+O(\tau\varepsilon^2)$$

$$= \mathrm{Re}\left\{\frac{i\tau\beta}{2G'(i\alpha)}\int_{-1}^1 (G'(i\alpha)\varepsilon h(z)+O(\varepsilon^2 h^2(z))+O(\varepsilon^2))z\,dz\right\}+O(\tau\varepsilon^2)$$

$$= \mathrm{Re}\left\{\frac{i\tau\beta\varepsilon}{2}\int_{-1}^1 h(z)z\,dz\right\}+O(\tau\varepsilon^2).$$

Let $v_1=v\chi_{[A,B]}$, $v_2=v-v_1$, and $h_i=\mathrm{PI}(v_i)$, $i=1,2$ (recall that $h=\mathrm{PI}(v)$). We have

$$\int_{-1}^{1} h_1(z)z\,dz = O\left(\int_{-1}^{1} |h_1(x)|\,dx\right) = O\left(\sqrt{\|v_1\|}\right),$$

by (3-4) and the Schwarz inequality. Also,

$$\int_{-1}^{1} h_2(z)\,z\,dz = \int_{(-\infty,A)\cup(B,\infty)} \left(\frac{i}{\pi}\int_{-1}^{1} P(z,t)\,z\,dz\right) v_2(t)\,dt.$$

Now, for $|t|>1$

$$\frac{i}{\pi}\int_{-1}^{1} P(z,t)\,z\,dz = \frac{i}{\pi}\int_{-1}^{1} \frac{z}{z-t}+\frac{tz}{1+t^2}\,dz = \frac{2i}{\pi}\int_{0}^{1}\frac{z^2}{z^2-t^2}\,dz.$$

From this it follows that

$$\operatorname{Re}\left\{\frac{i\tau\beta\varepsilon}{2}\int_{-1}^{1} h_2(z)z\,dz\right\} = \operatorname{Re}\left\{\frac{-\tau\beta\varepsilon}{\pi}\int_{(-\infty,A)\cup(B,\infty)}\left(\int_{0}^{1}\frac{z^2}{z^2-t^2}\,dz\right) v_2(t)\,dt\right\}$$
$$= \Omega\left(\tau\varepsilon\operatorname{Re}\{\beta\}\,\|v_2\|\right) = \Omega(\tau\varepsilon\|v_2\|),$$

since $v_2(t)\geqq 0$ a.e. on $\mathbf{R}$, and by our assumption that the curvature of ∂D is positive, the real part of the expression β given in (4-6) is strictly positive. Thus

$$\operatorname{Re}\left\{\frac{2i\tau\beta}{(B-A)D(A,B,\alpha)}\int_{A}^{B}\left(G(\varepsilon h(z)+i\alpha)-Z(\alpha)\right)z\,dz\right\}$$
$$= \Omega(\tau\varepsilon\|v_2\|)+O\left(\tau\varepsilon\sqrt{\|v_1\|}+\tau\varepsilon^2\right).$$

Since $\int_A^B \left|G(\varepsilon h(x)+i\alpha)-Z(\alpha)\right|dx=O(\varepsilon)$ by (1-10), we have from (4-10) that

$$I = \frac{\mathbf{G}(h,\alpha)}{B-A}+\Omega(\tau\varepsilon\|v_2\|)+O\left(\tau\varepsilon\sqrt{\|v_1\|}+\tau\varepsilon^2+\tau^2\varepsilon\right). \tag{4-11}$$

By (4-9) there exists a number $\mu>0$ such that if $\|v\|\leqq\mu$, then $I=\Omega(\varepsilon^2)$. Since $\mathbf{G}(h,\alpha)\geqq 0$ by (3-1), (4-11) implies that there is a $\varphi>0$ such that if $\|v\|\geqq\mu$ and $\|v_1\|\leqq 2\varphi$, then $I=\Omega(\tau\varepsilon)$. Finally, if $\|v_1\|\geqq 2\varphi$, then for ε sufficiently small $\|v_1\chi_{[-1,1]}\|\geqq\varphi$, so that by (3-14), $\|v_1\|\geqq 2\varphi$ implies that $I=\Omega(\varepsilon)$. This shows that $I>0$ for any f provided that $\varepsilon>0$ is sufficiently small. This finishes the proof of (0-4).

Index of notation

$$A_M = \{z\colon 1<|z|<M\}.$$

conv (R) denotes the convex hull of R.
$D(A,B,\alpha)$ - See definition at the beginning of Section 1.

$E(D,R)=\{f[b,a]\colon f'(D)\subseteq R,\ a,b\in D\}$.

$E(R)=E(\Delta,R)$.

$G(z)$ - See discussion preceding (0-3).
$\mathbf{G}(\eta, \alpha)$ - See definition in (0-5).
H denotes the upper half-plane.
$N^+(A, B, \alpha)$, $N^-(A, B, \alpha)$ - See definitions at the beginning of Section 1.
$O(Y)$ - See explanation given in the next to the last paragraph of the introduction.

$$P(z, t) = \frac{1+zt}{(z-t)(1+t^2)} = \frac{1}{z-t} + \frac{t}{1+t^2}.$$

$$\mathrm{PI}(u) = \frac{i}{\pi} \int_{-\infty}^{\infty} P(z, t) u(t)\, dt.$$

$$r(\alpha) = \mathrm{Re}\left\{\frac{G''(i\alpha)}{G'(i\alpha)}\right\}.$$

$S = \{z\colon 0 < \mathrm{Re}\, z < 1\}$.

$V(t, \alpha)$ - See definition just after (0-6).

$$\|u\| = \int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2}\, dt.$$

$$\gamma(\varepsilon) = \gamma(\varepsilon, \alpha) = \exp\left(\pi \sqrt{\frac{2}{\varepsilon r(\alpha)}}\right).$$

$\Delta = \{z\colon |z| < 1\}$.

$$k(t) = \int_{-1}^{1} \frac{\ln \dfrac{|x-1|}{|x+1|}}{x-t}\, dx.$$

$$\xi(z) = \frac{i}{\pi} \ln \frac{z-1}{z+1} + 1.$$

Ξ is the contour defined in (1-4).
χ_X denotes the characteristic function of the subset X of $\mathbf{R}$.
$\Omega(Y)$ - See explanation given in the next to the last paragraph of the introduction.

References

[1] Gevirtz, J.: On sharp first order univalence criteria. - Indiana Univ. Math. J. 36, 1987, 189—209.
[2] Heins, M.: Complex function theory. - Pure and Applied Mathematics 28. Academic Press, New York—London, 1968.
[3] John, F.: On quasi-isometric mappings. I. - Comm. Pure Appl. Math. 21, 1968, 77—110.

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