Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 12, 1987, 329—333

ON THE HAUSDORFF DIMENSION OF QUASICIRCLES

J. BECKER and CH. POMMERENKE

1. Introduction

For a bounded set Γ in $\boldsymbol{C}$, let $N(\varepsilon, \Gamma)$ denote the minimal number of disks of radius $\varepsilon>0$ that are needed to cover Γ. The Hausdorff dimension [4, p. 7] satisfies

$$\dim \Gamma \leqq \liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon, \Gamma)}{\log(1/\varepsilon)}. \tag{1.1}$$

For sufficiently regular compact sets, the limit exists and is equal to $\dim \Gamma$, for instance if Γ is a self-similar fractal curve [8, p. 736], [11, p. 29].

We shall study the case that Γ is a K-quasicircle where $1\leqq K<\infty$. This means that Γ is the image of the unit circle under a K-quasiconformal mapping of $\boldsymbol{C}$. Let $c_1, c_2, \ldots$ denote suitable positive absolute constants.

Theorem 1. *Let* $1\leqq K<\infty$ *and define*

$$\beta(K) = \sup_{\Gamma} \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon, \Gamma)}{\log(1/\varepsilon)} \tag{1.2}$$

where Γ *ranges over all K-quasicircles in* $\boldsymbol{C}$. *Then*

$$\beta(K) \leqq 2-c_1 K^{-3.41} \tag{1.3}$$

and furthermore, for K *close to* 1,

$$1+0.36\varkappa^2 \leqq \beta(K) \leqq 1+37\varkappa^2, \quad \varkappa = \frac{K-1}{K+1}. \tag{1.4}$$

It follows from (1.1) and (1.2) that

$$\dim \Gamma \leqq \beta(K)$$

for all K-quasicircles. Gehring and Väisälä [5] were the first to show that $\beta(K)<2$. If Lehto's form [9] of the Bojarski integrability theorem is used in their proof [5, p. 507/508] one obtains

$$\beta(K) \leqq 2-c_2 K^{-c} \tag{1.5}$$

doi:10.5186/aasfm.1987.1206

with some unspecified $c \geqq 1$. It has been conjectured that $c=1$ which would be best possible.

The lower estimate in (1.4) follows from Ruelle's asymptotic expansion [17, p. 107]

$$\dim J(\lambda) = 1+\frac{|\lambda|^2}{4\log 2}+O(|\lambda|^3) \quad (\lambda \to 0), \tag{1.6}$$

where $J(\lambda)$ is the Julia set of the polynomial $w^2+\lambda w$.

2. Univalent functions with quasiconformal extension

Let Σ denote the family of all functions

$$g(z) = z+b_0+b_1 z^{-1}+\dots \tag{2.1}$$

that are analytic and univalent in $\{1<|z|<\infty\}$. For $1 \leqq K<\infty$, let Σ_K denote the subfamily of functions that have a K-quasiconformal extension to $\mathbf{C}$. We shall deduce Theorem 1 from the following result.

Theorem 2. *Let $g \in \Sigma_K$ and $1<r<\infty$. Then*

$$\int_0^{2\pi} |g'(re^{it})|^p \, dt \leqq \frac{c_1}{(1-1/r)^{p-1}} \tag{2.2}$$

holds with

$$p = 2-c_2 K^{-a}, \quad a = 1+\frac{2}{\pi}\arctan 2 < 1.705. \tag{2.3}$$

If $1 \leqq K<1+c_3$ *then* (2.2) *also holds with*

$$p = 1+9.1\varkappa^2, \quad \varkappa = (K-1)/(K+1). \tag{2.4}$$

It seems probable that any $a>1$ can be chosen in (2.3). Our proof would, with suitable modifications, give this if, for $g \in \Sigma$,

$$\int_0^{2\pi} |\exp[2e^{i\theta}\log g'(re^{it})]| \, dt \leqq c(\varepsilon)(1-1/r)^{-1-\varepsilon} \quad (1<r<2)$$

is true for real θ and each $\varepsilon>0$, with $c(\varepsilon)$ depending only on ε. This is a strengthened version of the Brennan conjecture [2], [15].

We need the following important result [10, p. 69].

Theorem A. *Let $g \in \Sigma_K$. Then there are functions*

$$g_\lambda \in \Sigma, \quad |\lambda| < 1 \tag{2.5}$$

that depend analytically on λ such that

$$g_0(z) = z, \quad g_\varkappa = g, \quad \varkappa = (K-1)/(K+1). \tag{2.6}$$

Proof of Theorem 2. (a) Let

(2.7) $$\varphi(\lambda) = a_{-1}\lambda^{-1}+a_0+a_1\lambda+\dots, \quad \varphi(1) = 2, \quad a_{-1} > 0$$

map $\{|\lambda|<1\}$ conformally onto the outer domain of the triangle Δ with the vertices 2, $1+i/2$ and $1-i/2$. We write

(2.8) $$\log g'_\lambda = u_\lambda+iv_\lambda, \quad z = re^{it}.$$

The function $\varphi(\lambda)\log g'_\lambda(z)$ is analytic in $\{|\lambda|<1\}$, by (2.6) and (2.7). Hence

(2.9) $$\psi(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} |g'_\lambda(z)^{\varphi(\lambda)}|\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{u_\lambda \operatorname{Re}\varphi - v_\lambda \operatorname{Im}\varphi}\,dt$$

is subharmonic in $\{|\lambda|<1\}$ for fixed $r>1$.

Since $g_\lambda\in\Sigma$, an inequality of Golusin [14, p. 65] shows that $|v_\lambda(z)|<\log 1/(1-r^{-1})$ for $|\lambda|<1$. Hence (2.9) shows that

$$\psi(\lambda) \leqq (1-1/r)^{-|\operatorname{Im}\varphi(\lambda)|}\frac{1}{2\pi}\int_0^{2\pi} e^{u_\lambda \operatorname{Re}\varphi}\,dt.$$

Since [14, p. 127]

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2u_\lambda(z)}\,dt = \frac{1}{2\pi}\int_0^{2\pi}|g'_\lambda(z)|^2\,dt < c_4(1-1/r)^{-1}$$

we therefore obtain that

(2.10) $$\psi(\lambda) \leqq \left(c_4(1-1/r)^{-1}\right)^{\operatorname{Re}\varphi(\lambda)/2}(1-1/r)^{-|\operatorname{Im}\varphi(\lambda)|}$$

if $|\lambda|<1$ and $0<\operatorname{Re}\varphi(\lambda)\leqq 2$. The choice of the triangle Δ shows that

$$\operatorname{Re}\varphi(\lambda)/2+|\operatorname{Im}\varphi(\lambda)| \leqq 1 \quad \text{for} \quad |\lambda| = 1.$$

Hence we obtain from (2.10) by the maximum principle for subharmonic functions that

(2.11) $$\psi(\lambda) \leqq c_5(1-1/r)^{-1} \quad \text{for} \quad |\lambda| < 1.$$

Since $\varphi(1)=2$ by (2.7), it follows from the Schwarz—Christoffel formula that

(2.12) $$\varphi'(\lambda) = -a_{-1}\lambda^{-2}(1-\lambda)^{\alpha}(1-\bar\lambda_1\lambda)^{\alpha_1}(1-\lambda_1\lambda)^{\alpha_1}$$

with $|\lambda_1|=1$ and $\alpha=(2/\pi)\arctan 2$ and therefore

$$\varphi(\lambda) = 2+c_6(1-\lambda)^{\alpha+1}+o\left((1-\lambda)^{\alpha+1}\right) \quad \text{as} \quad \lambda \to 1-0.$$

Hence (2.3) holds if we set $p=\varphi(\varkappa)/(\varphi(\varkappa)-1)$. Furthermore, by (2.6) and (2.9),

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|g'|^p\,dt\right)^{1/p} \leqq \left(\frac{1}{2\pi}\int_0^{2\pi}|g'|^{\varphi(\varkappa)}\,dt\right)^{1/\varphi(\varkappa)} = \psi(\varkappa)^{1/\varphi(\varkappa)}$$

and (2.11) therefore shows that

$$\int_0^{2\pi}|g'(z)|^p\,dt \leqq c_1(1-1/r)^{-p/\varphi(\varkappa)} = c_1(1-1/r)^{-p+1}.$$

(b) Since $g\in\Sigma_K$ it follows from Golusin's estimate [7, p, 132] for the class Σ and from Theorem A by Lehto's majorant principle that

$$\left|\frac{g''(z)}{g'(z)}\right| \leqq \frac{6\varkappa}{r^2-1} \leqq \frac{3\varkappa}{r-1} \quad \text{for} \quad z = re^{it}, \quad r > 1. \tag{2.13}$$

We conclude as in [13] or [6] that

$$\int_0^{2\pi} |g'(z)|^p\, dt \leqq c_7(1-1/r)^{-9p^2\varkappa^2} \tag{2.14}$$

and this implies (2.2) with p given by (2.4) because $9p^2\varkappa^2 \leqq p-1$ if $0\leqq\varkappa<c_8$.

3. Proof of Theorem 1

(i) Let Γ be a K-quasicircle in $\boldsymbol{C}$. Then there is a K-quasiconformal mapping w of $\hat{\boldsymbol{C}}$ mapping $\{|z|=1\}$ onto Γ. Let g be a conformal mapping of $\{|z|>1\}$ onto the outer domain of Γ; we may assume that g has the form (2.1). If we define

$$g(z) = w\left(1/\overline{w^{-1}\circ g(1/\bar{z})}\right) \quad \text{for} \quad |z| \leqq 1$$

then $g\in\Sigma_{K^*}$ with $K^*=K^2$; see e.g. [10, p. 39/40].

It follows from Theorem 2 that (2.2) holds with

$$p = 2 - c_2 K^{-2a}$$

and, for K close to 1, also with

$$p = 1 + 9.1\varkappa^{*2} \leqq 1 + 37\varkappa^2.$$

Hence our upper estimates are an immediate consequence of the following result which is the analogue of [16, Corollary 2] for the class Σ.

Theorem B. *Let $g\in\Sigma$. If $\{g(z)\colon |z|>1\}$ is bounded by a quasicircle Γ and if*

$$\int_0^{2\pi} |g'(re^{it})|^p\, dt = O\left(\frac{1}{(1-1/r)^{p-1}}\right) \quad \text{as} \quad r \to 1+0$$

then

$$\limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon, \Gamma)}{\log(1/\varepsilon)} \leqq p.$$

(ii) We need some known facts from complex iteration theory to prove the lower estimate; see e.g. [3] or [12] for an overview. We give a brief indication of the proofs.

Let $|\lambda|<1$ and let $J(\lambda)$ be the Julia set associated with $w^2+\lambda w$; this is a Jordan curve. If

$$g_\lambda(z) = b_{-1}(\lambda)z + \sum_{n=0}^{\infty} b_n(\lambda) z^{-n}$$

maps $\{|z|>1\}$ conformally onto the outer domain of $J(\lambda)$, then

$$g_\lambda(z^2) = g_\lambda(z)^2 + \lambda g_\lambda(z) \quad \text{for} \quad |z| > 1.$$

It follows that $b_{-1}(\lambda)=1$ and, by induction, that $b_n(\lambda)$ is a polynomial in λ. Since $|b_n(\lambda)|<1$ for $n\geqq 1$ by the area theorem, we conclude that g_λ depends analytically on λ.

Since $g_\lambda(z)$ is univalent in $\{|z|>1\}$ and since $g_0(z)=z$, a deep result of Bers and Royden [1, Theorem 1] shows that

$$g_\lambda \in \Sigma_K, \quad \varkappa = |\lambda|.$$

Hence $J(\lambda)$ is a K-quasicircle, and $\beta(K)\geqq 1+0.36\varkappa^2$ (for K close to 1) follows from (1.1), (1.2) and Ruelle's estimate (1.6).

References

[1] Bers, L., and H. L. Royden: Holomorphic families of injections - Acta Math. 157, 1986, 259—286.

[2] Brennan, J. E.: The integrability of the derivative in conformal mapping. - J. London Math. Soc. (2) 18, 1973, 261—272.

[3] Brolin, H.: Invariant sets under iteration of rational functions. - Ark. Mat. 6, 1965, 103—144.

[4] Falconer, K. J.: The geometry of fractal sets. - Cambridge University Press, 1985.

[5] Gehring, F. W., and J. Väisälä: Hausdorff dimension and quasiconformal mapping. - J. London Math. Soc. (2) 6, 1973, 504—512.

[6] Göktürk, Z.: Estimates for univalent functions with quasiconformal extensions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 589, 1974, 1—21.

[7] Golusin, G. M.: Geometrische Funktionentheorie. - VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

[8] Hutchinson, J. E.: Fractals and self similarity. - Indiana Univ. Math. J. 30, 1981, 713—747.

[9] Lehto, O.: Remarks on the integrability of the derivatives of quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 371, 1965, 1—8.

[10] Lehto, O.: Univalent functions and Teichmüller spaces. - Springer-Verlag, New York, 1987.

[11] Martio, O., and M. Vuorinen: Whitney cubes, p-capacity, and Minkowski content. - Exposition Math. 5, 1987, 17—40.

[12] Peitgen, H.-O., and P. H. Richter: The beauty of fractals. - Springer-Verlag, Berlin, 1986.

[13] Pommerenke, Ch.: On the growth of the coefficients of analytic functions. - J. London Math. Soc. (2) 5, 1972, 624—628.

[14] Pommerenke, Ch.: Univalent functions. - Vandenhoeck & Ruprecht, Göttingen, 1975.

[15] Pommerenke, Ch.: On the integral means of the derivative of a univalent function, II. - Bull. London Math. Soc. 17, 1985, 565—570.

[16] Pommerenke, Ch.: On boundary size and conformal mapping. - Preprint TU Berlin, 1987.

[17] Ruelle, D.: Repellers for real analytic maps. - Ergodic Theory Dynamical Systems 2, 1982, 99—107.

Technische Universität Berlin
Fachbereich Mathematik
D 1000 Berlin 12
Bundesrepublik Deutschland

Received 20 May 1987