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ON REPRESENTABLE PAIRS

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Introduction

This paper is organised in five parts. We first consider some properties of semigroups in Chapter 1 and prove structural results which might be interesting as such. In Chapter 2 we define our central notion, namely, that of a representable pair (S, f) where S is a semigroup and f is a mapping from S into the class of non-zero cardinal numbers. We also give here some necessary conditions for a pair (S, f) to be representable by a groupoid.

Chapter 3 contains a representation criterion. By using this criterion we are able to prove in Chapter 4 several sufficient conditions for a pair (S, f) to be representable by a groupoid. Finally, Chapter 5 contains a special treatment of a semigroup of order five.

We assume that the reader is familiar with the rudiments of the theory of abstract algebraic systems. The background can be obtained e.g. from [1], [2] and [3].

1. Preliminaries

Let S be a semigroup. We denote $S^2 = SS = \{ab\colon a, b \in S\}$ and $S^n = SS^{n-1}$ for every positive integer $n \geq 3$. We also need the following sets:

$$\begin{aligned}
\mathrm{Id}(S) &= \{a \in S\colon a = a^2\},\\
L(S) &= \{a \in S\colon a \in Sa\},\\
R(S) &= \{a \in S\colon a \in aS\},\\
\mathrm{Li}(S) &= \{a \in S\colon a \in \mathrm{Id}(S)a\},\\
\mathrm{Ri}(S) &= \{a \in S\colon a \in a\,\mathrm{Id}(S)\},\\
K(S) &= \bigcap_{n=1}^{\infty} S^n.
\end{aligned}$$

We shall now formulate some easy observations.

Lemma 1.1. (i) *The set $L(S)$ (or $R(S)$) is either empty or a right (or left) ideal of S.*

(ii) *The set $\mathrm{Li}(S)$ (or $\mathrm{Ri}(S)$) is either empty or a right (or left) ideal of S.*

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(iii) *The set* $K(S)$ *is either empty or an ideal of* S.

(iv) $\mathrm{Id}(S) \subseteq \mathrm{Li}(S) \subseteq L(S) \subseteq K(S)$ *and* $\mathrm{Id}(S) \subseteq \mathrm{Ri}(S) \subseteq R(S) \subseteq K(S)$.

If there exists an integer $n \geq 0$ *such that* $S^{n-1} \neq S^n = K(S)$, *we say that the number* $n = nc(S)$ *is the class number of* S *(now* S^0 *means a one-element semigroup and* $S^{-1} = \emptyset$*).*

Lemma 1.2. *If* S *is finite, then* $\mathrm{Id}(S)$ *is non-empty,* $L(S) = \mathrm{Li}(S)$ *and* $R(S) = \mathrm{Ri}(S)$. *Furthermore,* $K(S)^2 = K(S)$.

Lemma 1.3. *Let* S *be finite and* $S = S^2$ *(i.e.,* $nc(S) \leq 1$*). Then* $S = R(S)L(S)$. *In particular,* $S = L(S)$, *provided that* S *is commutative.*

Proof. Put $I = R(S)L(S)$ and define a relation r on S by $(a,b) \in r$ if and only if $a \in bS$. Now I is an ideal of S, r is transitive and $a \in R(S)$ if and only if $(a,a) \in r$. Then assume that $a_1 \in S - I$. There are elements $a_2, b_1 \in S$ such that $a_1 = a_2 b_1$, also $a_3, b_2 \in S$ such that $a_2 = a_3 b_2$ etc. Now $(a_1, a_2) \in r, (a_2, a_3) \in r$ etc., so that $(a_i, a_j) \in r$ whenever $1 \leq i < j$. Since I is an ideal and $a_1 \notin I$, we conclude that I contains none of the elements a_2, a_3, ... As S is finite, it follows that there are positive integers $i < j$ such that $a_i = a_j$. Thus $(a_i, a_i) \in r$, $a_i \in R(S)$, and since $R(S) \subseteq I$ by 1.2(i), we get $a_i \in I$, a contradiction. The proof is complete.

Lemma 1.4. *Suppose that* S *contains at most four elements and* $S = S^2$. *Then* $S = L(S) \cup R(S)$.

Proof. Suppose that $a \in S$ and $a \notin L(S) \cup R(S)$. By 1.3, $a = bc$, where $b \in R(S)$ and $c \in L(S)$. Clearly, $b \notin L(S)$ and $c \notin R(S)$. Now the elements a, a^2, b and c are pair-wise different, hence $\mathrm{card}(S) = 4$. If $ba = b$, then $a = bc = bac = b^2c^2 = b^3c^3 = \ldots$; now $b^n \in \mathrm{Id}(S)$ for some $n \geq 1$, the equation $a = b^n c^n$ implying $a \in L(S)$, a contradiction. Similarly, if $ba^2 = b$, we get a contradiction. Consequently, $ba \neq b$ and $ba^2 \neq b$. The inequalities $bb \neq b$ and $bc \neq b$ are obvious; thus we have proved that $b \notin bS$. Hence $b \notin R(S)$ and we again have a contradiction. We conclude $S = L(S) \cup R(S)$.

Example 1.5. Consider the following five-element semigroup T:

	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	a	b
c	0	0	0	0	0
d	0	0	0	c	d

Now $T = T^2$ and $a \notin L(T) \cup R(T)$.

Lemma 1.6. *Let S be a five-element semigroup such that $S = S^2$ and $S \neq L(S) \cup R(S)$; then S is isomorphic to the semigroup T constructed in 1.5.*

Proof. Let $a \in S - \big(L(S) \cup R(S)\big)$. Now $a = bc$, where $b \in R(S)$ and $c \in L(S)$. Furthermore, $b \notin L(S)$ and $c \notin R(S)$. The elements a, a^2, b and c are pair-wise different and, as in the proof of Lemma 1.4, one can show that $b \notin \{ba, ba^2, bb, bc\}$. Since $b \in R(S), b = bd$, where $d \in S$ and thus $S = \{a, a^2, b, c, d\}$. By using a similar type of argument, we get $c = dc$.

Now we know that $b = bd$ and $c = dc$ and we try to compute the rest of the multiplication table for S. It is easy to see that $ab \neq a, ab \neq b, ab \neq c$ and $ab \neq d$; hence $ab = a^2$. We also have $ba = a^2, ac = a^2$ and $ca = a^2$. Further, it is easy to see that $b^2 = a^2$ and $c^2 = a^2$. Clearly, $ad \neq a$ and $ad \neq b$. If $ad = c$, then $a = bc = bad = b^2ad^2 = \cdots$, a contradiction. If $ad = d$, then $b = bd = bad$ and $a = bc = badc = b^2a(dc)^2 = \cdots$, again a contradiction. Consequently, $ad = a^2$ and, similarly, $da = a^2$. Since $a \notin R(S) \cup L(S)$, $b \notin L(S)$ and $c \notin R(S)$, we must have $cb = a^2$, $cd = a^2$ and $db = a^2$. Clearly, $a^3 \neq a$, $a^3 \neq b$ and $a^3 \neq c$. If $a^3 = d$, then $b = bd = ba^3$ and $a = bc = ba^3c$, which is not possible. Thus $a^3 = a^2$, and it follows that $a^4 = a^2$, $ba^2 = a^2b = ca^2 = a^2c = da^2 = a^2d = a^2$. Finally, we have to see that $d^2 = d$. If we denote $a^2 = 0$, we have the same multiplication table as in the example. The proof is complete.

2. Groupoids and semigroups

By a groupoid we mean a non-empty set together with a binary operation denoted multiplicatively.

Let ζ be a groupoid. Denote by $r(\zeta)$ the intersection of all congruences r such that the corresponding factor groupoid is a semigroup. Now $r(\zeta)$ is a congruence on ζ, $\zeta/r(\zeta)$ is a semigroup and $r(\zeta)$ is the least congruence with this property. Clearly, $r(\zeta)$ is the congruence on ζ generated by the ordered pairs $\big(a(bc), (ab)c\big)$, where $a, b, c \in \zeta$.

Throughout the paper, let C denote the class of non-zero cardinal numbers. Consider a semigroup S and a mapping $f\colon S \to C$. We say that the pair (S, f) is representable by a groupoid if there exist a groupoid ζ and a homomorphism g from ζ onto S such that $\ker(g) = r(\zeta)$ and $\operatorname{card}\big(g^{-1}(a)\big) = f(a)$ for every $a \in S$.

We immediately have

Lemma 2.1. *Let S be a semigroup and $f\colon S \to C$ a mapping such that the pair (S, f) is representable by a groupoid. Then $f(a) = 1$ for every $a \in S - S^3$.*

Next we establish

Lemma 2.2. *Let S be a semigroup and $f\colon S \to C$ a mapping such that the pair (S, f) is representable by a groupoid. Let $a \in S^2$ and*

$$A = \{\,(b, c) \mid b, c \in S \text{ and } a = bc\,\}.$$

Then $f(a) \leq \sum_{(b,c)\in A} f(b)f(c)$.

Proof. Suppose that

$$f(a) > \sum_{(b,c)\in A} f(b)f(c).$$

Now there exist a groupoid ζ and a surjective homomorphism $g\colon \zeta \to S$ such that $\ker(g) = r(\zeta)$ and $\operatorname{card}(g^{-1}(u)) = f(u)$ for every $u \in S$. Put $H = g^{-1}(a)$ and

$$L = \{xy \mid x \in g^{-1}(b), y \in g^{-1}(c), (b,c) \in A\}.$$

Then $L \subseteq H$ and $L \neq H$. Now we define a relation d on ζ as follows:

$$d = \big(r(\zeta) - (H \times H)\big) \cup (L \times L) \cup \big((H - L) \times (H - L)\big).$$

Clearly, d is an equivalence relation, $d \subseteq r(\zeta)$ and $d \neq r(\zeta)$. In fact, it is not difficult to see that d is a congruence on ζ and ζ/d is a semigroup. However, this is a contradiction. The required result follows.

3. A representation criterion

Let S be a semigroup and $f\colon S \to C$ a mapping. For each $a \in S$, define a mapping $f_a\colon S \to C$ by $f_a(a) = f(a)$ and $f_a(b) = 1$ for every $b \neq a$, $b \in S$.

Theorem 3.1. *Suppose that the pair (S, f_a) is representable by a groupoid for any $a \in S$. Then the pair (S, f) is also representable by a groupoid.*

Proof. Now there exist pair-wise disjoint groupoids ζ_a (their operations are denoted by $\star$) and surjective homomorphisms

$$g_a\colon \zeta_a \to S$$

such that

$$\ker(g_a) = r(\zeta_a),$$
$$\operatorname{card}(g_a^{-1}(a)) = f(a) \quad \text{and}$$
$$\operatorname{card}(g_a^{-1}(b)) = 1 \quad \text{for every } b \in S,\ b \neq a.$$

Now denote $H_a = g_a^{-1}(a)$ and $\zeta = \bigcup_{a\in S} H_a$. We shall define a binary operation on ζ as follows:

(1) If $x, y \in H_a$ and $a = aa$, we put $xy = x \star y \in H_a$.

(2) If $x \in H_a, y \in H_b$ and $ab = c$ where $a \neq c \neq b$, then $xy = g_c^{-1}(a) \star g_c^{-1}(b) \in H_c$.

(3) If $x \in H_a$ and $y \in H_b$ $(a \neq b)$ and $ab = a$, then $xy = x \star g_a^{-1}(b) \in H_a$.

(4) If $x \in H_a$ and $y \in H_b$ $(a \neq b)$ and $ab = b$, we put $xy = g_b^{-1}(a) \star y \in H_b$.

Now we define a mapping g from ζ onto S by

$$g(H_a) = a \qquad \text{for each } a \in S.$$

It is obvious that g is a homomorphism from ζ to S.

We still have to show that $r(\zeta) = \ker(g)$. Clearly, $r(\zeta) \subseteq \ker(g)$. We shall now construct two equivalence relations for our proof. Let $a \in S$ and denote

$$d = \big(\ker(g) - (H_a \times H_a)\big) \cup \big(r(\zeta) \cap (H_a \times H_a)\big)$$

and

$$s = \{(x,x) \mid x \in \zeta_a\} \cup (r(\zeta) \cap (H_a \times H_a)).$$

We are going to prove that d is a congruence on ζ and s is a congruence on ζ_a.

Let $x, y, z \in \zeta$ and $(x,y) \in d$. We have to distinguish between the following cases:

(1) $x, y \in H_b$ for some $b \in S, b \neq a$. Then $(zx, zy) \in \ker(g)$ and $(zx, zy) \in d$ unless $zx \in H_a$. If $zx \in H_a$, then $zy \in H_a$, too. Then there exists $c \in S$ such that $z \in H_c$ and $a = cb$. If $a \neq c$, then

$$zx = g_a^{-1}(c) \star g_a^{-1}(b) = zy,$$

hence $(zx, zy) \in d$. If $a = c$, then

$$zx = z \star g_a^{-1}(b) = zy$$

and again $(zx, zy) \in d$.

(2) $x, y \in H_a$ and $(x,y) \in r(\zeta)$. If $zx \notin H_a, zy \notin H_a$, then $(zx, zy) \in \ker(g)$, hence $(zx, zy) \in d$. Now consider the case where $zx \in H_a$ and $zy \in H_a$. Then, clearly, $(zx, zy) \in r(\zeta) \cap (H_a \times H_a)$, hence $(zx, zy) \in d$.

Now we have proved that $(zx, zy) \in d$ (in a similar way we could prove that $(xz, yz) \in d$). Thus d is a congruence on ζ.

After this we shall have a look at the relation s. Let $x, y, z \in \zeta_a$ and $(x,y) \in s$. Now we have to consider the following three cases:

(1) $x \notin H_a$. Then $y \notin H_a, x = y$ and $(z \star x, z \star y) \in s$.

(2) $x \in H_a$ and $z \star x \notin H_a$. Now $y \in H_a, (x,y) \in \ker(g_a), (z\star x, z\star y) \in \ker(g_a)$ and thus $z \star x = z \star y$ implying that $(z \star x, z \star y) \in s$.

(3) $x \in H_a$ and $z \star x \in H_a$. Then $y \in H_a, z \star y \in H_a$, and naturally $(x,y) \in r(\zeta)$. Now put $b = g_a(z)$ so that $a = ba$. If $b \neq a$ (this means that $z \notin H_a$), then $(ux, uy) \in r(\zeta)$ for any $u \in H_b$, and, moreover, $ux = z \star x$ and $uy = z \star y$, hence $(z \star x, z \star y) \in s$.

If $b = a$ (then $z \in H_a$), we have $(zx, zy) \in r(\zeta)$ and now $zx = z \star x$ and $zy = z \star y$. Once again $(z \star x, z \star y) \in s$, and we thus have shown that s is a congruence on ζ_a.

Now it is clear that $r(\zeta) \subseteq d \subseteq \ker(g)$. There exist the projective natural homomorphisms

$$p\colon \zeta \to \zeta/r(\zeta),$$
$$q\colon \zeta/r(\zeta) \to \zeta/d$$

and a homomorphism

$$k\colon \zeta/d \to S$$

such that $g = kqp$.

Since $s \subseteq \ker(g_a)$ we also have the projective natural homomorphism

$$f\colon \zeta_a \to \zeta_a/s$$

and a homomorphism

$$v\colon \zeta_a/s \to S$$

such that $g_a = vf$.

Finally, define a mapping $h\colon \zeta \to \zeta_a$ by

$$h(x) = \begin{cases} x & \text{if } x \in H_a, \\ g_a^{-1}(b) & \text{if } x \in H_b,\ b \neq a. \end{cases}$$

The mapping h thus defined is a homomorphism from ζ onto ζ_a, and we now have the following commutative diagram:

$$\begin{tikzcd}
\zeta \arrow[r, "p"] \arrow[dr, "h"'] & \zeta/r(\zeta) \arrow[r, "q"] & \zeta/d \arrow[r, "k"] & S \\
& \zeta_a \arrow[r, "f"] & \zeta_a/s \arrow[ur, "v"'] &
\end{tikzcd}$$

It is easily verified that $\ker(fh) = d = \ker(qp)$, from which it follows that the groupoids ζ/d and ζ_a/s are isomorphic. Since ζ/d is a homomorphic image of $\zeta/r(\zeta)$, it is a semigroup and it follows that ζ_a/s is a semigroup, too.

We first conclude that $s = \ker(g_a)$ and then $r(\zeta) \cap (H_a \times H_a) = H_a \times H_a$. This implies that $r(\zeta) \supseteq H_a \times H_a$, hence $r(\zeta) = \ker(g)$. The proof is complete.

4. Some representable pairs

We shall now establish some representable pairs by using the results of the preceding chapters. However, we first prove two preliminary lemmas.

Lemma 4.1. *Let M be a non-empty set. Then there exists a mapping t from M onto M such that for all $x, y \in M$ there exist positive integers m, n such that $t^m(x) = t^n(y)$.*

Proof. (1) Let k be a positive integer. Now the permutation $t(1) = 2$, $t(2) = 3$, ..., $t(k-1) = k$, $t(k) = 1$ on the set $\{1, 2, \dots, k\}$ has the desired property.

(2) If we consider $\mathbf{N}$, we define $t(k) = k - 1$ for every $k \geq 2$ and $t(1) = 1$.

(3) Let a be an infinite cardinal number, A be a set with $\operatorname{card}(A) = a$ and B be the set of all mappings $f\colon A \to \mathbf{N}$ with $f(x) \neq 1$ only for a finite number of elements x from A. Define a mapping t from B onto B by $t(f)(x) = 1$ if $f(x) = 1$ and $t(f)(x) = f(x) - 1$ if $f(x) \geq 2$. Again, t has the desired property.

Lemma 4.2. *Let S be a semigroup, $a \in L(S)$ (or $a \in R(S)$) and let $f\colon S \to C$ be a mapping such that $f(b) = 1$ for every $b \in S - \{a\}$. Then the pair (S, f) is representable by a groupoid.*

Proof. We shall now exceptionally denote the operation of S by $(\star)$. Denote further $R = S - \{a\}$ and let M be a set with $\operatorname{card}(M) = f(a)$ and $S \cap M = \emptyset$. Finally, let $\zeta = R \cup M$.

Since $a \in L(S)$, there exists $e \in S$ such that $e \star a = a$. We first assume that $a \notin \operatorname{Id}(S)$ and define a multiplication on ζ as follows:

(1) $ex = (e \star e) = t(x)$ for every $x \in M$, where t is a mapping from M onto M as given in Lemma 4.1,

(2) $bc = b \star c$ for all $b, c \in R$ with $b \star c \neq a$,

(3) $bc =$ any element of M if $b, c \in R$ and $b \star c = a$,

(4) $bx = b \star a$ if $b \in R$, $x \in M$ and $b \star a \neq a$,

(5) $bx =$ any element of M if $b \in R$, $b \neq e$, $b \neq e \star e$, $x \in M$ and $b \star a = a$,

(6) $xb = a \star b$ if $b \in R$, $x \in M$ and $a \star b \neq a$,

(7) $xb =$ any element of M if $b \in R$, $x \in M$ and $a \star b = a$,

(8) $xy = a \star a$ for all $x, y \in M$.

After this we define a mapping from ζ onto S by $g(x) = x$ if $x \in R$ and $g(x) = a$ if $x \in M$. Clearly, g is a homomorphism. It remains to show that $\ker(g) = r(\zeta)$. Once again, it is obvious that $r(\zeta) \subseteq \ker(g)$. Put $s = r(\zeta) \cap (M \times M)$. If $(x, y) \in s$, then $\big(t(x), t(y)\big) = (ex, ey) \in s$, which means that s is a congruence on the algebra (M, t) with one unary operation. If $x \in M$, then, by the definition of $r(\zeta)$, $\big(e(ex), (ee)x\big) \in r(\zeta)$. Now $e(ex) = t^2(x)$ and $(ee)x = t(x)$, hence $\big(t^2(x), t(x)\big) \in s$. In fact, if n is a positive integer, then $\big(t^n(x), t(x)\big) \in s$. Let $(u, v) \in M \times M$. Now there exist elements $a, b \in M$ such that $u = t(a)$ and $v = t(b)$. By Lemma 4.1, there also exist positive integers m, n such that

$t^m(a) = t^n(b)$. On the other hand, $(t^m(a), t(a)) \in s$ and $(t^n(b), t(b)) \in s$, hence $(t(a), t(b)) = (u, v) \in s$. It follows that $s = M \times M$. Then certainly $r(\zeta) \supseteq M \times M$ and $\ker(g) = r(\zeta)$.

Then assume that $a \in \mathrm{Id}(S)$. Choose an element $w \in M$ and define a multiplication on ζ as follows:

(1) $xy = w$ for all $x, y \in M, y \neq w$,
(2) $xw = x$ for every $x \in M$,
(3) $bc = b \star c$ for all $b, c \in R$ with $b \star c \neq a$,
(4) $bc = w$ if $b, c \in R$ and $b \star c = a$,
(5) $bx = b \star a$ if $b \in R$, $x \in M$ and $b \star a \neq a$,
(6) $bx = w$ if $b \in R$, $x \in M$ and $b \star a = a$,
(7) $xb = a \star b$ if $b \in R$, $x \in M$ and $a \star b \neq a$,
(8) $xb = w$ if $b \in R$, $x \in M$ and $a \star b = a$.

Then we define a mapping g from ζ onto S as in the first part of the proof. Naturally, g is a homomorphism. Suppose that $(x, y) \in M \times M$. By the definition of $r(\zeta)$, $(x(wx), (xw)x) \in r(\zeta)$, i.e., $(x, w) \in r(\zeta)$. Similarly, $(y, w) \in r(\zeta)$, hence $(x, y) \in r(\zeta)$. Now again $r(\zeta) \supseteq M \times M$ and $\ker(g) = r(\zeta)$. This completes the proof.

We are now able to give our first theorem about representable pairs.

Theorem 4.3. *Let S be a semigroup such that $S = L(S) \cup R(S)$. Then the pair (S, f) is representable by a groupoid for any mapping $f\colon S \to C$.*

Proof. Just combine Theorem 3.1 with Lemma 4.2.

By using Lemmas 1.3 and 1.4 we immediately have

Corollary 4.4. *The pair (S, f) is representable by a groupoid for any mapping $f\colon S \to C$ if*

(i) S is finite, commutative and $nc(S) \leq 1$, or
(ii) S contains at most four elements and $nc(S) \leq 1$.

The rest of this chapter is devoted to the investigation of the situation where $f(S) = \{1, 2\}$. We first prove

Theorem 4.5. *Let S be a semigroup, $a \in S$ and let $f\colon S \to C$ be a mapping such that $f(a) = 2$ and $f(b) = 1$ for every $b \in S$, $b \neq a$. Then the pair (S, f) is representable by a groupoid if and only if at least one of the following two conditions is satisfied:*

(i) $a \in L(S) \cup R(S)$,
(ii) there exist $x, y, z \in S$ such that $a = xyz$ and either $x \neq xy$ or $z \neq yz$.

Proof. If (i) holds, the result follows from Lemma 4.2.

Then assume that (ii) holds: i.e., $a = xyz$ with $x \neq xy$. Now take an element $e \notin S$ and put $\zeta = S \cup \{e\}$. An operation $(\star)$ on ζ is defined in the following manner:

(1) $u \star v = uv$ for all $u, v \in S$ with $uv \neq a$,
(2) $u \star v = a$ for all $u, v \in S$ with $uv = a$, and either $u \neq x$ or $v \neq yz$,
(3) $x \star yz = e$,
(4) $e \star u = a \star u$ and $u \star e = u \star a$ for every $u \in S$,
(5) $e \star e = a \star a$.

Define a mapping g from ζ onto S by $g(e) = a$ and $g(x) = x$ for every $x \in \zeta - \{e\}$. Clearly, g is a homomorphism and $\ker(g) = r(\zeta)$.

The case where (ii) holds, $a = xyz$ and $z \neq yz$ proceeds in a similar way.

Now we prove the converse statement. Suppose that the pair (S, f) is representable by a groupoid. This means that we have a groupoid ζ and a homomorphism $g: \zeta \to S$ such that $\ker(g) = r(\zeta), \operatorname{card}(g^{-1}(a)) = 2$ and $\operatorname{card}(g^{-1}(b)) = 1$ for every $b \in S, b \neq a$. Assume that neither (i) nor (ii) is true. Let u, v, w be elements of ζ and $x = uv, y = vw$. If $g(uy) \neq a$, then also $g(xw) \neq a$, hence $uy = xw$. If $g(uy) = a$, then $g(xw) = a$ and we have $a = g(u)g(v)g(w)$. Since (ii) does not hold, $g(u) = g(u)g(v) = g(x)$ and $g(w) = g(v)g(w) = g(y)$. Since $a \notin L(S) \cup R(S), g(u) \neq a \neq g(w)$, yielding $u = x$ and $w = y$. But then $u(vw) = uy = uw = xw = (uv)w$, and we have shown that ζ is a semigroup, a contradiction. Thus either (i) or (ii) is true as required.

By combining Theorems 3.1 and 4.5 we get

Corollary 4.6. *Let S be a semigroup such that for every $a \in S - \big(L(S) \cup R(S)\big)$ there exist elements $x, y, z \in S$ with $a = xyz$ and $(x, yz) \neq (xy, z)$. Then the pair (S, f) is representable by a groupoid for any mapping $f: S \to \{1, 2\}$.*

It is also easy to see that the following result (which is in fact partially converse to Lemma 2.1) is true.

Corollary 4.7. *Let S be a commutative semigroup and $f: S \to \{1, 2\}$ a mapping. Then the pair (S, f) is representable by a groupoid if $f(a) = 1$ for every $a \in S - S^3$.*

5. An example

In Lemma 2.2 we proved that if the pair (S, f) is representable by a groupoid and $a \in S^2$, then

$$f(a) \leq \sum f(b)f(c),$$

where we go through all elements $b, c \in S$ such that $bc = a$. In what follows we consider the semigroup $T = \{0, a, b, c, d\}$ from Example 1.5 and show that (T, f) is representable by a groupoid if $f(a) \leq f(b)f(c)$.

Let us assume that $f(a) \leq f(b)f(c)$. Then take five pair-wise disjoint sets P, A, B, C and D such that $\operatorname{card}(P) = f(0)$, $\operatorname{card}(A) = f(a)$, $\operatorname{card}(B) = f(b)$, $\operatorname{card}(C) = f(c)$ and $\operatorname{card}(D) = f(d)$. Put $\zeta = P \cup A \cup B \cup C \cup D$ and let $p: B \to B$ and $q: C \to C$ be mappings described in Lemma 4.1. From our assumption it follows that there exists a mapping h from $B \times C$ onto A. Now choose elements $z \in P$ and $w \in D$ and define a multiplication on ζ as follows:

(1) $xy = yx = z$ for every $x \in P$ and $y \in A \cup B \cup C \cup D$,
(2) $xy = z$ for all $x, y \in A \cup B$,
(3) $xy = yx = z$ for every $x \in A$ and $y \in C \cup D$,
(4) $xy = z$ for every $x \in C$ and $y \in B$,
(5) $xy = z$ for all $x, y \in C$,
(6) $xy = z$ for every $x \in C$ and $y \in D$,
(7) $xy = z$ for every $x \in D$ and $y \in B$,
(8) $xy = z$ for all $x, y \in P \quad (y \neq z)$,
(9) $xz = x$ for every $x \in P$,
(10) $xy = w$ for all $x, y \in D \quad (y \neq w)$,
(11) $xw = x$ for every $x \in D$,
(12) $xy = p(x)$ for every $x \in B$ and $y \in D$,
(13) $xy = q(y)$ for every $x \in D$ and $y \in C$,
(14) $xy = h(x, y)$ for every $x \in B$ and $y \in C$.

Then define a mapping g from ζ onto T by $g(P) = 0$, $g(A) = a$, $g(B) = b$, $g(C) = c$ and $g(D) = d$. It is easy to check that g is a homomorphism. We now have to show that $r(\zeta) = \ker(g)$.

First, $(x(xx), (xx)x) \in r(\zeta)$ for any $x \in P$. Since $x(xx) = xz = x$ and $(xx)x = zx = z$, it follows that $(x, z) \in r(\zeta)$, hence $P \times P \subseteq r(\zeta)$. Similarly, one can prove that $D \times D \subseteq r(\zeta)$. The inclusions $B \times B \subseteq r(\zeta)$ and $C \times C \subseteq r(\zeta)$ can be proved as in Lemma 4.2 (now the mappings p and q have the role of t). Finally, if $(x, y) \in B \times B$ and $(u, v) \in C \times C$, then also $(x, y) \in r(\zeta)$ and $(u, v) \in r(\zeta)$, hence $(xu, yv) \in r(\zeta)$. By definition $\big(h(x, u), h(y, v)\big) \in r(\zeta)$, and thus $A \times A \subseteq r(\zeta)$. We conclude that $\ker(g) = r(\zeta)$, and the proof is complete.

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