HOMEOMORPHISMS OF BOUNDED LENGTH DISTORTION

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1. Introduction

1.1. Let $D$ and $D'$ be domains in the plane $\mathbb{R}^2$, and let $f : D \to D'$ be a homeomorphism. We let $l(x)$ denote the length of a path $x$. If $L \geq 1$ and if

$$l(x)/L \leq l(fx) \leq Ll(x) \quad (1.2)$$

for all paths $x$ in $D$, we say that $f$ is of $L$-bounded length distortion, abbreviated $L$-BLD. In a joint article [MV] of O. Martio and the author, we consider more general BLD maps: discrete open maps of domains of $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying (1.2). For homeomorphisms and, more generally, for immersions, (1.2) is equivalent to the following condition: Every point in $D$ has a neighborhood $U$ such that $f|U$ is $L$-bilipschitz, that is,

$$|x-y|/L \leq |f(x)-f(y)| \leq L|x-y| \quad (1.3)$$

for all $x, y \in U$. For this reason, the BLD immersions are often called locally bilipschitz maps or local quasi-isometries or just quasi-isometries [Jo], [Ge].

The BLD property can also be defined in terms of upper and lower derivatives. Let $L_f(x)$ and $l_f(x)$ be the upper and lower limits of $|f(x+h)-f(x)|/|h|$ as $h \to 0$. Then a homeomorphism $f$ is $L$-BLD if and only if $l_f(x) \equiv 1/L$ and $L_f(x) \equiv L$ for all $x \in D$. In particular, if $f$ is differentiable at $x$, this means

$$|h|/L \leq |f'(x)h| \leq L|h| \quad (1.4)$$

for all $h \in \mathbb{R}^2$.

Every $L$-BLD homeomorphism is $L^2$-quasiconformal, but a quasiconformal map is BLD only if its derivative is a.e. bounded away from 0 and $\infty$.

The purpose of this paper is to identify the domains $D \subset \mathbb{R}^2$ which are BLD homeomorphic to a disk or to a half plane. The corresponding problem for bilipschitz maps was solved in the early eighties by Tukia [Tu_1], [Tu_2], Jerison—Kenig [JK] and Latfullin [La]; see also [Ge]. Their results can be stated as follows: A bounded domain $D$ is bilipschitz homeomorphic to a disk if and only if its boundary $\partial D$ is a rectifiable Jordan curve satisfying the chord-arc condition: There is $c \equiv 1$ such that

$$\sigma(x, y) \equiv c|x-y| \quad (1.5)$$

for all $x, y \in \partial D$; here $\sigma(x, y)$ is the length of the shorter arc of $\partial D$ between $x$ and $y$. The half plane case is similar; then $\partial D$ is a locally rectifiable Jordan curve through $\infty$ satisfying (1.5).

We show that the BLD homeomorphic images of the disk and the half plane can be characterized by a somewhat similar condition. However, the euclidean distance $|x - y|$ in (1.5) must be replaced by the internal distance $\lambda_D(x, y)$, which is the infimum of the lengths of all arcs joining $x$ and $y$ in $D$. Moreover, $\partial D$ need not be a Jordan curve. Hence we shall replace $\partial D$ by the prime end boundary $\partial^* D$. Alternatively, the condition can be expressed in terms of the neighborhood system of $\partial D$ in $D$. An equivalent condition has been considered by Pommerenke [Po].

In a forthcoming paper I shall apply the results of the present paper to show that a bounded domain is BLD homeomorphic to a disk if and only if $D \times R^1$ is quasiconformally equivalent to a ball.

1.6. Notation. If $x \in R^n$ and $r > 0$, $B(x, r)$ is the open ball with center $x$ and radius $r$, and $S(x, r)$ is its boundary sphere. We shall write

$$B(r) = B(0, r), \quad B^n = B(1), \quad S(r) = S(0, r), \quad S^{n-1} = S(1).$$

We let $d(A)$ denote the euclidean diameter of a set $A \subset R^n$, and $d(x, A)$ is the distance between $A$ and a point $x \in R^n$.

## 2. Preliminaries

In this section we introduce the internal chord-arc condition for simply connected domains in $R^2$. In 2.9 we give a modulus estimate needed in Section 3. Since it may have independent interest, it is formulated for an arbitrary dimension $n$.

2.1. Jordan domains. A domain $D \subset R^2$ is a Jordan domain if its boundary $\partial D$ in the extended plane $\tilde{R}^2 = R^2 \cup \{\infty\}$ is a Jordan curve (homeomorphic to a circle). Suppose that $D$ is a Jordan domain and that $\partial D$ is locally rectifiable, that is, every compact arc in $\partial D \setminus \{\infty\}$ is rectifiable. If $a, b \in \partial D \setminus \{\infty\}$, we let $\sigma_D(a, b)$ denote the length of the shorter subarc of $\partial D$ with end points $a$ and $b$. If $x, y \in \tilde{D} \setminus \{\infty\}$, $\lambda_D(x, y)$ will denote the infimum of the lengths of all paths joining $x$ and $y$ in $D$. We say that $D$ has the internal chord-arc property with parameter $c \geq 1$ if

$$\sigma_D(a, b) \leq c \lambda_D(a, b)$$

(2.2)

for all finite boundary points $a, b$ of $D$. We abbreviate this by saying that $D$ is $c$-ICA.

The ordinary chord-arc condition (1.5) clearly implies (2.2). One can show that a Jordan curve through $\infty$ satisfies (1.5) if and only if both components of its complement have the property (2.2). The domain $D = \{(x, y) \in R^2: x < 0 \text{ or } y > |x|^2\}$ satisfies (2.2) but not (1.5).
2.3. Prime ends. We give a brief summary on some known facts on prime ends. In what follows, we assume that $D$ is a simply connected proper subdomain of $\mathbb{R}^2$ which is finitely connected on the boundary. This means that every boundary point of $D$ has arbitrarily small neighborhoods $U$ such that $D \cap U$ has only a finite number of components. Equivalently, $\partial D$ is locally connected. Still equivalently, every QC map $f: B^2 \to D$ has a continuous extension $\tilde{f}: \overline{B}^2 \to \overline{D}$. See [Nä, 3.2] and [Po, 9.8].

The prime ends of such a domain $D$ are always of the first kind and can be defined as equivalence classes of tails. By a tail of $D$ we mean a path $\alpha: [a, b) \to D$ such that $\alpha(t) \to z \in \partial D$ as $t \to b$. The point $z$ is written as $h(\alpha)$. A subtail of $\alpha$ is a restriction to a subinterval $[a_1, b_1)$. If $U$ is a neighborhood of $h(\alpha)$, there is a unique component $W(U, \alpha)$ of $U \cap D$ containing a subtail of $\alpha$. Two tails $\alpha$ and $\beta$ are equivalent if $h(\alpha) = h(\beta)$ and if $W(U, \alpha) = W(U, \beta)$ for every neighborhood $U$ of $h(\alpha)$. The equivalence class $\bar{x}$ of a tail $x$ is a boundary element of $D$, and their collection $\partial^* D$ is the prime end boundary of $D$. The set $D^* = D \cup \partial^* D$ has a natural topology such that $(D^*, \partial^* D)$ is homeomorphic to $(\overline{B}^2, S^1)$. In fact, every QC homeomorphism $f: B^2 \to D$ has a unique extension to a homeomorphism $f^*: \overline{B}^2 \to D^*$. There is a natural continuous impression map $i: D^* \to \overline{D}$, defined by $i(\bar{z}) = h(\alpha)$ for $\bar{z} \in \partial^* D$ and by $i|D = id$. If $D$ is locally connected at a boundary point $z$, $i^{-1}(z)$ consists of a single point, which is often identified with $z$. In particular, if $D$ is a Jordan domain, we can identify $\partial D = \partial D^*$ and $D^* = \overline{D}$.

Suppose that $\alpha$ is a subarc of $\partial^* D$. Then $i|\alpha$ is a path in $\mathbb{R}^2$ and has a well-defined length $l(\alpha) \in (0, \infty]$, called the length of $\alpha$. If $i(u) = \infty$ for at most one $u \in \partial^* D$, then also written as $\infty$, and if $l(\alpha) = \infty$ for every compact arc $\alpha \subset \partial^* D \setminus \{\infty\}$, we say that $\partial^* D$ is locally rectifiable. Equivalently, $l(\alpha)$ can be defined as the infimum of all numbers $\lambda$ such that there is a sequence of arcs $\alpha_j \subset D$ such that (1) $\alpha_j \to \alpha$ in the natural topology of the space of all arcs of $D^*$ and (2) $l(\alpha_j) \to \lambda$.

2.4. The ICA property. Suppose that $D$ is as in 2.3 and that $\partial^* D$ is locally rectifiable. If $u$ and $v$ are finite points in $\partial^* D$, we let $\sigma_D(u, v)$ denote the length of the shorter arc of $\partial^* D$ between $u$ and $v$. Furthermore, let $\lambda_D(u, v)$ be the infimum of the lengths of all paths $\alpha$ joining $u$ and $v$ in $D$. By this we mean that $\alpha$ is an open path which has subpaths representing both $u$ and $v$. One has always $\lambda_D(u, v) \equiv \sigma_D(u, v)$. If there is a constant $c \equiv 1$ such that

\[
(2.5) \quad \sigma_D(u, v) \leq c\lambda_D(u, v)
\]

for all finite $u, v \in \partial^* D$, we say that $D$ is c-ICA.

For Jordan domains $D$, this definition is equivalent to that given in 2.1. The complement of a ray and a disk with a radial slit are ICA non-Jordan domains.

2.6. Remarks. 1. Pommerenke [Po, Theorem 2] considered domains $D$ satisfying the condition

\[
(2.7) \quad \sigma_D(u, v) \leq c\delta_D(u, v)
\]
where \( \delta_D(u, v) \) is the infimum of the diameters \( d(|x|) \) of all paths \( x \) joining \( u \) and \( v \) in \( D \). Since \( \delta_D \equiv \lambda_D \), (2.7) implies (2.5). Conversely, (2.5) implies that (2.7) is true with \( c \) replaced by a constant \( c_1 = c_1(c) \). This follows easily from the results in Section 3. The half plane with an orthogonal slit is \( 2^{1/2} \)-ICA, but satisfies (2.7) only for \( c \geq 2 \).

2. It is possible to characterize the ICA property without mentioning prime ends: Let \( D \subset \mathbb{R}^2 \) be simply connected, \( D \neq \mathbb{R}^2 \). Then \( D \) is \( c \)-ICA if and only if for each pair \( a, b \in \partial D \backslash \{ \infty \} \) and for each \( \varepsilon > 0 \) there is \( r > 0 \) such that if a path \( x \) joins \( x \in D \cap B(a, r) \) and \( y \in D \cap B(b, r) \) in \( D \), there is a path \( y \) joining \( x \) and \( y \) in \( D \cap (\partial D + \varepsilon B^2) \) with \( l(y) \equiv c l(x) + \varepsilon \).

3. One can also show that \( D \) is \( c \)-ICA if and only if the chord-arc condition

\[
\sigma_D(u, v) \equiv c |i(u) - i(v)|
\]

is valid for all \( u, v \in \partial^* D \) which are the ends points of a segmental crosscut of \( D \), that is, the open line segment with end points \( i(u) \), \( i(v) \) lies in \( D \) and represents both \( u \) and \( v \).

2.8. Path families. In Section 3 we shall consider paths \( y \) joining a boundary point \( a \in \partial D \) to a point \( b \in \bar{D} \) in \( D \). Such a path defines an element \( u \in \partial^* D \) with \( i(u) = a \), and we can as well consider \( y \) as a path joining \( u \) to \( b \). If \( \Gamma \) is a family of such paths, the modulus \( M(\Gamma) \) is always well defined. If \( y \) is a path, we let \( |y| \) denote its locus im \( y \).

2.9. Lemma. Let \( t > 0 \) and let \( A \subset \mathbb{R}^n \) with \( d(A) \equiv t \). Let \( \lambda > 0 \) and let \( \Gamma \) be a family of paths in \( \mathbb{R}^n \) such that \( l(y) \equiv \lambda t \) and \( \lambda |y| \cap A \neq \emptyset \) for all \( y \in \Gamma \). Then \( M(\Gamma) \equiv \mu_n(\lambda) \), where \( \mu_n(\lambda) \to 0 \) as \( \lambda \to \infty \).

Proof. We may assume that \( t = 1 \) and that \( A \subset \overline{B}^n \). We may also assume that \( \lambda > 1 \), since otherwise [Vä1, 7.1] gives \( M(\Gamma) \equiv m(B(2))/\lambda^n \). Define \( q_1, q_2 : \mathbb{R}^n \to \mathbb{R}^1 \) by

\[
q_1(x) = \frac{2}{(\ln \lambda)|x|} \quad \text{for} \quad 1 < |x| < \lambda^{1/2}, \quad q_2(x) = 1/\lambda \quad \text{for} \quad |x| < \lambda^{1/2},
\]

and \( q_j(x) = 0 \) for other \( x \in \mathbb{R}^n \). We show that \( q = \max(q_1, q_2) \) belongs to \( F(\Gamma) \), that is, the line integral of \( q \) along any rectifiable \( y \in \Gamma \) is at least one.

If \( |y| \subset B(\lambda^{1/2}) \), we have

\[
\int_y q_2 \, ds \geq l(y)/\lambda \geq 1.
\]

If \( |y| \subset B(\lambda^{1/2}) \), \( |y| \) meets the spheres \( S^{n-1} \) and \( S(\lambda^{1/2}) \), and hence

\[
\int_y q_1 \, ds \geq \int_1^{\lambda^{1/2}} \frac{2 \, dr}{r \ln \lambda} = 1.
\]
Thus \( q \in F(\Gamma) \), which implies
\[
M(\Gamma) \leq \int_{R^n} q^\alpha dm.
\]
Letting \( \Omega \) and \( \omega \) denote the volume of \( B^n \) and the area of \( S^{n-1} \) we have
\[
\int_{R^n} q^\alpha dm \leq 2^{n-1} \omega (\ln \lambda)^{1-n}, \quad \int_{R^n} q^\alpha dm = \Omega \lambda^{-n/2},
\]
and the lemma follows. \( \square \)

3. Main results

3.1. In this section we characterize the BLD homeomorphic images of \( B^2 \) and \( H^2 \). The half plane case is given in 3.4 and the disk case in 3.8. We recall from 1.1 that a homeomorphism \( f : D \to D' \) is L-BLD if
\[
(3.2) \quad l(x)/L \leq l(fx) \leq Ll(x)
\]
for every path in \( D \) or, equivalently, \( f \) is locally L-bilipschitz.

3.3. Theorem. Let \( D \subset R^2 \) be a convex Jordan domain, and let \( f : D \to D' \subset R^2 \) be an L-BLD homeomorphism. Then:

1. \( f \) is L-Lipschitz in the euclidean metric.
2. \( D' \) is finitely connected on the boundary.
3. \( f \) is L-bilipschitz in the metric \( \lambda_{D'} \).
4. \( \partial D' \) is locally rectifiable.
5. \( f \) has a unique extension to a homeomorphism \( f^* : \overline{D} \to \overline{D}^* \), which is L-bilipschitz outside \( \infty \) in the metric \( \lambda_{D'} \).
6. \( f^*|\partial D \) is L-bilipschitz outside \( \infty \) in the metrics \( \sigma_D \) and \( \sigma_{D'} \).

If, in addition, \( D \) has the c-chord-arc property, \( D' \) is \( L^2c \)-ICA.

Proof. Observe that since \( D \) is convex, \( \lambda_D \) is the euclidean metric. The condition (1) follows at once from convexity. Hence \( f \) has a continuous extension \( \tilde{f} : \overline{D} \to D' \). Then (2) follows from [Nä, 3.2]. Since \( D \) is convex, \( \partial D \) is locally rectifiable. The rest of the theorem follows easily from (3.2) and from the considerations in 2.3 and 2.4. \( \square \)

3.4. Theorem. A simply connected domain \( D \subset R^2 \) is BLD homeomorphic to the half plane \( H^2 = \{(x, y) \in R^2 : y \geq 0\} \) if and only if (1) \( D \neq R^2 \), (2) \( D \) is finitely connected on the boundary, (3) \( D \) is ICA, and (4) \( D \) is unbounded.

Proof. Suppose that \( f : H^2 \to D \) is an L-BLD homeomorphism. Since the image of the segment \( \{0\} \times (0, 1] \) has length at most \( L \), (1) is true. Since \( H^2 \) is convex and I-ICA, (2) and (3) follow from 3.3. Since \( f^{-1} \) is locally L-Lipschitz, \( \infty = m(H^2) \leq L^2 m(D) \), which implies (4).
The converse part is considerably harder. We first give an outline of the proof. Suppose that \( D \) satisfies the conditions (1)–(4). Choose a conformal map \( f_1: H^2 \to D \). It has a homeomorphic extension, still written as \( f_1: \overline{H^2} \to D^* \). We may assume that \( f_1(\infty) = \infty \). Choose a homeomorphism \( g: R^1 \to \partial D \setminus \{\infty\} \) such that \( \sigma_D(g(x), g(y)) = |x-y| \) for all \( x, y \in R^1 \) and such that the homeomorphism \( s: R^1 \to R^1 \) defined by \( s(x) = g^{-1}(f_1(x)) \) is increasing. Extend \( s \) by the Beurling–Ahlfors construction [Ah, p. 69] to a homeomorphism \( f_2: \overline{H^2} \to \overline{H^2} \). Then \( f = f_1 f_2^{-1}: \overline{H^2} \to D^* \) is a homeomorphism, and \( f|\overline{H^2} \) will be the desired BLD homeomorphism.

**Step 1.** We show that \( s \) is quasisymmetric (QS). Let \( x \in R^1 \) and \( t > 0 \). Let \( \Gamma \) be the family of all paths joining the intervals \([x-t, x]\) and \([x+t, \infty]\) in \( H^2 \). Then \( M(\Gamma) = 1 \). If \( \gamma \) belongs to the image \( \Gamma' \) of \( \Gamma \) under \( f_1 \), \( \gamma \) has end points \( a, b \) with \( a \in A = f_1[x-t, x] \) and \( b \in B = f_1[x+t, \infty] \). The \( \sigma \)-diameter of \( A \) is at most its length \( s(x) - s(x-t) \), and hence \( d(iA) \leq s(x) - s(x-t) \). Furthermore, \( \sigma_D(a, b) \equiv s(x+t) - s(x) \). Since \( D \) is c-ICA, this implies \( s(x+t) - s(x) \equiv c \gamma \). From 2.8 we obtain the estimate \( M(\Gamma') \equiv \mu_2(R) \) with

\[
Rc = \frac{s(x+t) - s(x)}{s(x) - s(x-t)}.\]

Since \( f \) is conformal, \( M(\Gamma) = M(\Gamma') = 1 \). Since \( \mu_2(R) \to 0 \) as \( R \to \infty \), we obtain an upper bound for \( Rc \). A lower bound is found similarly, changing the roles of \( x-t \) and \( x+t \). Hence \( s \) is \( H \)-QS with a constant \( H \) depending only on \( c \).

Let \( f_2: \overline{H^2} \to \overline{H^2} \) be the Beurling–Ahlfors extension of \( s \). Then \( f_2|H^2 \) is K-QC and \( L \)-bilipschitz in the hyperbolic metric of \( H^2 \) [Ah, p. 73] with \( L = L(c) \) and \( K = L^3 \). Then \( f = f_1 f_2^{-1}: \overline{H^2} \to D^* \) is a homeomorphism, and \( f|H^2 = g; \) thus

\[
(3.5) \quad \sigma_D(f(x), f(y)) = |x-y|
\]

for all \( x, y \in R^1 \).

**Step 2.** We write \( \delta(w) = d(w, \partial D) \) for \( w \in D \) and show that there is a constant \( M = M(c) \) such that

\[
(3.6) \quad y/M \equiv \delta(f(z)) \equiv My
\]

for every \( z = (x, y) \in \overline{H^2} \).

Let \( T \) be the line through \( b = i(f(x)) \) and \( f(z) \), let \( R \) be the component of \( T \setminus \{f(z)\} \) not containing \( b \) and let \( C' \) be the component of \( R \cap D \) with end point \( f(z) \). Let \( \Gamma \) be the family of all paths joining the real segment \([x, x+y]\) to \( C = f^{-1}C' \) in \( H^2 \). Then well known modulus estimates show that \( M(\Gamma) \equiv q_0 \) with a universal constant \( q_0 > 0 \), cf. [GV, Lemma 3.3, p. 13]. Assume that \( \delta(f(z)) = \delta > y \). Since (3.5) implies \( \sigma_D(f(x+y), f(x)) = y \), the members of \( \Gamma' = f\Gamma' \) meet the circles \( S(b, y) \) and \( S(b, \delta) \). Hence \( M(\Gamma') \equiv 2\pi/|\ln (\delta/y)| \). Since \( M(\Gamma) \equiv KM(\Gamma') \), we obtain the second inequality of (3.6) with \( M = e^{2\pi k/q_0} \).

We turn to the first inequality of (3.6). Fix \( z = (x, y) \in \overline{H^2} \) and set \( \delta = \delta(f(z)) \). Choose \( w_0 \in \partial D \) with \( |w_0 - f(z)| = \delta \). The segment \([f(z), w_0]\) defines an element
Let $u_0\in\partial^*D$ with $i(u_0) = w_0$. Let $C'_0$ be the arc on $\partial^*D$ such that $u_0$ divides $C'_0$ to two subarcs of length $7c\delta$. Let $C_1$ be the vertical ray with end point $z$. Then $C'_1 = fC_1$ joins $f(z)$ to $\infty$ in $D$. Let $J$ be the subarc of $C'_1$ joining $f(z)$ and a point $w_1 \in S(f(z), \delta)$ in $B(f(z), \delta)$.

**Case 1.** $|w_1 - w_0| \geq \delta$. Set $w_2 = (w_0 + f(z))/2$. For every $r \in [\delta/2, 3^{1/2}\delta/2]$ we can choose an arc $\alpha_r$ of $S(w_2, r)$ with end points $a_r \in J$, $b_r \in \partial D$ and with $\alpha_r \setminus \{b_r\} \subset D$. Let $\Gamma'$ be the family of all these arcs $\alpha_r$. A standard estimate gives $M(\Gamma') \leq (\ln 3)/4\pi = q_1$. The arc $\alpha'_r = \alpha_r \setminus \{b_r\}$ defines an element $u_r \in \partial^*D$ with $i(u_r) = b_r$. Moreover, $\alpha_r \cup \{a_r, w_0\}$ joins $u_r$ and $u_0$ in $D$. Since $D$ is $c$-ICA, we have

$$\sigma_p(u_r, u_0) \leq c(l(\alpha_r) + |a_r - w_0|) \leq c(2\pi r + |a_r - w_2| + |w_2 - w_0|) \leq 3^{1/2}c\delta(2\pi + 3^{1/2}/2 + 1/2)/2 < 7c\delta.$$

Hence $b_r \in C'_0$. Consequently, the members of $\Gamma = f^{-1}\Gamma'$ join $C_0 = f^{-1}C'_0 = [x - 7c\delta, x + 7c\delta]$ and $C_1$. Thus either $y \geq 7c\delta$ or $M(\Gamma) \geq 2\pi/\ln(y/7c\delta)$. Since $M(\Gamma') \equiv KM(\Gamma)$, we obtain

$$y \leq 7c\delta e^{2\pi K/q_1},$$

which yields the first inequality of (3.6).

**Case 2.** $|w_1 - w_0| = t < \delta$. We repeat the argument of Case 1 replacing $w_2$ by $w_3 = (w_0 + w_1)/2$. Since $S(w_3, r)$ meets $J$ and $\partial D$ whenever $t/2 \leq r \leq 3^{1/2}t/2$, we obtain the same estimate as in Case 1.

**Step 3.** We prove that the homeomorphism $f|H^2: H^2 \to D$ is BLD. Since $f = f_1 f_2^{-1}$ where $f_1|H^2$ is conformal and $f_2|H^2$ $L$-bilipschitz in the hyperbolic metric, the diffeomorphism $f|H^2$ is $L$-bilipschitz in the hyperbolic metrics of $H^2$ and $D$. Hence

$$|h|/Ly \leq Q(f(z))|f'(z)h| \leq L|h|/y$$

for all $z = (x, y) \in H^2$ and $h \in \mathbb{R}^2$, where $Q$ is the density of the hyperbolic metric in $D$. It is well known that

$$1/4\delta(w) \equiv Q(w) \equiv 1/\delta(w)$$

for all $w \in D$. Together with (3.6), these inequalities show that $f$ is $L_4$-BLD with $L_4 = 4LM = L_4(c)$.

**3.7. Remark.** The proof above shows that the quantitative version of 3.4 is also true: If $f: H^2 \to D$ is an $L$-BLD homeomorphism, $D$ is $c$-ICA with $c = L^2$. If $D$ is $c$-ICA and unbounded, there is an $L$-BLD homeomorphism $f: H^2 \to D$ with $L = L(c)$.

**3.8. Theorem.** A simply connected domain $D \subset \mathbb{R}^2$ is BLD homeomorphic to the unit disk $B^2$ if and only if (1) $D$ is finitely connected on the boundary, (2) $D$ is ICA, and (3) $D$ is bounded.
Proof. Suppose that \( f : B^2 \to D \) is a BLD homeomorphism. Then \( f \) is \( L \)-Lipschitz and hence \( d(D) \leq 2L \). The conditions (1) and (2) follow from 3.3. More precisely, since \( B^2 \) is \( \pi \)-ICA, \( D \) is \( \pi L^2 \)-ICA.

The converse part is proved by modifying the proof of 3.4. Suppose that \( D \) satisfies (1) and (3) and that \( D \) is \( c \)-ICA. Then \( \partial^* \overline{D} \) is rectifiable. We normalize the situation by assuming \( l(\partial^* \overline{D}) = 2\pi \). Then there is a lengthpreserving homeomorphism \( g : \overline{B^2} \to \overline{D} \). Let \( \overline{f_1} : \overline{B^2} \to \overline{D} \) be a conformal map. It has an extension to a homeomorphism, still written as \( \overline{f_1} : \overline{B^2} \to \overline{D} \). Then \( g \circ \overline{f_1} \) is a self homeomorphism of \( \overline{G} \), and \( l(\alpha) = l(\beta) \) for every arc \( \alpha \subset \overline{S^1} \). We may assume that \( s|N_3 = \text{id} \) where \( N_3 = \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \).

Step 1. We show that \( f_1|S^1 \) has the following quasisymmetry property: if \( \alpha \) and \( \beta \) are adjacent arcs of \( S^1 \) with \( l(\alpha) = l(\beta) \), then

\[(3.9) \quad l(\overline{f_1} \beta) \leq c_1 l(\overline{f_1} \alpha) \]

with some constant \( c_1 = c_1(e) \).

Assume first that \( l(\alpha) \leq \pi/3 \). Then we may assume that \( \alpha \cup \beta \) does not meet the arc \( A = \{e^{i\theta} : 2\pi/3 < \theta < 4\pi/3\} \). Let \( a \) be the end point of \( A \) which has the greater distance from \( \alpha \cup \beta \). Using the terminology of [LV, I.3.2] we consider the quadrilateral \( Q \) consisting of the domain \( B^2 \), the three end points of \( \alpha \) and \( \beta \), and the point \( a \). There are two path families \( \Gamma_1, \Gamma_2 \) associated with \( Q \) with moduli \( M(\Gamma_1) = 1/M(\Gamma_2) \).

The length of a path in either family is at least \( d(\alpha) = t \). Hence 2.9 implies \( M(\Gamma_1) = \mu_2(1) \) and thus \( M(\Gamma_1) \equiv \mu(1) \). Let \( \Gamma_1 \) be the family joining \( \alpha \) to the opposite side of \( Q \), and suppose that \( g \in f_1 \Gamma_1 = \Gamma_1' \). The end points of \( g \) divide \( \partial^* \overline{D} \) into two arcs. One of these contains \( f_1 \beta \) and the other \( f_1 \alpha \). Since \( s|A = A \), we have \( l(f_1 \alpha) = 2\pi/3 \). Since \( D \) is \( c \)-ICA, this implies \( c \leq l\overline{f_1} \alpha \) and \( 2l(\overline{f_1} \alpha) \). Since \( l(\overline{f_1} \alpha) \leq l(\overline{f_1} \beta) \), 2.9 gives \( M(\Gamma_1') \equiv \mu_2(R) \) with

\[
R = \min(l(\overline{f_1} \beta), 2\pi/3) \leq l(\overline{f_1} \alpha) \ ;
\]

Since \( M(\Gamma_1') = M(\Gamma) \equiv 1/\mu(1) \) and since \( \mu(1) \to 0 \) as \( t \to \infty \), \( R \) is bounded by a universal constant \( c_0 \). Hence either (3.9) holds with \( c_1 = c_0 \) or \( 2\pi/3 \leq c_0 l(\overline{f_1} \alpha) \).

In the latter case (3.9) holds with \( c_1 = 2c_0 \).

The case \( l(\alpha) > \pi/3 \) reduces to the case above by dividing \( \alpha \) and \( \beta \) to three subarcs, cf. [LV, II.7.1].

Step 2. We want to extend \( s : S^1 \to S^1 \) to a QC homeomorphism \( f_2 : \overline{B^2} \to \overline{B^2} \). To this end we choose an auxiliary Möbius map \( h \) with \( hB^2 = H^2 \) and \( h(1) = \infty \). Then \( s_1 = hsh^{-1}|R^1 \) is an increasing homeomorphism onto \( R^1 \). Moreover, \( s_1 \) is (weakly) \( H \)-QS with \( H = H(c) \). This can be seen for example as follows: Since \( l(\overline{s_1} \alpha) = l(\overline{f_1} \alpha) \), (3.9) implies that \( s : S^1 \to S^1 \) is weakly \( H \)-QS in the arc metric, hence in the euclidean metric, cf. [TV, p. 113]. Since \( S^1 \) is of \( \pi \)-bounded turning, \( s \) is \( \eta \)-QS with \( \eta = \eta_\epsilon \) [TV, 2.16]. Hence \( s \) is \( \theta \)-quasimöbius with \( \theta = \theta_\epsilon \) by [Vä]
3.2]. Consequently, $s_1$ is $\theta$-quasimöbius. Since $s_1(\infty) = \infty$, $s_1$ is $\theta$-QS and hence (weakly) $H$-QS with $H = \theta(1)$.

Let $g: \overline{H^2} \to \overline{H^2}$ be the Beurling—Ahlfors extension of $s_1$. It induces a homeomorphism $f_2 = h^{-1}g: \overline{B^2} \to \overline{B^2}$. Then $f_2|S^1 = s$, and $f_2|B^2$ is $K$-QC and $L$-bilipschitz in the hyperbolic metric of $B^2$ with $L = L(c)$, $K = L^2$.

**Step 3.** The map $f = f_1f_2^{-1}: \overline{B^2} \to D^*$ is the desired map. This follows as in the proof of 3.3 from the inequalities

$$
(3.10) 
(1 - |z|)/M \subseteq \delta(f(z)) \subseteq M(1 - |z|)
$$

where $z \in B^2$, $M = M(c)$, $\delta(w) = d(w, \partial D)$. This is proved by a rather obvious modification of the proof of the corresponding inequalities (3.6) of the half plane case. Omitting other details, we describe the construction of the arcs $C'_0 = f_1C_0$ and $C'_1 = f_1C_1$. We may assume that $7c\delta < 1 - |z|$. As in the proof of (3.6), $C'_0$ will be a subarc of $\partial^*D$ with $l(C'_0) = 14c\delta$. This is possible, since $14c\delta < 2(1 - |z|) \leq 2 < 2\pi = l(\partial^*D)$. Then $C_1$ is chosen to be the line segment with end points $z$ and $-f_1^{-1}(w_0)$. \(\square\)

3.11. **The quantitative version of 3.8.** If $f: B^2 \to D$ is an $L$-BLD homeomorphism, $D$ is $c$-ICA with $c = \pi L^2$. If $D$ is $c$-ICA and bounded with $l(\partial^*D) = r$, there is an $L$-BLD homeomorphism $f: B(r) \to D$ with $L = L(c)$.

**References**


[La] **Laflullin, T. G., (Laflullin T. G.):** О геометрических условиях на образы прямой и окружности при квазиизометрии плоскости. - Материалы XVIII всесоюзной научной студенческой конференции, Новосибирск, 1980, 18–22.


[MV] **Martio, O., and J. Väisälä:** Elliptic equations and maps of bounded length distortion. - To appear.


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