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# ON SPECTRAL PREDICTION ERROR FORMULAS FOR STATIONARY RANDOM FIELDS ON $Z^2$

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## 1. Introduction

We are concerned with analytical expressions for the prediction errors of second order stationary random fields $x_{m,n}$, $(m,n) \in Z^2$. The study of prediction theory of stationary random fields goes back to Chiang Tse-Pei [1] and Helson and Lowdenslager [2], [3]. More recently several authors have treated different kinds of prediction theoretical problems for stationary random fields, cf. e.g. [4]–[11] and [13].

Let $x_{m,n}$, $(m,n) \in Z^2$, be a stationary random field. Mainly the following prediction problems have been treated in literature

(i) the *half-plane prediction error*

$$\|e^1(x)\|^2 = \|x_{0,0} - Proj_{\overline{sp}\{x_{j,k}:j<0,k\in Z\}}x_{0,0}\|^2, \tag{1.1}$$

(ii) the *lexicographic prediction error*

$$\|e^2(x)\|^2 = \|x_{0,0} - Proj_{\overline{sp}\{x_{j,k}:j<0,k\in Z \text{ or } j=0,k<0\}}x_{0,0}\|^2, \tag{1.2}$$

(iii) the *extended half-plane prediction error*

$$\|e^3(x)\|^2 = \|x_{0,0} - Proj_{\overline{sp}\{x_{j,k}:j\le 0,k\in Z,(j,k)\neq(0,0)\}}x_{0,0}\|^2, \tag{1.3}$$

(iv) the *quarter-plane prediction error*

$$\|e^4(x)\|^2 = \|x_{0,0} - Proj_{\overline{sp}\{x_{j,k}:j<0,k<0\}}x_{0,0}\|^2. \tag{1.4}$$

Analytical expressions for $\|e^1(x)\|^2$ were obtained independently in [5] and [8] (cf. [1]) and, respectively, for $\|e^2(x)\|^2$ in [2]. Corresponding results for $\|e^3(x)\|^2$ have been obtained in [6] and [11].

Our main result is an analytical expression for $\|e^4(x)\|^2$ under the strong commutation condition, introduced in [4] (cf. Theorem 3.9 and 3.10). Our results are based on the four-fold Wold decomposition for stationary random fields having the strong commutation property obtained by Kallianpur and Mandrekar [4] and its spectral counterpart obtained by Korezlioglu and Loubaton [8] (cf. [5]). We

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also make use of the spectral representation theorems for the horizontal and, respectively, vertical innovation fields of $x_{m,n}$, $(m,n) \in Z^2$, obtained by Korezlioglu and Loubaton [8].

As noted earlier, our main results are derived under the strong commutation condition. Sufficient spectral conditions for the strong commutation condition to hold have been obtained by Soltani [13] and Miamee and Niemi [10].

## 2. Geometrical interpretation

Let $\{x_{m,n}\}$ be a stationary random field. The information sets generated by observations $x_{m,n}$, $(m,n) \in \mathcal{S}$ $(\subset Z^2)$, are defined as closed linear subspaces of $L^2(\Omega,\mathcal{A},P)$ as follows:

$$H_x = \overline{sp}\{x_{j,k} : (j,k) \in Z^2\},$$

$$H_x^1(m) = \overline{sp}\{x_{j,k}\colon\ j \le m,\ \ k \in Z\}, \quad H_x^1(-\infty) = \bigcap\nolimits_{m\in Z} H_x^1(m),$$

$$H_x^2(n) = \overline{sp}\{x_{j,k}\colon\ j \in Z\,,\ k \le n\}, \quad H_x^2(-\infty) = \bigcap\nolimits_{n\in Z} H_x^2(n),$$

$$H_x^{1+}(m,n) = H_x^1(m) \bigvee \overline{sp}\{x_{m+1,k}\colon\ k \le n\},$$

$$H_x^{2+}(m,n) = H_x^2(n) \bigvee \overline{sp}\{x_{j,n+1} : j\ \le m\},$$

$$H_x^*(m,n) = \overline{sp}\{x_{j,k}\colon\ j \le m \text{ or } k \le n\}\ , H_x(m,n) = \overline{sp}\{x_{j,k}\colon\ j \le m, k \le n\}$$

and, in general, for an arbitrary $\mathcal{S} \subset Z^2$

$$H_x(\mathcal{S}) = \overline{sp}\{x_{j,k}\colon\ (j,k) \in \mathcal{S}\}.$$

Furthermore, for any closed linear subspace $M \subset H_x$ we define

$$x_{m,n}/M = Proj_M x_{m,n}, \quad (m,n) \in Z^2.$$

**2.1. Definition.** A stationary random field $\{x_{m,n}\}$ is
(a) *horizontally deterministic*, if $H_x^1(-\infty) = H_x$,
(b) *horizontally purely non-deterministic*, if $H_x^1(-\infty) = \{0\}$,
(c) *vertically deterministic*, if $H_x^2(-\infty) = H_x$,
(d) *vertically purely non-deterministic*, if $H_x^2(-\infty) = \{0\}$,
(e) *strongly purely non-deterministic*, if $H_x^1(-\infty) = H_x^2(-\infty) = \{0\}$.

Recall that any stationary random field $\{x_{m,n}\}$ admits two-fold Wold decompositions of the form

$$x_{m,n} = R^i_{m,n}(x) + S^i_{m,n}(x),$$

with

$$S^i_{m,n}(x) = x_{m,n}/H^i_x(-\infty), \quad R^i_{m,n}(x) = x_{m,n} - S^i_{m,n}(x), \quad i = 1,2.$$

The component $\{R^1_{m,n}(x)\}$ (respectively $\{R^2_{m,n}(x)\}$) is *horizontally* (respectively *vertically*) *purely non-deterministic* and $\{S^1_{m,n}(x)\}$ (respectively $\{S^2_{m,n}(x)\}$) is *horizontally* (respectively *vertically*) *deterministic*. The stationary random fields

$$W^1_{m,n}(x) = x_{m,n} - x_{m,n}/H^1_x(m-1)$$

respectively

$$W^2_{m,n}(x) = x_{m,n} - x_{m,n}/H^2_x(n-1)$$

are the *horizontal* (respectively *vertical*) *innovations* of $\{x_{m,n}\}$. It is well-known that

$$H^i_{R^i(x)}(m) = H^i_{W^i(x)}(m), \quad i = 1,2. \tag{2.2}$$

**2.3. Remark.** Our method to obtain an analytical expression for the prediction error (1.4) is based on the spectral representation of the innovation fields $\{W^i_{m,n}(x)\}$, $i = 1,2$. The formula (2.4), going back to Korezlioglu and Loubaton [8], p. 155, shows that $e^2(x)$ can be obtained as the one-step prediction error of the stationary sequence $\{W^1_{0,n}\}_{n \in Z}$.

**2.4. Proposition.** *Let $\{x_{m,n}\}$ be a stationary random field. Then*

$$e^2(x) = W^1_{0,0}(x) - W^1_{0,0}(x)/H_{W^1(x)}(\{(j,k) \in Z^2 : j = 0, k < 0\}). \tag{2.4}$$

The next commutativity property was introduced by Kallianpur and Mandrekar [4].

**2.5. Definition.** A stationary random field $\{x_{m,n}\}$ has the *strong commutation property*, if

$$Proj_{H^1_x(m)} Proj_{H^2_x(n)} = Proj_{H_x(m,n)}, \quad (m,n) \in Z^2.$$

**2.6. Remark.** Each of the conditions (2.6.a) and (2.6.b) is equivalent to the strong commutation property:

$$\begin{aligned} &Proj_{H^1_x(m)} Proj_{H^2_x(n)} = Proj_{H^2_x(n)} Proj_{H^1_x(m)} \quad \text{and} \\ &H_x(m,n) = H^1_x(m) \cap H^2_x(n), \quad (m,n) \in Z^2, \end{aligned} \tag{2.6.a}$$

$$H^1_x(m) \ominus H_x(m,n) \perp H^2_x(n) \ominus H_x(n,n), \quad (m,n) \in Z^2. \tag{2.6.b}$$

**2.7. Lemma.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. Then*

$$(2.7.a)\quad H_x^*(m,n) = H_x(m,n) \oplus \big[H_x^1(m) \ominus H_x(m,n)\big] \oplus \big[H_x^2(n) \ominus H_x(m,n)\big],$$

$$(2.7.b)\quad H_x\big(\{(j,k) \in Z^2 : j \le m, k \le n\} \setminus \{(m,n)\}\big) = H_x(m-1,n-1) \oplus \big[H_x(m,n-1) \ominus H_x(m-1,n-1)\big] \oplus \big[H_x(m-1,n) \ominus H_x(m-1,n-1)\big],$$

$$(2.7.c)\quad H_x^{1+}(m,n) = H_x(m,n) \oplus \big[H_x^1(m) \ominus H_x(m,n)\big] \oplus \big[H_x(m+1,n) \ominus H_x(m,n)\big],$$

$$(2.7.d)\quad H_x^{2+}(m,n) = H_x(m,n) \oplus \big[H_x^2(n) \ominus H_x(m,n)\big] \oplus \big[H_x(m,n+1) \ominus H_x(m,n)\big],$$

$$(2.7.e)\quad H_x(m,n) \ominus H_x(m,n-1) = H_{W^2(x)}\big(\{(j,k) \in Z^2 : j \le m, k = n\}\big),$$

$$(2.7.f)\quad H_x(m,n) \ominus H_x(m-1,n) = H_{W^1(x)}\big(\{(j,k) \in Z^2 : j = m, k \le n\}\big).$$

*Proof.* The statements (2.7.a–d) are obvious. By symmetry it is enough to present a proof only to the first one of the statements (2.7.e–f). Denote,

$$\mathcal{S}(m-,n) = \big\{(j,k) \in Z^2 : j \le m, k = n\big\},$$

and

$$M = \big\{z - Proj_{H_x^2(n-1)} z : z \in H_x\big(\mathcal{S}(m-,n)\big)\big\}.$$

It is obvious that $M = H_{W^2(x)}\big(\mathcal{S}(m-,n)\big)$. Moreover, by the strong commutativity

$$\begin{aligned} Proj_{H_x^2(n-1)} z &= Proj_{H_x^2(n-1)} Proj_{H^1(m)} z \\ &= Proj_{H_x(m,n-1)} z, \quad z \in H_x\big(\mathcal{S}(m-,n)\big), \end{aligned}$$

showing that

$$M = \big\{z - Proj_{H_x(m,n-1)} z : z \in H_x(\mathcal{S}(m-,n))\big\}.$$

Since for all $z \in H_x(m,n-1)$ , $z - Proj_{H_x(m,n-1)} z = 0$, it is then obvious that

$$M = \big\{z - Proj_{H_x(m,n-1)} z : z \in H_x(m,n)\big\} = H_x(m,n) \ominus H_x(m,n-1).$$

The next result shows that the $*$-prediction problem, introduced in [13], reduces to the lexicographical one when $\{x_{m,n}\}$ has the strong commutation property. The fact that

$$x_{m,n}/H_x^{1+}(m-1,n-1) = x_{m,n}/H_x^{2+}(m-1,n-1), \quad (m,n) \in Z^2,$$

for any stationary random field $\{x_{m,n}\}$ having the strong commutation property has been proved already by Korezlioglu and Loubaton [7; Proposition 2.1.4] (under the assumption $x_{m,n}/H_x^{1+}(m-1,n-1) \neq 0$).

**2.8. Proposition.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. Then*

$$\begin{aligned}(2.9) \qquad & x_{m,n}/H_x^*(m-1,n-1) = x_{m,n}/H_x^{1+}(m-1,n-1) \\ & = x_{m,n}/H_x^{2+}(m-1,n-1) \\ & = x_{m,n}/H_x(\{(j,k) \in Z^2 : j \le m, k \le n\} \setminus \{(m,n)\}).\end{aligned}$$

*Proof.* By Lemma 2.7

$$\begin{aligned} x_{m,n}/H_x^*(m-1,n-1) = & \, Proj_{H_x(m-1,n-1)} x_{m,n} \\ & + [Proj_{H_x^1(m-1)} - Proj_{H_x(m-1,n-1)}] x_{m,n} \\ & + [Proj_{H_x^2(n-1)} - Proj_{H_x(m-1,n-1)}] x_{m,n}. \end{aligned}$$

Furthermore, by the strong commutativity

$$Proj_{H_x^1(m-1)} x_{m,n} = Proj_{H_x^1(m-1)} Proj_{H_x^2(n)} x_{m,n} = Proj_{H_x(m-1,n)} x_{m,n}$$

and by symmetry

$$Proj_{H_x^2(n-1)} x_{m,n} = Proj_{H_x(m,n-1)} x_{m,n},$$

yielding together with (2.7.b–d) all the equalities in (2.9).

In what follows we make heavy use of the four-fold Wold decompositon theorem obtained by Kallianpur and Mandrekar [4]. According to Theorems 2.1 and 2.2 in [4] any stationary random field having the strong commutation property admits a representation of the form

$$(2.10.a) \qquad x_{m,n} = \xi_{m,n} + \zeta^1_{m,n} + \zeta^2_{m,n} + \eta_{m,n}, \quad (m,n) \in Z^2,$$

where all the components are mutually orthogonal stationary random fields having the strong commutation property and

$$\begin{aligned}(2.10.b) \quad H_x(m,n) = & \, H_\xi(m,n) \oplus H_{\zeta^1}(m,n) \\ & \oplus H_{\zeta^2}(m,n) \oplus H_\eta(m,n), \quad (m,n) \in Z^2.\end{aligned}$$

Moreover,

$$(2.10.c) \qquad \begin{cases} H_\xi^1(-\infty) = H_\xi^2(-\infty) = \{0\}, & H^1_{\zeta^1}(-\infty) = H^2_{\zeta^2}(-\infty) = \{0\}, \\ H^2_{\zeta^1}(-\infty) = H_{\zeta^1}, & H^1_{\zeta^2}(-\infty) = H_{\zeta^2}, \\ H_\eta^1(-\infty) = H_\eta^2(-\infty) = H_\eta. & \end{cases}$$

**2.11. Theorem.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. If $\{x_{m,n}\}$ is strongly purely non-deterministic, then*

(2.11.a) $e^4(x) = W^1_{0,0}(x) + W^2_{0,0}(x) - d_{0,0}(x)$

*with*

$$d_{0,0}(x) = x_{0,0} - x_{0,0}/H^{1+}_x(-1,-1) = x_{0,0} - x_{0,0}/H^{2+}_x(-1,-1);$$

*and*

(2.11.b) $\|e^4(x)\|^2 = \|W^1_{0,0}(x)\|^2 + \|W^2_{0,0}(x)\|^2 - \|e^2(x)\|^2.$

**2.12. Theorem.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. Then*

(2.12.a) $$e^4(x) = \xi_{0,0}(x) - \xi_{0,0}(x)/H_\xi(-1,-1) + \zeta^1_{0,0}(x) - \zeta^1_{0,0}(x)/H^1_{\zeta^1}(-1) + \zeta^2_{0,0}(x) - \zeta^2_{0,0}(x)/H^2_{\zeta^2}(-1)$$

*with*

(2.12.b) $\xi_{0,0} - \xi_{0,0}/H_\xi(-1,-1) = W^1_{0,0}(\xi) + W^2_{0,0}(\xi) - d_{0,0}(\xi);$

*and*

(2.12.c) $$\|e^4(x)\|^2 = \|W^1_{0,0}(\xi)\|^2 + \|W^2_{0,0}(\xi)\|^2 - \|e^2(\xi)\|^2 + \|\zeta^1_{0,0}(x) - \zeta^1_{0,0}(x)/H^1_{\zeta^1}(-1)\|^2 + \|\zeta^2_{0,0}(x) - \zeta^2_{0,0}(x)/H^2_{\zeta^2}(-1))\|^2.$$

*Proof of Theorem 2.11.* We first notice that by the strong commutativity

$$\begin{aligned} x_{0,0} - x_{0,0}/H_x(-1,-1) &= (I - Proj_{H^1_x(-1)} Proj_{H^2_x(-1)})x_{0,0} \\ &= (I - Proj_{H^2_x(-1)} Proj_{H^1_x(-1)})x_{0,0}. \end{aligned}$$

Furthermore, for any projections $P_1$ and $P_2$ one has $I - P_1P_2 = (I-P_1) + P_1(I - P_2)$. Thus, in the present case

$$x_{0,0} - x_{0,0}/H_x(-1,-1) = W^1_{0,0}(x) + W^2_{0,0}(x)/H^1_x(-1).$$

Moreover, by the strong commutativity

$$W^2_{0,0}(x)/H^1_x(-1) = Proj_{H^1_x(-1)} Proj_{H^2_x(0)} W^2_{0,0}(x) = Proj_{H_x(-1,0)} W^2_{0,0}(x),$$

and since $W^2_{0,0}(x) \perp H_x(-1,-1)$,

$$Proj_{H_x(-1,0)} W^2_{0,0}(x) = Proj_{H_x(-1,0)\ominus H_x(-1,-1)} W^2_{0,0}(x).$$

By (2.7.e), with $\mathcal{S}\big((-1)-,0\big) = \big\{(j,k) \in Z^2 : j \le -1, k = 0\big\}$,

$$H_x(-1,0) \ominus H_x(-1,-1) = H_{W^2(x)}\big(\mathcal{S}\big((-1)-,0\big)\big).$$

This, together with Proposition 2.4, gives

$$W^2_{0,0}(x)/H^1_x(-1) = W^2_{0,0}(x)/H_{W^2_x}(\mathcal{S}((-1)-,0)) = x_{0,0}/H^{1+}_x(-1,-1).$$

Thus, by applying (2.4) together with (2.9) we obtain

$$\begin{aligned} x_{0,0} - x_{0,0}/H^1_x(-1,-1) &= W^1_{0,0}(x) + W^2_{0,0}(x) \\ &\quad - \left(W^2_{0,0}(x) - W^2_{0,0}(x)\right)/H_{W^2_x}(\mathcal{S}((-1)-,0)) \\ &= W^1_{0,0}(x) + W^2_{0,0}(x) - d_{0,0}(x). \end{aligned}$$

The proof of (2.11.b) is obvious.

*Proof of Theorem 2.12.* It clearly follows from the orthogonality property (2.10.b) of the four-fold decomposition (2.10.a) that

$$\begin{aligned} x_{0,0} - x_{0,0}/H_x(-1,-1) &= \xi_{0,0} - \xi_{0,0}/H_\xi(-1,-1) + \zeta^1_{0,0} - \zeta^1_{0,0}/H_{\zeta^1}(-1,-1) \\ &\quad + \zeta^2_{0,0} - \zeta^2_{0,0}/H_{\zeta^2}(-1,-1) + \eta_{0,0} - \eta_{0,0}/H_\eta(-1,-1). \end{aligned}$$

Since $H^1_\eta(-\infty) = H^2_\eta(-\infty) = H_\eta$ (cf. (2.10.c)) and since $\{\eta_{m,n}\}$ has the strong commutation property it is obvious that $H_\eta(-1,1) = H_\eta$, yielding

$$\eta_{0,0} - \eta_{0,0}/H_\eta(-1,-1) = 0.$$

Since $\{\zeta^1_{m,n}\}$ has the strong commutation property and is vertically deterministic,

$$H_{\zeta^1}(-1,-1) = H^1_{\zeta^1}(-1).$$

By symmetry, $H_{\zeta^2}(-1,-1) = H^2_{\zeta^2}(-1)$; finishing the proof of (2.12.a).

Since $\{\xi_{m,n}\}$ has the strong commutation property and is strongly purely non-deterministic one can apply Theorem 2.11 to $\{\xi_{m,n}\}$, giving (2.12.b).

The proof of (2.12.c) is obvious.

## 3. Analytical solution

Our method to obtain an analytical expression for the prediction error $\|e^4(x)\|^2$ is based on the spectral representation for the covariance kernel of the innovation field $\{W^1_{m,n}(x)\}$ obtained by Korezlioglu and Loubaton [8]. According to Proposition II.2 and (IV.8) in [8] the covariance spectral measure of $\{W^1_{m,n}(x)\}$ has the properties

$$d\nu_{W^1(x)}(u,v) = \frac{1}{2\pi}\, du\, d\rho^1_x(v) \tag{3.1.a}$$

with

$$(3.1.b) \qquad d\rho_x^1(v) = \int_{u=-\pi}^{\pi} d\nu_{W^1(x)}(u,v)$$

and

$$(3.1.c) \qquad \frac{d\rho_x^1(v)}{dv} = \frac{1}{2\pi} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_x(u,v)\, du \right\}$$

as the absolutely continuous part of $d\rho_x^1$ with respect to the normalized Lebesgue measure $dv$ on $[-\pi, \pi)$. By symmetry, the same properties hold for $d\nu_{W^2(x)}$.

**3.2. Remark.** It clearly follows from (3.1.a) that the covariance spectral measure of the stationary sequence $\{W_{0,n}^1(x)\}_{n\in Z}$ is $d\rho_x^1$. Furthermore, by the well-known prediction theoretical results on stationary sequences only $d\rho_x^1(v)/dv$ is needed in calculating the prediction error of $\{W_{0,n}^1(x)\}$ needed in Proposition 2.4 (see e.g. [12], pp. 63-71). The spectral counterpart of (2.4) is well-known [2]. However, (2.4) combined with (3.1.c) gives a simple method to obtain

$$(3.3) \qquad \|e^2(x)\|^2 = \exp\left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log f_x(u,v)\, du\, dv \right\}.$$

The next example shows that $d\rho_x^1$ need not be absolutely continuous with respect to $dv$.

**3.4. Example.** Let $\{x_{m,n}\}$ be a bivariate and, respectively, $\{f_m\}_{m\in Z}$, a univariate white noise. Assume, in addition $f_k \perp x_{m,n}$, $k, m, n \in Z$. Define

$$(3.4.a) \qquad z_{m,n} = x_{m,n} + f_m, \quad (m,n) \in Z^2.$$

Then, $W_{m,n}^1(z) = z_{m,n}$, $(m,n) \in Z^2$, i.e., $\{W_{0,n}^1(z)\}_{n\in Z}$, contains a deterministic component. Moreover, obviously

$$(3.4.b) \qquad d\nu_z(u,v) = d\nu_{W^1(x)} = \frac{1}{(2\pi)^2}\, du\, dv + \frac{1}{2\pi}\, du \otimes \delta_0,$$

where $\delta_0$ is the Dirac measure concentrated at the origin.

Our method to derive an analytical expression for the prediction error $\|e^4(x)\|^2$ is based on the spectral counterpart of the four-fold decomposition (2.10.a) obtained by Korezlioglu and Loubaton [8; Corollary III.13] and, under the weak commutation condition

$$(3.5) \qquad Proj_{H_x^1(m)} Proj_{H_x^2(n)} = Proj_{H_x^2(n)} Proj_{H_x^1(m)}, \quad m, n \in Z,$$

independently in [5; Theorem II.12]. According to Corollary III.13 [8], for any stationary random field $\{x_{m,n}\}$ one has $\xi_{m,n}(x) \neq 0$, if and only if

$$\text{(3.6)} \qquad \begin{cases} \int_{-\pi}^{\pi} \log f_x(u,v)\,du > -\infty \\ \int_{-\pi}^{\pi} \log f_x(u,v)\,dv > -\infty \end{cases}$$

and, under this condition,

$$\text{(3.7.a)} \qquad d\nu_{\xi(x)} = \frac{1}{(2\pi)^2} \log f_x(u,v)\,du\,dv,$$

$$\text{(3.7.b)} \qquad d\zeta^1(x) = \frac{1}{2\pi} \frac{d\nu_x(u,v)}{du\,d\rho_x^{1,s}(v)}\,du\,d\rho_x^{1,s}(v),$$

$$\text{(3.7.c)} \qquad d\zeta^2(x) = \frac{1}{2\pi} \frac{d\nu_x(u,v)}{d\rho_x^{2,s}(u)dv}\,d\rho_x^{2,s}(u)\,dv,$$

where $d\rho_x^{1,s}(v)$ and $d\rho_x^{2,s}(u)$ are the singular parts with respect to the Lebesgue measure of $d\rho_x^1(v)$ and $d\rho_x^2(u)$, respectively. For brevity we state the spectral counterparts of Theorems 2.11 and 2.12 only under the assumption (3.6).

**3.8. Remark.** Let $\{x_{m,n}\}$ be a stationary random field. In what follows we use the notation

$$\begin{aligned} d^4(x^a) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_x(u,v)\,du \right\} dv \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f_x(u,v)\,dv \right\} du \\ &- \exp\left\{ \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log f_x(u,v)\,du\,dv \right\}. \end{aligned}$$

**3.9. Theorem.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. If $\{x_{m,n}\}$ is strongly purely non-deterministic, then*

$$\text{(3.9.a)} \qquad d\nu_x \ll du\,dv \quad \text{and (3.6) holds,}$$

*and*

$$\text{(3.9.b)} \qquad \|e^4(x)\|^2 = d^4(x^a).$$

**3.10. Theorem.** *Let $\{x_{m,n}\}$ be a stationary random field having the strong commutation property. If (3.6) holds, then*

$$\begin{aligned} \text{(3.10.a)} \qquad \|e^4(x)\|^2 &= \|e^4(\xi(x))\|^2 + \|\zeta^1_{0,0}(x) - \zeta^1_{0,0}(x)/H^1_{\zeta^1(x)}(-1)\|^2 \\ &+ \|\zeta^2_{0,0}(x) - \zeta^2_{0,0}(x)/H^2_{\zeta^2(x)}(-1)\|^2 \end{aligned}$$

*with*

(3.10.b) $\|e^4(\xi(x))\|^2 = d^4(x^a)$,

(3.10.c) $\|\zeta^1_{0,0}(x) - \zeta^1_{0,0}(x)/H^1_{\zeta^1(x)}(-1)\|^2 =$

$$\int_{-\pi}^{\pi} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{d\nu_x(u,v)}{du\, d\rho_x^{1,s}(v)}\right] du\right\} d\rho_x^{1,s}(v),$$

(3.10.d) $\|\zeta^2_{0,0}(x) - \zeta^2_{0,0}(x)/H^2_{\zeta^2(x)}(-1)\|^2 =$

$$\int_{-\pi}^{\pi} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{d\nu_x(u,v)}{d\rho_x^{2,s}(u)\, dv}\right] dv\right\} d\rho_x^{2,s}(u).$$

Before presenting proofs of Theorems 3.9 and 3.10 we continue Example 3.4.

**3.11. Example.** Let $\{z_{m,n}\}$ be defined according to (3.4). It is obvious that $\{z_{m,n}\}$ has the strong commutation property and $\xi_{m,n}(z) = x_{m,n}$, $\zeta^1_{m,n}(z) = f_m$, $\zeta^2_{m,n}(z) = 0$, $\eta_{m,n}(z) = 0$, $(m,n) \in Z^2$. Moreover,

$$d\rho_x^1(v) = \int_{u=-\pi}^{\pi} d\nu_{W^1(x)}(u,v) = \frac{1}{2\pi}\, dv + \delta_0.$$

It is obvious, that

$$\zeta^1_{m,n}(x)/H^1_{\zeta^1(x)}(m-1) = 0 \quad \text{and} \quad \zeta^1_{m,n}(x) - \zeta^1_{m,n}(x)/H^1_{\zeta^1(x)}(m-1) = f_m.$$

*Proof of Theorem 3.9.* Theorem III.12 together with Corollary III.13 in [8] imply that the properties (3.9.a) hold for any strongly purely non-deterministic stationary random field. The expression (3.9.b) then follows straightforwardly from (2.11.b) by using (3.1.c) and (3.3).

*Proof of Theorem 3.10.* The formula (3.10.a) is just a reformulation of (2.12.c). Since $\{\xi_{m,n}(x)\}$ has, by Theorem 2.1 [4], the strong commutation property (3.10.b) follows from (3.7.a) combined with Theorem 3.9.

By symmetry it is enough to justify (3.10.c) to finish the proof. It follows from Theorem II.1 [5] (cf. Proposition II.11 [7]) that

$$d\nu_{W^1(\zeta^1(x))}(u,v) = \frac{1}{2\pi}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{d\nu_x(u,v)}{du\, d\rho_x^{1,s}(v)}\right] du\right\} \rho_x^{1,s}(v).$$

The formula (3.10.c) follows then immediately.

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