MORI'S THEOREM FOR n-DIMENSIONAL QUASICONFORMAL MAPPINGS

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1. Introduction

In this paper we shall study distortion properties of quasiconformal mappings in two cases. The first case deals with quasiconformal mappings of the unit ball $B^n$ in $\mathbb{R}^n$ for which we generalize a classical theorem of Akira Mori (see \cite{2}, p. 47, or \cite{16}, p. 66). The second case deals with quasiconformal mappings of the whole space $\mathbb{R}^n$ which keep the $x_1$-axis pointwise fixed. In both cases our results will have the correct limiting behavior as $K \to 1$. Furthermore, all the estimates involved are explicitly computable. We shall also study conformal mappings onto quasidisks.

In 1956 the following theorem of A. Mori appeared \cite{18}.

1.1. Theorem. A $K$-quasiconformal mapping $f$ of the unit disk $B^2$ onto itself with $f(0) = 0$ satisfies

\begin{equation}
|f(x) - f(y)| \leq 16|x - y|^{1/K}
\end{equation}

for all $x, y \in B^2$. Furthermore, the constant 16 in (1.2) cannot be replaced by any smaller constant independent of $K$.

The main result of this paper is the following generalization of Theorem 1.1.

1.3. Theorem. Let $f$ be a $K$-quasiconformal mapping of $B^n$ onto $B^n$, $n \geq 2$, with $f(0) = 0$. Then

\begin{equation}
|f(x) - f(y)| \leq M_1(n, K)|x - y|^{\alpha}
\end{equation}

for all $x, y \in B^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M_1(n, K)$ has the following three properties:

(1) $M_1(n, K) \to 1$ as $K \to 1$, uniformly in $n$;

(2) $M_1(n, K)$ remains bounded for fixed $K$ and varying $n$;

(3) $M_1(n, K)$ remains bounded for fixed $n$ and varying $K$.

An $n$-dimensional version of Mori's theorem has already been given in \cite{20}. In \cite{11}, Remark 1 on p. 235, it is said that this theorem holds with a constant

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satisfying (3) (namely \( M_1(n, K) \leq 4\lambda_n^2 \) in our notation of Section 2), and in [14] the inequality (1.4) is also proved, but with a constant that does not satisfy any of these three properties. In Section 2 (Theorem 2.28) we shall give explicit bounds from above for the constant \( M_1(n, K) \) which actually hold in a wider class of mappings of the unit ball (cf. (2.15)). For an extension of Mori’s theorem to more general domains the reader is referred to [12], Corollary 3.30, and, for a recent application of it, to [8], Section 10.

In Section 3 we shall prove

1.6. Theorem. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a \( K \)-quasiconformal mapping which keeps the \( x_1 \)-axis pointwise fixed. If \( K > 1 \), then

\[
|f(x)| \leq \lambda_n^{2\beta-2} \frac{\beta^\beta}{(\beta - 1)^{\beta-1}} |x|
\]

for all \( x \in \mathbb{R}^n \) where \( \beta = K^{1/(n-1)} \) and \( \lambda_n \) is the Grötzsch ring constant (see Section 2).

This theorem is a sharpened version of Corollary 2.17 in [4]. Observe that the constant in (1.7) tends to one as \( K \to 1 \). Finally, in the last section, we apply these results to plane conformal mappings of the unit disk onto bounded \( K \)-quasidisks, again paying attention to the limiting behavior as \( K \to 1 \).

It is conjectured (cf. [16], p. 68) that the best constant in (1.2) is \( 16^{1-1/K} \), in place of 16. E. Reich has kindly informed us that his student G.P. Schwartz proved Mori’s theorem (1.2) with the constant \( 360^{1-1/K} \) in place of 16, in an unpublished Ph.D. thesis in 1970. Schwartz’ work relies heavily on the parametric representation of plane quasiconformal mappings and is therefore restricted to the two-dimensional case. A further improvement in the plane case has also been given in [19].

We shall adopt the relatively standard notation of [22], i.e., \( e_1, \ldots, e_n \) denote the orthogonal unit basis vectors, \( B^n(x, r) \) the ball with center \( x \) and radius \( r > 0 \), \( S^{n-1}(x, r) = \partial B^n(x, r) \), \( B^n(r) = B^n(0, r) \), \( S^{n-1}(r) = \partial B^n(r) \), \( B^n = B^n(1) \), \( S^{n-1} = \partial B^n \) and \( \omega_{n-1} \) the \( (n-1) \)-dimensional Lebesgue measure of \( S^{n-1} \). In particular, we employ the definition of \( K \)-quasiconformal mapping given in [22], p. 42.

2. Mori’s theorem

We shall next introduce some notation and some estimates necessary for the sequel.

A domain \( R \) in \( \mathbb{R}^n \) is called a ring or a ring domain if its complement in \( \mathbb{R}^n \) consists of two components. Its conformal capacity is denoted by \( \text{cap} R \). By \( R_{G,n}(t) \), \( t > 1 \), we denote the Grötzsch ring whose complementary components consist of the closed unit ball \( B^n \) and the ray \( \{ te_1, \infty \} = \{ se_1 : s \geq t \} \), and
by $R_{T,n}(t)$, $t > 0$, the Teichmüller ring whose complementary components are $[-e_1, 0] = \{ se_1: -1 \leq s \leq 0 \}$ and $[te_1, \infty]$. For their capacities we write

\[ \gamma_n(t) = \text{cap } R_G,n(t), \]
\[ \tau_n(t) = \text{cap } R_{T,n}(t). \]

These functions are related by the functional identity

\[ \gamma_n(t) = 2^{n-1} \tau_n(t^2 - 1) \]  

(cf. [9], Lemma 6). Later we shall also use the estimation ([9], Lemma 8)

\[ \gamma_n(t) \geq \omega_{n-1}(\log \lambda_n)^{1-n}, \quad t > 1, \]

where $\lambda_n \in [4, 2e^{n-1}]$ is the Grötzsch ring constant (cf. [10]; for these estimations from above see [3] and from below [7], [13]; note also that $\lambda_2 = 4$ [16]).

For $K > 0$ we define a homeomorphism $\varphi_{K,n}: [0,1] \to [0,1]$ with $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$ and

\[ \varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K \gamma_n(1/t))}, \quad 0 < t < 1. \]

Throughout this paper we use $\alpha$ and $\beta$ to denote the following numbers

\[ \alpha = K^{1/(1-n)}, \quad \beta = 1/\alpha. \]

The following important estimates (due to Wang [26] for $n = 2$ and generalized to $n \geq 2$ in [4]) are essential for the sequel

\[ \varphi_{K,n}(t) \leq \lambda_n^{1-\alpha} t^\alpha, \]
\[ \varphi_{1/K,n}(t) \geq \lambda_n^{1-\beta} t^\beta, \]

where $K \geq 1$. For $n = 2$, (2.4) is given also in [16] p. 65.

The Poincaré metric $\varrho(x,y)$ on $B^n$ is defined by (cf. [6])

\[ \tanh^2 \frac{1}{2} \varrho(x,y) = \frac{|x - y|^2}{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}. \]

It is easy to show (see [5], 3.2) that

\[ |x - y| \leq 2 \tanh \frac{1}{4} \varrho(x,y) \]

for all $x, y \in B^n$.

The following theorem, a quasiconformal counterpart of the Schwarz lemma, is a conformally invariant formulation of Theorem 3.1 in [17] (cf. [23], 3.3).
2.8. **Theorem.** Let $f$ be a $K$-quasiregular mapping of the unit ball $B^n$ into $B^n$. Then

$$\tanh \frac{1}{2} g(f(x), f(y)) \leq \varphi_{K,n}(\tanh \frac{1}{2} g(x,y))$$

for all $x, y \in B^n$.

2.10. **Corollary.** Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a $K$-quasiconformal mapping with $f(0) = 0$, $f(\infty) = \infty$ and $fB^n \subset B^n$. If $s > 1$ and $|x| \leq s$, then

$$|f(x)| \leq \gamma_n^{-1} (\gamma_n(s)/K).$$

**Proof.** This inequality follows easily by inversion, application of Theorem 2.8 to the inverse mapping and formula (2.3).

As in [4] we define

$$H_n(K) = \sup \frac{|f(x)|}{|f(y)|}$$

where the supremum is taken over all $K$-quasiconformal mappings $f: \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ and over all pairs of points $x, y$ in $\mathbb{R}^n$ with $|x| = |y| > 0$. From [24] and (2.5) we have

$$H_n(K) \leq 1/\varphi_{1/K,n}(1/\sqrt{2})^2 \leq \lambda_n^{2\beta-2\beta}.$$  

Since $\lambda_n \leq 2e^{n-1}$ we get as in [4] a dimension-free bound for $H_n(K)$, namely $\lambda_n^{1-\alpha} \leq 2^{1-1/K}K$, and hence $H_n(K) \leq 2^{1-1}K^{2K}$. Therefore this number remains bounded for fixed $K$ and varying $n$. We also observe that $\lambda_n^{1-\alpha} \to 1$ as $K \to 1$, uniformly in $n$. Next we shall use the fact that

$$\lim_{K \to 1} H_n(K) = 1$$

for every $n \geq 2$. This can be concluded by a normal family argument. A quantitative inequality with this property has been given in [25], namely

$$H_n(K) \leq \lambda_n^{2(\beta^2-1)} \exp(3K(K + 1)\sqrt{K-1}),$$

for all $K \geq 1$ and $n \geq 2$. Hence, as $\lambda_n^{1-\alpha}$ and $\alpha$, also $H_n(K)$ tends to one for $K \to 1$, uniformly in $n$.

Taking into account that a $K$-quasiconformal mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ and $f(\infty) = \infty$ maps the ball $B^n(s)$ into $B^n(H_n(K)|f(se_1)|)$, we note that by Corollary 2.10 we get
2.14. Corollary. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $K$-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$. If $|x| \leq |y|$, $s > 1$, then

$$|f(x)| \leq H_n(K) \gamma_n^{-1}(\gamma_n(s)/K)|f(y)|.$$  

We observe that this estimation is similar to Theorem 2.12 in [4] where the constant is

$$1 + \tau_n^{-1}(\tau_n(s)/K) = (\gamma_n^{-1}(\gamma_n(\sqrt{1 + s})/K))^2,$$

which is better in general and applies to all values $s > 0$. However, the constant in Corollary 2.14 has the advantage that it tends to $s$ for $K \rightarrow 1$, it is hence sharp. Finally, we want to add the remark that in [1] it is shown that for $n = 2$ the sharp constant is $\tau_2^{-1}(\tau_2(s)/K)$ for all $s \geq 1$.

Every mapping satisfying the assumptions of Theorem 1.3 can be extended by reflection to a $K$-quasiconformal mapping of the whole space $\mathbb{R}^n$. This leads us to the

2.15. Definition. $M_1(n, K)$ is the smallest number such that (1.4) holds for all $x, y \in B^n$ and for all $K$-quasiconformal mappings $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(0) = 0$, $f(\infty) = \infty$ and $fB^n \subset B^n$.

We prove now that $M_1(n, K)$ satisfies (1.5) from which Theorem 1.3 then follows. Let $f$ be as in the definition above and fix $x, y \in B^n$. First we prove part (3) of (1.5) (and in particular that $M_1(n, K) \leq 3\lambda_n^2$). To this end we employ a fairly straightforward generalization of the 2-dimensional argument in [16], p. 66.

Proof of part (3). The proof is divided into two cases. Consider first the case when

$$|x - y|^2 + (1 - |x|^2)(1 - |y|^2) \geq 1/16.$$

Then by (2.6)

$$(2.16) \quad \tanh \frac{1}{2} \varrho(x, y) \leq 4|x - y|.$$  

Furthermore, by (2.7) and Theorem 2.8

$$|f(x) - f(y)| \leq 2 \tanh \frac{1}{2} \varrho(f(x), f(y)) \leq 2 \tanh \frac{1}{2} \varrho(f(x), f(y)) \leq 2 \varphi_{K,n}(\tanh \frac{1}{2} \varrho(x, y)),$$

and by (2.4) and (2.16)

$$(2.17) \quad |f(x) - f(y)| \leq 2\lambda_n^{1-\alpha} (\tanh \frac{1}{2} \varrho(x, y))^{\alpha} \leq 2\lambda_n^{1-\alpha} 4^{\alpha}|x - y|^{\alpha}.$$  

In the remaining second case we have $|x - y| \leq 1/4$ and $(1 - |x|^2)(1 - |y|^2) \leq 1/16$. We may assume that $1 - |x|^2 \leq 1/4$, so $|x| \geq \sqrt{3}/2 > 0.85$. Hence

$$\frac{1}{2}|x + y| = |x + \frac{1}{2}(y - x)| \geq |x| - \frac{1}{2}|x - y| > 0.7.$$
Then the ring domain

\[ A = \{ z \in \mathbb{R}^n : \frac{1}{2}|x-y| < |z - \frac{1}{2}(x+y)| < \frac{1}{2} \} \]

separates the origin and infinity from \( x \) and \( y \), so \( fA \) separates 0 and \( \infty \) from \( f(x) \) and \( f(y) \). By performing a spherical symmetrization we obtain by [9] (Lemma 2.6 in [4])

\[
\text{cap} \ fA \geq \tau_n \left( \frac{|f(y)|}{|f(x) - f(y)|} \right).
\]

Furthermore, we have

\[
\text{cap} \ fA \leq K \text{ cap} \ A = K \omega_{n-1} \left( \log(1/|x-y|) \right)^{1-n}.
\]

The functional identity (2.1) gives

\[
2^{n-1}K\omega_{n-1} \left( \log(1/|x-y|) \right)^{1-n} \geq \gamma_n \left( \frac{|f(x) - f(y)| + |f(y)|}{|f(x) - f(y)|} \right).
\]

Then we use \(|f(x)|, |f(y)| \leq 1\), the fact that \( \gamma_n \) is decreasing and (2.2) to infer that the right side is larger than

\[
\omega_{n-1} \left( \log \left( \lambda_n \sqrt{\frac{3}{|f(x) - f(y)|}} \right) \right)^{1-n}.
\]

Hence

\[
\log \frac{1}{|x-y|} \leq 2\beta \log \left( \lambda_n \sqrt{\frac{3}{|f(x) - f(y)|}} \right)
\]

and finally

\[
(2.18) \quad |f(x) - f(y)| \leq 3\lambda_n^2|x-y|^{\alpha}.
\]

Since \( 4 \leq \lambda_n \), the inequality (2.18) holds in both cases (cf. (2.17)). Hence part (3) of (1.5) is proved with \( M_1(n,K) \leq 3\lambda_n^2 \).

**Proof of part (1) and (2).** Fix \( s > 1 \). Corollary 2.10 implies that \( f \) maps the ball \( B^n(s) \) into \( B^n(c) \) where \( c = \gamma_n^{-1}(\gamma_n(s)/K) \). We define \( g(z) = f(sz)/c \) and note that \( g \) maps \( B^n(1/s) \) into \( B^n(1/c) \). We put

\[
a = \frac{1}{2} \phi \left( \frac{x}{s}, \frac{y}{s} \right).
\]
By (2.6) and \(|x/s|, |y/s| \leq 1/s\) we have

\[
\tanh a \leq \frac{|x/s - y/s|}{\sqrt{(1 - |x/s|^2)(1 - |y/s|^2)}} \leq \frac{s|x - y|}{s^2 - 1}.
\]

Application of Theorem 3.4 in [5] to the mapping \(g\) gives

\[
|f(x) - f(y)| = c|g(x/s) - g(y/s)| \leq c \frac{2\varphi_{K,n}(\tanh a)}{1 + \sqrt{1 - \varphi^2_{K,n}(\tanh a)}}.
\]

From (2.3) and (2.5) we use

\[
c \leq \lambda_n^{-1}s^\beta,
\]

from (2.4) and (2.19)

\[
\varphi_{K,n}(\tanh a) \leq \min \left\{1, \lambda_n^{1-\alpha} \left(\frac{s}{s^2 - 1}\right)^\alpha |x - y|^\alpha\right\}
\]

and hence we get

\[
|f(x) - f(y)| \leq \lambda_n^{\beta-1}s^\beta \frac{2\lambda_n^{1-\alpha}(s/(s^2 - 1))^{\alpha|x - y|^\alpha}}{1 + \sqrt{1 - \min\{1, \lambda_n^{2-2\alpha}(s/(s^2 - 1))^{2\alpha}|x - y|^{2\alpha}\}}.
\]

This inequality holds for all \(s > 1\). We choose \(s\) (which depends on \(K\)) such that \(s^{\beta+\alpha}/(s^2 - 1)^\alpha\) becomes minimal. This amounts to putting

\[
s = \sqrt{\frac{\beta^2 + 1}{\beta^2 - 1}}.
\]

A straightforward computation shows that we have proved that

\[
M_1(n, K) \leq \theta(n, K)\lambda_n^{\beta-\alpha}(\beta^2 + 1)^{\beta/2 + \alpha/2}
\]

where

\[
\theta(n, K) = \frac{2}{1 + \sqrt{1 - \min\{1, \lambda_n^{2-2\alpha}(\beta^4 - 1)^\alpha\}}}.
\]

For \(K \to 1\), \(\lambda_n^{1-\alpha}\) tends to one, uniformly in \(n\), as well as \(\alpha\) and \(\beta\) do. Hence \(\theta(n, K) \to 1\) uniformly in \(n\) and so does \(M_1(n, K)\), since

\[
(\beta^2 - 1)^{\beta^2-\alpha/2} = (\beta + 1)^{\beta^2-\alpha/2}\beta^{1/2-\alpha/2}\sqrt{(\beta - 1)^{\beta^2-1}(1 - \alpha)^{1-\alpha}}.
\]

Part (1) is proved and part (2) is now evident, since the following bounds do not depend on \(n\):

\[
\theta(n, K) \leq 2, \quad \lambda_n^{\beta-\alpha} \leq (2^{1-1/K}K)^{K+1},
\]

\[
2^{-\alpha}(\beta^2 + 1)^{\beta/2 + \alpha/2} \leq (K^2 + 1)^{K/2+1/2}, \quad (\beta^2 - 1)^{\alpha/2-\beta/2} \leq \exp(-1/e).
\]

The proof is complete.
Finally we derive a

2.26. Corollary. Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be \( K \)-quasiconformal with \( f(0) = 0 \), \( f(e_1) = e_1 \) and \( f(\infty) = \infty \). Then

\[
|f(x) - f(y)| \leq M_2(n, K)|x - y|^\alpha
\]

for all \( x, y \in B^n \) where the constant \( M_2(n, K) \) has the properties

1. \( M_2(n, K) \leq H_n(K)M_1(n, K) \);
2. \( M_2(n, K) \to 1 \) as \( K \to 1 \), uniformly in \( n \);
3. \( M_2(n, K) \) remains bounded for fixed \( K \) and varying \( n \).

Proof. Since such a mapping \( f \) maps \( B^n \) into \( B^n(0, H_n(K)) \), the former result applied to \( f(z)/H_n(K) \) yields (2.27) and (1). From (1), the properties of \( M_1(n, K) \), (2.12) and (2.13) we get (2) as well as (3).

Remark. For any constant \( M_2(n, K) \) satisfying (2.27) clearly \( M_2(n, K) \to \infty \) for fixed \( n \) and \( K \to \infty \) as the example \( f_0 \) shows where \( f_0 \) is the identity in the right half space and an affine stretching in the left half space. A quantitative better lower bound is obtained by observing that (2.27) with \( x = 0 \), \( |y| = 1 \) implies that

\[
M_2(n, K) \geq H_n(K) \geq \lambda(\beta)
\]

where the last inequality is (1.14) in [4]. Here \( \lambda(K) \) is a well-known transcendental function (cf.[16], p. 81). In fact

\[
\lambda(K) = \left( \frac{\varphi_{K,2}(1/\sqrt{2})}{\varphi_{1/K,2}(1/\sqrt{2})} \right)^2
\]

and hence \( \lambda(K) \to 1 \) as \( K \to 1 \) and \( \lambda(K) \to \infty \) as \( K \to \infty \). For the constant \( M_1(n, K) \) we have the lower bound \( 4^{1-\alpha} \). This follows by rotation of the extremal plane quasiconformal mapping between the extremal Grötzsch ring domains (as it is done in the proof of Theorem 4.9 in [4]) and the fact that \( \lim_{r \to 0} r^\alpha \varphi_{1/2,2}(r) = 4^{1-\alpha} \) (see p. 65 in [16]). In [16], p. 68, it is also shown that \( M_1(2, K) \geq 16^{1-1/K} \).

We collect our results:

2.28. Theorem. The constant \( M_1(n, K) \) satisfies \( M_1(n, K) \leq 3\lambda_n^2 \) and

\[
M_1(n, K) \leq \theta(n, K)\lambda_n^{\beta-\alpha}(\beta^2 + 1)^{\beta/2+\alpha/2} \frac{2^\alpha(\beta^2 - 1)^{\beta/2-\alpha/2}}{2^n}
\]

where \( \theta(n, K) \) is given by (2.25), in particular, \( \theta(n, K) \in [1, 2] \) and \( \theta(n, K) \to 1 \) for \( K \to 1 \).
**Remark.** It is not known if there exists an upper bound for $M_1(n, K)$ which is independent of $n$ and $K$.

We close this section by considering the special case $n = 2$ and improving Theorem 2.28 slightly.

**2.29. Theorem.** The constant $M_1(2, K)$ satisfies $M_1(2, K) \leq 16$ and

$$M_1(2, K) \leq \left(1 + \varphi_{K,2} \left(\frac{K^2 - 1}{K^2 + 1}\right)\right) 2^{2K-3/K} \frac{(K^2 + 1)^{(K+1/K)/2}}{(K^2 - 1)^{(K-1/K)/2}}.$$

**Proof.** $M_1(2, K) \leq 16$ is the original content of Theorem 1.1, and from its proof in [16], p. 67, it is clear that this constant also holds for our definition (2.15) of the constant $M_1(2, K)$. For the second part we use the same notation as in the preceding proof and recall (2.20) for $n = 2$:

\begin{equation}
(2.30) \quad |f(x) - f(y)| \leq c \frac{2\varphi_{K,2}(\tanh a)}{1 + \sqrt{1 - \varphi_{K,2}^2(tanh a)}}.
\end{equation}

For two points $u, v \in B^n(r)$ with $0 \leq r \leq 1$ we have

\begin{equation}
(2.31) \quad \tanh \frac{1}{2} \varrho(u, v) \leq \frac{2r}{1 + r^2},
\end{equation}

because $\varrho(u, v)$ is maximized in $\bar{B}^n(r)$ for opposite points on $S^{n-1}(r)$ (where its value is $2\log((1 + r)/(1 - r))$ and $\tanh \log((1 + r)/(1 - r)) = 2r/(1 + r^2)$). Next we use the functional identity

\begin{equation}
(2.32) \quad \frac{2}{1 + \sqrt{1 - \varphi_{K,2}^2(2r/(1 + r^2))}} = 1 + \varphi_{K,2}(r^2).
\end{equation}

This can be derived from the identities

\begin{equation}
(2.33) \quad \varphi_{K,2}(r) = \sqrt{1 - \varphi_{1,K,2}^2(2\sqrt{1 - r^2})},
\end{equation}

\begin{equation}
(2.34) \quad \varphi_{K,2}(r) = \frac{1 - \varphi_{1,K,2}(1 - r)/(1 + r)}{1 + \varphi_{1,K,2}((1 - r)/(1 + r))}
\end{equation}

which follow from [16], (2.7) and (2.9) on p. 61, by applying the function $\mu(r) = 2\pi/\gamma_2(1/r)$ to (2.33) and (2.34) and recalling that $\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$.

Since $|x/s|, |y/s| \leq 1/s$ we have by (2.31) with $r = 1/s$

$$\varphi_{K,2}(\tanh a) \leq \varphi_{K,2} \left(\frac{2/s}{1 + 1/s^2}\right).$$
and with (2.30) and (2.32)

$$|f(x) - f(y)| \leq c(1 + \varphi_{K,2}(1/s^2))\varphi_{K,2}(\tanh a),$$

finally (2.4), (2.19) and $c \leq 4^{K-1}s^K$ give

$$|f(x) - f(y)| \leq 4^{K-1}s^K (1 + \varphi_{K,2}(1/s^2)) \frac{1}{4^{1-1/K}} \left(\frac{s}{s^2 - 1}\right)^{1/K} |x - y|^{1/K}$$

and, as before, the choice $s = \sqrt{(K^2 + 1)/(K^2 - 1)}$ yields the desired bound.

3. Mappings keeping an axis pointwise fixed

In this section we study the distortion of $K$-quasiconformal mappings $f$: $\mathbb{R}^n \to \mathbb{R}^n$ with the property

$$(3.1) \quad f(te_1) = te_1 \quad \text{for all } t \in \mathbb{R}.$$ 

**Proof of Theorem 1.6.** Let $f$ be a $K$-quasiconformal mapping of $\mathbb{R}^n$ satisfying (3.1). In order to study the quantity $|f(x)|/|x|$ we may evidently assume that $|x| = 1$ and that $f(x)$ is in the right half space (first co-ordinate non-negative). Then we fix $s > 0$ and consider the ring $R'$ whose complement consists of $[-se_1, 0]$ and $\{ f(x) + t(f(x) + e_1): t \geq 0 \}$. We put $a = |f(x) + se_1|$, and hence $a^2 \geq |f(x)|^2 + s^2$. By Lemma 2.58 in [23], which is due to Gehring (Lemma 2.7 in [4]), we have

$$(3.2) \quad \text{cap } R' \leq \tau_n \left( \frac{a}{s} - 1 \right).$$

On the other hand, we put $R = f^{-1}(R')$ and conclude by [9] (Lemma 2.6 in [4])

$$(3.3) \quad \text{cap } R \geq \tau_n \left( \frac{1}{s} \right).$$

(3.2), (3.3) and $\text{cap } R \leq K \text{ cap } R'$ then yield

$$(3.4) \quad 1 + \tau_n^{-1} \left( \tau_n \left( \frac{1}{s} \right) / K \right) \geq \frac{a}{s} \geq \sqrt{1 + |f(x)|^2/s^2}.$$ 

From the functional identity (2.1) and the definition (2.3) we infer that the left side in (3.4) is equal to

$$c := 1/\varphi^2_{1/K, n}(1/\sqrt{1 + 1/s}).$$
Hence

\[(3.5) \quad |f(x)| < s\sqrt{c^2 - 1}.\]

If we choose \(s = 1\), then this proof reduces to the one given in [4], and (3.5) reduces to the bound given there. To get a bound that gives the right behavior for \(K \to 1\) we use from (3.5)

\[|f(x)| \leq sc\]

and then (2.5) to get

\[|f(x)| \leq \lambda_n^{2\beta - 2} s^{1+\frac{1}{s}}.\]

This holds for any \(s > 0\). The best choice is \(s = \beta - 1\) which yields

\[|f(x)| \leq \lambda_n^{2\beta - 2} \frac{\beta^\beta}{(\beta - 1)^{\beta - 1}}\]

and Theorem 1.6 is proved.

In the special case \(n = 2\) the set of values which can be taken by \(K\)-quasiconformal mappings satisfying (3.1) is known for any subset of \(\mathbb{R}^2\). Namely, \(f\) then maps the upper half plane onto itself keeping the boundary points fixed, so Teichmüller's Verschiebungssatz [21] then provides the answer. This result easily shows (see [15]) that the set of values \(f(x)\) of such mappings \(f\) at a given point \(x\) is a hyperbolic disk with center \(x\) and radius

\[\varrho(K) = 2 \arctan \mu^{-1} \left( \log \left( \frac{\sqrt{K} + 1}{\sqrt{K} - 1} \right) \right),\]

where \(\mu(r) = 2\pi / \gamma_2(1/r)\) as above. Hence the possible set of values attained on \(S^1\) is the set of all points \(x\) in the upper half plane with hyperbolic distance to \(S^1\) less or equal \(\varrho(K)\) as well as its mirror image in the lower half plane and the points 1 and \(-1\). The euclidean distance of \(x\) and \(f(x)\) is hence maximal for \(x = i\) and \(f(x) = i \exp g(K)\). Therefore (1.7) holds with the sharp constant

\[\exp g(K) = (1 + \mu^{-1}(t)) / (1 - \mu^{-1}(t))\]

instead of \(\lambda_n^{2\beta - 2} \beta^\beta / (\beta - 1)^{\beta - 1}\) where \(t = \log((\sqrt{K} + 1) / (\sqrt{K} - 1))\).

4. Conformal mappings of the unit disk onto quasidisks

A plane domain \(D\) is called a \(K\)-quasidisk if there exists a \(K\)-quasiconformal mapping \(g: \overline{D} \to \overline{D}\) with \(gD = B^2\). We first prove

4.1. Lemma. Let \(D\) be a \(K\)-quasidisk with \(0 \in D\) and \(\max \{|z|: z \in \partial D\} = 1\). If \(r = \min \{|z|: z \in \partial D\}\) then there is a number \(K_1 = K_1(r, K)\) such that there is a \(K_1\)-quasiconformal mapping \(g_1: \overline{D} \to \overline{D}\) with \(g_1(0) = 0\), \(g_1(\infty) = \infty\) and \(g_1D = B^2\) where \(K_1 \to 1\) as \(K \to 1\) and \(r \to 1\).
Remark. To achieve $K_1 \rightarrow 1$ it is necessary to let $r$ tend to one, too.

Proof. Let $g$ be the $K$-quasiconformal mapping in the definition of the bounded $K$-quasidisk $D$. By a Möbius transformation we may assume that $g(\infty) = \infty$. First we find an upper bound for $|g(0)|$ (and may hence assume that $g(0) \neq 0$).

Let $R'$ be the ring with complementary components $[g(0), g(0)/|g(0)|]$ and $[-\infty, -g(0)/|g(0)|]$. Then

$$\text{cap } R' = \tau_2 \left( \frac{1 + |g(0)|}{1 - |g(0)|} \right).$$

Next we put $R = g^{-1}(R')$ and conclude as in the preceding section by [9] or Lemma 2.6 in [4]

$$\text{cap } R \geq \tau_2 \left( \frac{1}{r} \right).$$

From $\text{cap } R \leq K \text{ cap } R'$, and (4.2) and (4.3) we derive that

$$\frac{1 + |g(0)|}{1 - |g(0)|} \leq \tau_2^{-1} \left( \frac{1}{K \tau_2} \left( \frac{1}{r} \right) \right).$$

By Teichmüller's Verschiebungssatz there is a $K^*$-quasiconformal mapping $g^*$: $B^2 \to B^2$ with $g^*(z) = z$ for $z \in \partial B^2$ and $g^*(g(0)) = 0$ with

$$\varrho(K^*) = \log \frac{1 + |g(0)|}{1 - |g(0)|},$$

where $\varrho(K)$ is as in Section 3. Hence the explicit formula is

$$K^* = \left( \frac{\exp \mu(|g(0)|) + 1}{\exp \mu(|g(0)|) - 1} \right)^2.$$

The desired mapping $g_1$ is now defined by $g_1(z) = g(z)$ for $z \in \overline{B}^2 \setminus D$ and $g_1(z) = g^*(g(z))$ for $z \in \overline{D}$. Its maximal dilatation $K_1 \leq KK^*$ has the required property by (4.4) and (4.5).

Remark. Explicit estimates for $K_1(r, K)$ can be derived from (4.4) and (4.5) and

$$\tau_2^{-1} \left( \frac{1}{K \tau_2} \left( \frac{1}{r} \right) \right) = \frac{1 - \varphi_{1/K,2}^2(\sqrt{r/(r+1)})}{\varphi_{1/K,2}^2(\sqrt{r/(r+1)})} = \frac{\varphi_{K,2}^2(\sqrt{1/(r+1)})}{\varphi_{1/K,2}^2(\sqrt{r/(r+1)})} \leq 4^{2(1-1/K)} \left( \frac{1}{r+1} \right)^{1/K} 4^{2(K-1)} \left( \frac{r+1}{r} \right)^K \frac{K-1/K}{rK}$$

where (2.1), (2.3), (2.33) and finally (2.4) and (2.5) have been used.
4.6. Theorem. Let $D$ be a bounded $K$-quasidisk, normalized such that $0 \in D$ and $1 = \max\{|z|: z \in \partial D\}$. Let $f: B^2 \to D$ be a conformal mapping with $f(0) = 0$. If $r = \min\{|z|: z \in \partial D\}$, then
\[
|f(x) - f(y)| \leq M_1(2, K_1^2)|x - y|^{1/K_1^2}
\]
for all $x, y \in B^2$ where $K_1 = K_1(r, K)$ is the constant from Lemma 4.1, in particular, the constant $M_1(2, K_1^2)$ tends to one for $K$ and $r$ tending to one.

Proof. Let $g_1$ be as in Lemma 4.1. Then $f$ has a $K_1^2$-quasiconformal extension to $\mathbb{R}^2$ which keeps $\infty$ fixed, namely $g_1^{-1} \circ i \circ g_1 \circ f \circ i$ where $i$ denotes inversion. By Definition 2.15 the inequality follows, since this extension fixes 0 and $\infty$ and sends $B^2$ into itself.

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