SOME COEFFICIENT ESTIMATIONS
IN THE CLASS $\Sigma'_b$ OF MEROMORPHIC
UNIVALENT FUNCTIONS

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1. Introduction

Let $\Sigma_b$ be the class of meromorphic univalent functions omitting a disc:

$$\Sigma_b = \{ H \mid H(z) = z + \sum_{\nu=0}^{\infty} A_\nu z^{-\nu}, \quad |z| > 1, \quad |H(z)| > b \in (0,1) \}.$$

This is closely connected with the class $S(b)$ of bounded univalent functions:

$$S(b) = \{ f \mid f(z) = b(z + \sum_{\nu=2}^{\infty} a_\nu z^\nu), \quad |z| < 1, \quad |f(z)| < 1, \quad b \in (0,1) \}.$$

The one-to-one connection between $f$ and $H$ reads

$$H(z) f(z^{-1}) = b, \quad |z| > 1.$$

From $w = H(z)$ we obtain the inverse relationship

$$z = I(w) = w + \sum_{\nu=0}^{\infty} E_\nu w^{-\nu}.$$

In [6] the inverse coefficients $E_n$ were all maximized by the radial-slit mapping $I_r$ defined by

$$I_r + I_r^{-1} = w - 2(1 - b) + b^2 w^{-1}.$$

The side-condition $A_0 = -a_2 = 0$ yields the subclass $\Sigma'_b \subset \Sigma_b$:

$$\Sigma'_b = \{ H \mid H(z) = z + \sum_{\nu=1}^{\infty} A_\nu z^{-\nu}, \quad |z| > 1, \quad |H(z)| > b \in (0,1) \}.$$

The special case $b = 0$, i.e., $\Sigma'_0 = \Sigma'$, has been extensively considered by Schober [4]. Especially the initial odd inverse coefficients were succesfully estimated.

In the present paper we are dealing with $\Sigma'_b$ and will generalize some of the results of [4] for odd inverse coefficients. Similarly, some initial $A_\nu$-coefficients will be considered.

The connections between $H$, $I$ and $f$ yield the corresponding coefficient connections. Thus

$$
\begin{align*}
A_1 &= -E_1, \\
A_2 &= -E_2, \\
A_3 &= -E_3 - E_1^2, \\
A_4 &= -E_4 - 3E_2E_1, \\
A_5 &= -E_5 - 4E_3E_1 - 2E_2^2 - 2E_1^2.
\end{align*}
$$

Clearly, the letters $A$ and $E$ can here also be interchanged. In $a_\nu$-coefficients we have further

$$
\begin{align*}
A_1 &= -a_3, & E_1 &= a_3, \\
A_2 &= -a_4, & E_2 &= a_4, \\
A_3 &= -a_5 + a_3^2, & E_3 &= a_5 - 2a_3^2, \\
A_4 &= -a_6 + 2a_4a_3, & E_4 &= a_6 - 5a_4a_3, \\
A_5 &= -a_7 + 2a_5a_3 + a_4^2 - a_3^2; & E_5 &= a_7 - 6a_5a_3 - 3a_4^2 + 7a_3^2.
\end{align*}
$$

2. $A_1$, $A_2$ by aid of coefficient bodies

For lower $a_\nu$-coefficients we have lots of connections and inequalities which yield sharp information also in the special case $A_0 = -a_2 = 0$ in question.

The first non-trivial coefficient body $(a_2,a_3)$ is studied in [9] (cf. p. 241 as well as pp. 264–265). Thus, for $a_2 = 0$ we have $|a_3| \leq 1 - b^2$, i.e.,

$$
|A_1| = |E_1| \leq 1 - b^2.
$$

The equality holds for the function $f$ of the type 2:2 (the notation is explained in [9], p. 149) with the image of two symmetric radial slits and defined by

$$
f(1 + f^2)^{-1} = b z (1 + z^2)^{-1}.
$$

The second coefficient body $(a_2,a_3,a_4)$ is also described in [9] with sufficient accuracy for our present purposes. However, there exist also direct estimations, given by Grunsky-type inequalities, which are sharp in the special case $a_2 = 0$. For example, from [7] there follows

$$
\Re a_4 \leq \frac{2}{3}(1 - b^3) - \frac{|a_3|^2}{4(1 - b)} \leq \frac{2}{3}(1 - b^3).
$$
The final equality holds for $a_2 = a_3 = 0$ and for a symmetric radial-slit mapping 3:3 for which
\[ f(1 - f^3)^{-2/3} = bz(1 - z^3)^{-2/3}. \]
Thus
\[ |A_2| = |E_2| \leq \frac{2}{3}(1 - b^3). \]

3. $E_{2\nu+1}$ ($\nu = 0, \ldots, 4$) by aid of FitzGerald–Launonen inequality

For inverse coefficients the FitzGerald inequality appears to be effective. It is advisable to apply it in the integral form given by Launonen [2]. In [5] this method is used in estimating the coefficients of functions inverse to odd $\Sigma_1$-functions—call them briefly "odd inverse coefficients". We are going to apply the Siejka-method to them in the case $\Sigma_3'$ and will thus test the possibilities of generalizing the estimations of Schober in [4].

For the inverse $\Sigma_3$-function the FitzGerald–Launonen condition (20) of [5] holds. This implies inequalities for odd $E_{b\nu}$-coefficients. These are obtained as in [5] from (22) and can be expressed by aid of the numbers $\alpha_{2\nu}$ of
\[ I(w)^{-1} = w^{-1} + \alpha_2 w^{-2} + \alpha_3 w^{-3} + \cdots, \]
where $\alpha_2 = -a_2 = 0$, $\alpha_3 = -E_1$, $\alpha_4 = -E_2$, $\alpha_5 = -E_3 + E_1^2$, $\alpha_6 = -E_4 + 2E_2E_1$, \ldots. The inequalities in question are those in (24) of [5] (write $D_{2\nu-1} = E_{2\nu-1}$). Thus, for $\nu = 1$ and $\nu = 2$ we obtain immediately
\[ |E_1| \leq 1 - b^2, \quad |E_3| \leq |\alpha_2|^2 + 1 - b^2 = 1 - b^2. \]

For the first condition the equality case was obtained from the coefficient body $(a_2, a_3)$ and was found to be the symmetric 2:2-case. In what follows, call the corresponding extremal $S(b)$-function $f_0$. The equality in the second condition is more problematic. In Section 4 we will show that again the previous mapping $f_0$ is the only extremal function.

For $\nu = 3$ the condition (22) of [5] yields
\[ |E_5| \leq |\alpha_3|^2 + (4 - b^2)|\alpha_2|^2 + 1 - b^2 \]
\[ = |E_1|^2 + 1 - b^2 \leq (1 - b^2)^2 + 1 - b^2 = 2 - 3b^2 + b^4. \]
The maximum is reached with that of $|E_1|$, i.e., by the function $f_0$.

The first non-trivial estimation occurs for $\nu = 4$:
\[ |E_7| = |\alpha_4|^2 + |2\alpha_3 + \alpha_2|^2 - b^2|\alpha_3|^2 + (9 - 4b^2)|\alpha_2|^2 + 1 - b^2 \]
\[ = |A_2|^2 + (4 - b^2)|A_1|^2 + 1 - b^2. \]
In order to estimate this we apply the area inequality for $S(b)$ ([9], p. 182):

$$\sum_{\nu=1}^{\infty} \nu |\alpha_\nu - b^2 a_\nu|^2 \leq 1,$$

where

$$bf(z)^{-1} = z^{-1} + \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu.$$  

Because of (1)

$$z^{-1} + \sum_{\nu=1}^{\infty} A_\nu z^\nu = H(z^{-1}) = bf(z)^{-1}.$$  

Thus $\alpha_\nu = A_\nu$ and the area inequality reads

(2)  

$$\sum_{\nu=1}^{\infty} \nu |A_\nu - b^2 a_\nu|^2 \leq 1.$$  

For $E_7$ we use the consequence

(3)  

$$|A_1 - b^2|^2 + 2|A_2|^2 \leq 1,$$

yielding

$$|E_7| = \frac{3}{2} - b^2 + (4 - b^2)|A_1|^2 - \frac{1}{2} |A_1 - b^2|^2$$

$$= \frac{3}{2} - b^2 - \frac{1}{2} b^4 + (\frac{7}{2} - b^2)|A_1|^2 + b^2 \text{Re} A_1.$$  

The rotated function $\tau^{-1}I(\tau w) = w + \Sigma_{\nu=1}^{\infty} \tau^{-\nu-1} E_\nu w^{-\nu}$, $|\tau| = 1$, preserves $|E_7|$ and allows the normalization $E_1 \geq 0$, i.e., $A_1 = -E_1 \leq 0$. In the variable

$$x = A_1 \in [-(1 - b^2), 0]$$

we thus have

$$|E_7| \leq \frac{3}{2} - b^2 - \frac{1}{2} b^4 + P(x),$$

$$P(x) = \left(\frac{7}{2} - b^2\right) x^2 + b^2 x.$$  

Require

$$P(-(1 - b^2)) = (1 - b^2)(b^4 - \frac{11}{2} b^2 + \frac{7}{2}) \geq P(0) = 0.$$  

This yields

$$\max P = P(-(1 - b^2)) \quad \text{for} \quad 0 \leq b \leq b_o.$$  

where
\[ b_0 = \frac{1}{2}(11 - \sqrt{65})^{1/2} = 0.856\,992\,160 \]
is the zero of \((\cdot)_0\). Hence for \(0 \leq b \leq b_0\) the sharp estimation holds:
\[ |E_7| \leq \frac{3}{2} - b^2 - \frac{1}{2} b^4 + P(-(1 - b^2)) = 5 - 10b^2 + 6b^4 - b^6. \]

Again, with \(|A_1|\) the coefficient \(|E_7|\) is maximized by the function \(f_0\).

The case \(\nu = 5\) can be treated similarly.

\[ |E_9| \leq |\alpha_5|^2 + (4 - b^2)|\alpha_4|^2 + (9 - 4b^2)|\alpha_3|^2 + 1 - b^2 \]

\[ = |E_1^2 - E_3|^2 + (4 - b^2)|A_2|^2 + (9 - 4b^2)|A_1|^2 + 1 - b^2. \]

Because \(|E_1| \leq 1 - b^2\), \(|E_3| \leq 1 - b^2\) we obtain, by using (3):

\[ |E_9| \leq [(1 - b^2)^2 + 1 - b^2]^2 + \frac{1}{2}(4 - b^2)[1 - |A_1 - b^2|^2] \]

\[ + (9 - 4b^2)|A_1|^2 + 1 - b^2 \]

\[ = (1 - b^2)^2(2 - b^2)^2 + 3 - \frac{3}{2}b^2 - (2 - \frac{1}{2}b^2) b^4 + Q(x); \]

\[ Q(x) = (7 - \frac{7}{2}b^2)x^2 + (4 - b^2)b^2x; \]

\[ x = A_1 \in \left[-(1 - b^2), 0\right]. \]

Require
\[ Q(-(1 - b^2)) = \frac{1}{2}(1 - b^2)(9b^4 - 29b^2 + 14) \geq Q(0) = 0. \]

This holds for
\[ 0 \leq b \leq b_0 = \left[\frac{1}{18}(29 - \sqrt{337})\right]^{1/2} = 0.768\,925\,667 \]

where the following sharp estimation is thus valid:
\[ |E_9| \leq (1 - b^2)^2(2 - b^2)^2 + 3 - \frac{3}{2}b^2 - (2 - \frac{1}{2}b^2) b^4 + Q(-(1 - b^2)) \]

\[ = 14 - 35b^2 + 30b^4 - 10b^6 + b^8. \]

Again, with \(|A_1|\) also \(|E_9|\) is maximized by \(f_0\).
The remaining interval \((b_0, 1)\) is left open by the above method. The structure of the FitzGerald inequalities shows that with increasing index the role of \(A_\nu\) coefficients increases. Already for \(A_2\) the function \(f_\omega\) is not the extremal one and, as will be seen, similar situation holds for \(A_3\). Hence, there is no hope to proceed very far in the \(E_{2\nu+1}\)-estimations by using the above method. It is not excluded that \(f_\omega\) actually loses its extremal role, at least for some \(b\)-intervals, for higher odd inverse coefficients.

For \(E_{11}\) and \(E_{13}\) the above estimation technique fails if \(b > 0\) and thus remains succesful only at \(b = 0\) as was proved by Schober in [4].

Collect the sharp results found:

**Theorem.** In \(\sum_b'\) the coefficients \(E_1\), \(E_3\), \(E_5\) are for the whole interval \(b \in [0, 1]\) maximized by the \(S(b)\)-function \(f_\omega\) of the symmetric 2 : 2-type defined by

\[
 f(1 + f^2)^{-1} = b z(1 + z^2)^{-1}.
\]

For \(E_7\), \(f_\omega\) preserves the extremal role at least for \(0 \leq b \leq 0.856 \, 992 \, 160\) and for the coefficient \(E_8\) the same holds at least for \(0 \leq b \leq 0.768 \, 925 \, 667\).

The coefficient \(|A_2| = |E_2|\) is maximized by the \(S(b)\)-function of the symmetric 3 : 3-type defined by

\[
 f(1 - f^3)^{-2/3} = b z(1 - z^3)^{-2/3}.
\]

The uniqueness of the extremal function for \(E_3\) is proved at the beginning of Section 4.

**4. \(E_3\) for \(b \in [0, 1]\) and \(A_3\) for \(b \in [e^{-2}, 1]\)**

In (85) p. 473 of [3] there is the Grunsky-type inequality for \(a_5\) which for \(a_2 = 0\) yields

\[
 2 \text{Re} (a_5 - 2a_3^2) - (1 - b^4) \leq - \text{Re} (a_3^2) - \frac{2(\text{Re} a_3)^2}{\ln b^{-1}}.
\]

Denote here \(a_3 = u + iv:\)

(4) \[
 2 \text{Re} (a_5 - 2a_3^2) - (1 - b^4) \leq v^2 - (1 + 2/\ln b^{-1})_0 u^2.
\]

Because \((\ )_0 > 0\) for \(b \in [0, 1]\) we have

\[
 2 \text{Re} (a_5 - 2a_3^2) - (1 - b^4) \leq v^2 \leq (1 - b^2)^2
\]

with the equality for \(u = 0, |v| = 1 - b^2\), because \(|a_3| \leq 1 - b^2\) for \(a_2 = 0\). Thus

\[
 \text{Re} (a_5 - 2a_3^2) \leq 1 - b^2,
\]
i.e., \(|E_3| \leq 1 - b^2\) where equality holds exactly for \(a_2 = 0, |a_3| = 1 - b^2\), i.e., for the function \(f_0\).

Turn to the combination \(-A_3 = a_5 - a_3^2\). For it (4) assumes the form

\[
2\Re(a_5 - a_3^2) - (1 - b^4) \leq (1 - 2/\ln b^{-1})u^2 - v^2 \leq 0
\]

provided \(1 - 2/\ln b^{-1} \leq 0\). This implies two cases

1) \(1 - 2/\ln b^{-1} < 0 \quad \Leftrightarrow \quad e^{-2} < b < 1; \)

2) \(1 - 2/\ln b^{-1} = 0 \quad \Leftrightarrow \quad b = e^{-2}. \)

In the case 1) the final equality in (5) holds for \(u = v = a_3 = 0\). In the case 2) equality requires \(v = 0\) but \(u\) is left as a free parameter.

Consider the equality cases more closely and apply the rotation \(\tau^{-1} f(\tau z), \quad |\tau| = 1,\) to yield \(a_5 - a_3^2 \Rightarrow \tau^4(a_5 - a_3^2), \quad a_3 \Rightarrow \tau^2 a_3.\) Thus we can normalize

\[
-A_3 = |A_3| = \Re(a_5 - a_3^2) \geq 0, \quad \Re a_3 \leq 0.
\]

In order to study equality in the cases 1)–2) put \(a_2 = 0\) in the original inequality (82) p. 472 of [3]:

\[
\Re(\ln b \cdot x_o^2 + a_3 x_1^2 + a_5 - 3/2a_3^2 + 2a_3 x_o + 2a_4 x_1) \leq (1 - b^2)|x_1|^2 + \frac{1}{2}(1 - b^4).
\]

Here \(x_o\) and \(x_1\) are free complex parameters. In the normalized equality case of (5) \(a_5 = 1(1 - b^4)/2 + u^2, \quad v = 0\), which implies

\[
\Re(\ln b \cdot x_o^2 + 2u x_o - \frac{u^2}{2} + u x_1^2 + 2a_4 x_1) \leq (1 - b^2)|x_1|^2.
\]

In the case 1) \(u = 0.\) By choosing \(x_o = 0\) we obtain

\[
2\Re(x_1 a_4) \leq (1 - b^2)|x_1|^2.
\]

Putting \(x_1 = |x_1|e^{i\phi}\) and letting \(0 < |x_1| \to 0\) we find

\[
2\Re(e^{i\phi} a_4) \leq (1 - b^2)|x_1| \quad \Rightarrow \quad \Re(e^{i\phi} a_4) \leq 0, \quad 0 \leq \phi \leq 2\pi
\]

which implies \(a_4 = 0.\)

In the case 2) \(b = e^{-2}, \quad u < 0\) and the above condition assumes the form

\[
\Re\left\{-2(x_o - \frac{u}{2})^2 + u x_1^2 + 2a_4 x_1\right\} \leq (1 - b^2)|x_1|^2.
\]
Choose now \( x_o = u/2 \) and put, as before, \( x_1 = |x_1| e^{i\phi}, \ 0 < |x_1| \to 0: \)

\[
\text{Re} \left( u|x_1|^2 e^{i2\phi} + 2 e^{i\phi} a_4 \right) \leq (1 - b^2)|x_1|
\]

\[
\Rightarrow \quad \text{Re} \left( e^{i\phi} a_4 \right) \leq 0, \quad 0 \leq \phi \leq 2\pi
\]

and hence \( a_4 = 0. \)

In the normalized extremal case all the coefficients up to \( a_5 \) are thus real. From the Power inequality it then follows that we may use the general condition derived for the extremal function in the real class. This is the condition (35) p. 488 in [8]:

\[
2x_o \ln f + b^2(f^2 - f^{-2}) = 2x_o \ln z + z^2 - z^{-2};
\]

\[
2x_o = a_3 = u \leq 0.
\]

In the case 1) in (6) \( u = x_o = 0. \) The image \( f(U), \ U : |z| < 1, \) is of the type 4:4 with four symmetrically located radial slits.

In the case 2), \( b = e^{-2}, \) the image \( f(U) \) can be studied by aid of the boundary correspondence. Thus, put

\[
z = e^{i\phi}, \quad f(e^{i\phi}) = r(\phi)e^{i\psi(\phi)}
\]

in (6):

\[
\cos 2\psi = -e^4 u \frac{\ln r}{r^2 - r^{-2}} \to -ue^4/4 \quad \text{for} \quad r \to 1;
\]

\[
uu \psi + e^{-4}(r^2 + r^{-2})\sin 2\psi = uu + 2\sin 2\phi.
\]

The first condition implies the limitation for \( u: \)

\[
(7) \quad -0.073 \ 262 \ 556 = -4e^{-4} \leq u \leq 0.
\]

(6) determines \( f': \)

\[
z \frac{f'}{f} = \frac{z^2 + x_o + z^{-2}}{bf^2 + x_o + b^2 f^{-2}}.
\]

Thus, the pre-image \( z = e^{i\phi} \) of the tip of the slit is determined by

\[
2 \cos 2\phi + x_o, \quad x_o = u/2
\]

and the starting point of \( f = e^{i\psi} \) of the slit satisfies

\[
2 \cos 2\psi + b^{-2} x_o, \quad b = e^{-2}.
\]

In Figure 1 there are some slits connected with \( f(U) \) of the type 4:4 in the case \( b = e^{-2}. \)
5. $A_3$ in the odd subclass of $\Sigma_b'$

On the interval $0 < b < e^{-2}$ the problem of maximizing $-A_3 = a_5 + a_3^2$ remains open in $\Sigma_b'$. The corresponding question in the odd subclass of $\Sigma_b'$ appears to be more reasonable and can be solved by aid of inequalities holding in $S(b)$.

As is well known $f \in S(b)$ determines an odd $\tilde{f} = \sqrt{f(z^2)} \in S(b^{1/2})$. Conversely, any odd $\tilde{f} = \tilde{f}(z) \in S(b)$ determines $f \in S(b^2)$ through the connection

$$\tilde{f}(z)^2 = f(z^2).$$

Any condition true for $f \in S(b)$ can be transformed to $\tilde{f}(z) = \tilde{b}(z + \tilde{a}_3 z^3 + \cdots) \in S(\tilde{b})$ by the alteration

$$z, f, b \quad \Rightarrow \quad z^2, \tilde{f}^2, \tilde{b}^2.$$

More closely, if $f(z) = b(z + a_2 z^2 + \cdots)$ we have the coefficient connections

$$b = \tilde{b}^2, \quad a_2 = 2\tilde{a}_3, \quad a_3 = 2\tilde{a}_5 + \tilde{a}_3^2.$$

Take the optimized Power inequality true for $S(b)$-functions, [10], p. 7:

$$\text{Re} (a_3 - a_2^2) \leq 1 - b^2 + U^2/\ln b, \quad U = \text{Re} a_2.$$  

Equality here can be reached if $|U| \leq 2b|\ln b|$. For odd $S(\tilde{b})$-functions this implies

$$2\text{Re} (\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \leq \text{Re} (\tilde{a}_3^2) + 4(\text{Re} \tilde{a}_3)^2/\ln \tilde{b}^2$$

$$= (1 + 4/\ln \tilde{b}^2)\tilde{u}^2 - \tilde{v}^2 \leq (1 + 4/\ln \tilde{b}^2)\tilde{u}^2 = M(\tilde{u}),$$

with equality for $\tilde{v} = 0$, where $\tilde{a}_3 = \tilde{u} + i\tilde{v}$. 

Figure 1.
If $e^{-2} \leq \tilde{b} < 1$ we obtain from this our former estimation $\text{Re} \, (\tilde{a}_5 - \tilde{a}_3^2) \leq (1 - \tilde{b}^4)/2$.

If $0 < \tilde{b} < e^{-2}$, $1 + 4/\ln \tilde{b}^2 > 0$ and $|\tilde{u}| \leq \tilde{b}^2 |\ln \tilde{b}^2|$ we obtain the non-sharp estimation

\[
(8) \quad \tilde{M}(\tilde{u}) \leq \left( 1 + 4/\ln \tilde{b}^2 \right) \tilde{b}^4 \ln^2 \tilde{b}^2 = \tilde{b}^4 \ln \tilde{b}^2 (4 + \ln \tilde{b}^2)
\]

which will be needed later.

The above inequality can be sharpened for the values $|U| \geq 2b |\ln b|$, [10], p. 17:

\[
\text{Re} \, (a_3 - a_3^2) \leq 1 - b^2 - 2|U|\sigma + 2(\sigma - b)^2; \\
\sigma \ln \sigma - \sigma + b + |U|/2 = 0, \quad U = \text{Re} \, a_2, \quad \sigma \in [b, 1].
\]

We shift this for $\tilde{f}$ and denote in this connection

\[
\sigma = \tilde{\sigma}^2 \in [\tilde{b}^2, 1] \Rightarrow \tilde{\sigma} \in [\tilde{b}, 1];
\]

\[
2\text{Re} \, (\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \leq \text{Re} \, (\tilde{a}_3^2) - 4|\text{Re} \, \tilde{a}_3|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2; \\
|\text{Re} \, \tilde{a}_3| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2).
\]

Denoting $\tilde{a}_3 = \tilde{u} + i\tilde{v}$ we obtain

\[
2\text{Re} \, (\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \leq \tilde{u}^2 - \tilde{v}^2 - 4|\tilde{u}|\sigma^2 + 2(\sigma^2 - \tilde{b}^2)^2 \\
\leq \tilde{u}^2 - 4|\tilde{u}|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2; \\
|\tilde{u}| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2),
\]

with the equality for $\tilde{\sigma} = 0$. This estimation is to be used for

\[
\tilde{b}^2 |\ln \tilde{b}^2| \leq |\tilde{u}| \leq 1 - \tilde{b}^2.
\]

For brevity, return to the variable $\sigma$ in estimating the upper bound found:

\[
2\text{Re} \, (\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \leq (\sigma \ln \sigma - \sigma + b)^2 + 4\sigma(\sigma \ln \sigma - \sigma + b) + 2(\sigma - b)^2 = M(\sigma),
\]

\[
\tilde{b}^2 = b \leq \sigma \leq 1.
\]

According to [10] p. 15 $\sigma = \sigma(\tilde{u}) \in [b, 1]$ is uniquely determined by $|\tilde{u}| \in [\tilde{b}^2 |\ln \tilde{b}^2|, 1 - \tilde{b}^2]$. Thus, $M$ as a function of $\tilde{u}$ is maximized by maximizing $M(\sigma)$ for $\sigma \in [b, 1]$. Because

\[
\frac{dM(\sigma)}{d\sigma} = 2\ln \sigma (\sigma \ln \sigma + 3\sigma + b)_1
\]
we see that the root $\sigma \in [e^{-4}, 1]$ of $(\sigma)_1 = 0$ yields the maximum in question. Because the maximum of $\bar{M}(\bar{u})$ in (8) equals $M(b)$ we have maximized $\text{Re} (\bar{a}_5 - \bar{a}_3^2)$ and hence $|\bar{a}_5 - \bar{a}_3^2|$. Collect the results concerning $A_3$. All the extremal functions to be mentioned are the essential ones, to which rotation can be added.

**Theorem.** In $\Sigma'_b$ the coefficient $A_3$ is maximized as follows.

$e^{-2} < b < 1 : |A_3| \leq (1-b^4)/2$. Equality holds for the symmetric 4:4-mapping and the corresponding $S(b)$-function $f$ is obtained from (6) for $x_0 = 0$.

$b = e^{-2}$: The above maximum remains to hold for a one-parametric family of 4:4-mappings the $S(b)$-function $f$ of which is determined by (6) with $u \in [-4e^{-4}, 0]$ as a parameter.

$0 \leq b < e^{-2}$: In the odd subclass of $\Sigma'_b$ $|A_3| \leq (1-b^4)/2 + (\sigma^2 - b^2)^2$ where $\sigma \in [e^{-2}, 1]$ is the root of

$$\sigma^2 \ln \sigma^2 + 3\sigma^2 + b^2 = 0.$$

In the limit case $b = 0$, $\sigma = e^{-3/2}$ and $\max |A_3| = 1/2 + e^{-6}$. The extremal $S(b)$-function $f$ is of the type 2:4 obtained (through $f_\sigma$) from,

$$-4\sigma^2 \ln f_\sigma + \sigma^2 (f_\sigma^2 - f_{\sigma^{-1}}^2) = -4\sigma^2 \ln z + z^2 - z^{-2},$$

$$f + f^{-1} = \sigma/b(f_\sigma + f_{\sigma^{-1}}).$$

The condition for the 2:4-function follows from [9], p. 72, where the extremal function of the type 1:2 for $a_3$ in $S(b)$ is defined.

In the limit cases $b = e^{-2}$ and $b = 0$ the maxima of odd functions are known to hold also in the general class $\Sigma'_b$. For $b = 0$ the result was proved by Garabedian and Schiffer [1]. Thus, it seems to be most probable that a similar state of things remains to hold also for the intermediate values $0 < b < e^{-2}$.

**References**


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