FINITELY GENERATED KLEINIAN GROUPS
An Introduction

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This paper, like the talk on which it is based, is directed not at workers in the theory of Kleinian groups but at mathematicians who want to find out what this theory is all about. An introductory talk cannot, of course, cover all aspects of a rapidly developing field. We stress those chapters of the theory in which the connections with Riemann surfaces and quasiconformal maps predominate; these are also the chapters in which the influence of Ahlfors is clearly visible. The reader will have to look elsewhere for an introductory account on Thurston's path-breaking work on hyperbolic three-manifolds. Neither can we report on Maskit's monumental classification of geometrically finite function groups.

For more information, and for references to original papers, I refer to the books and expository papers listed in the selected bibliography.

1. Basic definitions

Every unimodular 2×2 matrix with complex entries,

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1,$$

defines a biholomorphic automorphism

$$(1.1) \qquad z \mapsto \frac{az + b}{cz + d}$$

of the Riemann sphere (= complex projective line) $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$; the same map is defined by the matrix $(-\alpha)$. The *Möbius group* $\mathbf{M}$ of all such maps can be identified with the group $\mathrm{PSL}(2, \mathbf{C})$.

Every Möbius transformation (1.1) can be extended to act on the hyperbolic three-space $\mathbf{H}_3$. Indeed, $\mathbf{H}_3$ can be represented as the space of all "special" quaternions $Z = x + iy + jt$, with x, y, t real and $t > 0$. The Möbius transformation (1.1) takes Z into the special quaternion

$$(aZ + b)(cZ + d)^{-1};$$

it preserves orientation and the Poincaré metric

$$ds^2 = (dx^2 + dy^2 + dt^2)/t^2.$$

Similarly, the real Möbius group $\mathbf{M_R}$, which can be identified with $\mathrm{PSL}(2, \mathbf{R})$, is the group of all projective self-mappings of the real projective line $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, and also the group of motions of the non-Euclidean plane $\mathbf{H}_2$, represented as the upper half-plane U in $\mathbf{C}$, with the Poincaré metric $ds^2 = (dx^2 + dy^2)/y^2$.

Now let G be a discrete subgroup of $\mathbf{M}$. A point $z_0 \in \hat{\mathbf{C}}$ is called a *point of discontinuity* (of G) if there is a neighborhood of z_0 in which the elements of G form a normal family. The set of all points of discontinuity is the *region of discontinuity* $\Omega(G)$ of G; its complement $\Lambda(G) = \hat{\mathbf{C}} \setminus \Omega(G)$ is called the *limit set*. The set of discontinuity is always open and G-invariant.

In every G-invariant open set $\omega \subset \Omega(G)$ the group G operates *properly discontinuously*; this means that every compact subset of ω intersects at most finitely many of its translates under G. Since every element of G is a conformal map, one concludes that, for every component Δ of $\Omega(G)$, the quotient

$$S_\Delta = \Delta / \operatorname{Stab}_G \Delta$$

is a Riemann surface and the canonical surjection $\Delta \to S_\Delta$ is a holomorphic covering, which is branched precisely at those points of Δ which are fixed under a non-trivial element of $\operatorname{Stab}_G \Delta$, the stabilizer of Δ in G. The quotient $\Omega(G)/G$ is the disjoint union

$$S_{\Delta_1} + S_{\Delta_2} + \cdots$$

where $\Delta_1, \Delta_2, \ldots$ is a complete list of nonconjugate (under G) components of Δ. We say that G *represents* S_{Δ_i} over Δ_i and $S_{\Delta_1} + S_{\Delta_2} + \cdots$ over $\Omega(G)$.

The group G is called *Kleinian* (by some authors, *Kleinian of the second kind*) if its limit set is not the whole Riemann sphere $\hat{\mathbf{C}}$. If so, the limit set has either no more than two points (in which case G is called *elementary*) or infinitely many points (in which case the limit set $\Lambda(G)$ is nowhere dense and perfect).

By a *Fuchsian group* we shall mean, in this paper, a Kleinian group G which lies in $\mathbf{M_R}$. Such a G is called of the first kind (or of the second kind) if $\Lambda(G) = \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ (or if $\mathbf{R}$ is a nowhere dense subset of $\hat{\mathbf{R}}$). A Fuchsian group G of the first kind represents 2 Riemann surfaces between which there exists an anti-conformal map, induced by the reflection $z \mapsto \bar{z}$. A Fuchsian group of the second kind represents one Riemann surface endowed by a canonical anticonformal involution, induced by the reflection $z \mapsto \bar{z}$. Any Kleinian group G acts properly discontinuously on the non-Euclidean space $\mathbf{H}_3$; this is so even if $\Omega(G) = \emptyset$. If G is torsion free, the quotient $\mathbf{H}_3/G$ is a manifold endowed with a smooth hyperbolic metric, induced by the Poincaré metric in $\mathbf{H}_3$. If G has torsion, the natural map $\mathbf{H}_3 \to \mathbf{H}_3/G$ has branch points.

The assumption that G has no torsion is often made for the sake of simplicity. This restriction is not too severe, in view of *Selberg's lemma*: a finitely generated matrix group is either torsion free or has a torsion free subgroup of finite index.

2. Examples

A discrete subgroup of $\mathbf{M}_3$ for which every point of $\hat{\mathbf{C}}$ is a limit point is the *Picard group* consisting of all maps

$$(2.1) \qquad z \mapsto \pm \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad AD - BC = 1$$

where A, B, C, D are Gaussian integers.

The simplest Fuchsian group of the first kind is the *elliptic modular group* Γ of maps (2.1) where A, B, C, D are rational integers. The *principal congruence subgroup modulo 2* (of Γ) is defined by the additional requirement that A and D be odd and B and C even. This group is denoted by $\Gamma(2)$ and is of the first kind (for instance, because it is of finite index in Γ).

The group $\Gamma(2)$ is torsion free and represents (over U) a thrice punctured sphere; the canonical map $U \to U/\Gamma(2)$ is a holomorphic universal covering.

The group Γ represents a once punctured sphere and the canonical map $U \to U/\Gamma$ is branched of order 2 over 2 points.

The following construction, going back to Poincaré, yields a Fuchsian group which represents (over U) a closed Riemann surface of genus $p > 1$. Let $\sqcap$ be a regular non-Euclidean $4p-$gon with all interior angles equal to $\pi/2p$. There is a nearly standard way of choosing $2p$ non-Euclidean motions $A_1, \ldots, A_{2p}$ (in U) such that each A_j identifies one edge of $\sqcap$ with another. The non-Euclidean motions $A_1, \ldots, A_{2p}$ satisfy the relation

$$A_1 \circ A_2 \circ A_1^{-1} \circ A_2^{-1} \circ \cdots \circ A_{2p-1}^{-1} \circ A_{2p}^{-1} = \mathrm{id},$$

and the polygon $\sqcap$, with the identifications by $A_1, \ldots, A_{2p}$ carried out, becomes a closed Riemann surface S of genus p, with complex structure compatible with that of U. The group G generated by $A_1, \ldots, A_{2p}$ is Fuchsian of the first kind and represents S over U.

Under these circumstances one says that $\sqcap$ is a *fundamental domain* for G (in U), which means that the set-theoretical boundary $\partial\sqcap$ of $\sqcap$ has measure 0, no two distinct points of $\mathrm{Int}(\sqcap)$ are G-equivalent, and every point of U is G-equivalent to some point of $\mathrm{Cl}(\sqcap)$. Since $\sqcap$ is also a non-Euclidean polygon we call it a *fundamental polygon*.

Extending the method used above one can construct a Fuchsian group of the first kind G which represents (over U) a closed Riemann surface S of some genus $p \geq 0$ from which one has removed $n \geq 0$ points, and such that the canonical map

$U \to U/G = S$ is branched over $k \geq 0$ points ζ_1, ..., ζ_k of S, of orders ν_1, ..., ν_k, respectively. The only condition to be satisfied is the inequality

$$(2.2) \qquad 2p - 2 + n + \left(1 - \frac{1}{\nu_1}\right) + \cdots + \left(1 - \frac{1}{\nu_k}\right) > 0.$$

The sequence

$$(p, n, k; \nu_1, \ldots, \nu_k)$$

is called the *signature* of G, and the pair (p, n) is called the *type* of G. (This terminology is not used, or is modified, by some authors.) Every finitely generated Fuchsian group of the first kind has a signature.

Non-Fuchsian Kleinian groups can be obtained by constructing fundamental polyhedra in non-Euclidean 3-space; this also goes back to Poincaré, cf. Maskit's book for a modern presentation.

3. Combination theorems; Schottky groups

Let G_1 and G_2 be finitely generated Kleinian groups. *Klein's combination theorem* gives a sufficient condition for the group G generated by G_1 and G_2 to be Kleinian and the free product of G_1 and G_2. The condition reads: G_1 and G_2 have fundamental domains Π_1 and Π_2, each bounded by one or several Jordan polygons, such that the boundary of Π_1 (of Π_2) is contained in Π_2 (in Π_1). Under this hypothesis the intersection $\Pi = \Pi_1 \cap \Pi_2$ is a fundamental domain for G.

One owes to Maskit a far-reaching extension of this result. *Maskit's first combination theorem* deals with two Kleinian groups, G_1 and G_2, which have a common subgroup G_0, and gives sufficient geometric conditions for $\langle G_1, G_2 \rangle$ to be the *free product* of G_1 and G_2, *amalgamated over* G_0, and Kleinian.

Maskit's second combination theorem deals with a Kleinian group G and two subgroups H_1 and H_2 of G, which are not conjugate in G but such that there is a $g \in \mathrm{M} \setminus G$ with $H_2 = g H_1 g^{-1}$. The theorem gives sufficient geometric conditions for $\langle G, g_1, g_2 \rangle$ to be the HNN*extension* of G by g, and Kleinian.

The theory of Kleinian groups may be said to have begun with Schottky's work on groups which bear his name.

Let C_1, C_1', C_2, ..., C_p' be $2p$ disjoint Jordan curves in $\hat{\mathbb{C}}$ and assume that $C_1 + \cdots + C_p'$ is the boundary of a domain of connectivity $2p$. Assume also that there are p Möbius transformations A_1, ..., A_p such that A_j maps the domain exterior to C_j onto the domain interior to C_j'. Note that A_j certainly exists if C_j and C_j' are circles.

The group G generated by A_1, ..., A_p is free and *Kleinian* and the domain exterior to all circles C_1, ..., C_p' is a fundamental domain for G. The proof is easy if $p = 1$ and proceeds by induction on p for $p > 1$; the induction argument uses Klein's combination theorem.

A group G obtained in this way is called a *Schottky group of genus p*. It is called a *geometric Schottky group* if it can be constructed choosing as curves $C_1, \ldots, C'_p$ only circles. (In the older literature one says "Schottky group" and "group of Schottky type" instead of "geometric Schottky group" and "Schottky group".)

The existence of *non-geometric* Schottky groups was proved by Marden.

It is known that the limit set of a Schottky group has area 0. But it has positive logarithmic capacity (Myrberg). The latter property extends to all Kleinian groups since every such group must have Schottky subgroups.

Maskit proved that a (finitely generated) Kleinian group is Schottky if and only if it is *free* and has *no parabolic elements*.

The young Schottky thought that *every* closed Riemann surface of genus p can be represented by a Schottky group of this genus. He did not publish or pursue this guess, following the advice of Weierstrass. The so-called *retrosection theorem* which asserts that he was right (as long as one does not restrict oneself to geometric Schottky groups) was established much later by Koebe and by Courant; we shall return to this in Section 6.

4. The finiteness theorem; area inequalities

From now on we assume that G, with or without subscripts, denotes a non-elementary finitely generated Kleinian group. A central result in the theory is the *Ahlfors finiteness theorem* which asserts that $\Omega(G)$ has at most finitely many non-conjugate components $\Delta_1, \ldots, \Delta_r$, that each Riemann surface $S_i = \Delta_i / \operatorname{Stab}_G \Delta_i$ is a closed (compact) surface with at most finitely many points removed, and that the natural map $\Delta_i \to S_i$ is branched at at most finitely many points.

There are today several proofs of the finiteness theorem; they differ not in the conceptual framework but in the "hard analysis" tool used to overcome a technical difficulty. In Ahlfors' original proof this tool is a delicate "mollifier", in Sullivan's proof the tool is the theory of prime ends, and in the author's proof the tool is a lemma about quasiconformal mappings.

The author's *area inequalities* are quantitative refinements of the finiteness theorem. Recall that for every component S_i of $\Omega(G)/G$ there is a branched covering map $\Delta_i \to S_i$. This map together with the Poincaré metric on Δ_i induces a metric, called again Poincaré metric, on S_i, and in this metric S_i has an area A_i. The following statement is classical: if S_i is obtained from a closed surface of genus p_i by removing n_i points, and if the map $\Delta_i \to S_i$ is branched over m_i points, of orders $\nu_{i1}, \ldots, \nu_{im_i}$, respectively, then

$$(4.1) \qquad A_i = 2\pi(2p_i - 2 + n_i + m_i - b_i)$$

where b_i is the sum of the reciprocal branch numbers,

$$b_i = (1/\nu_{i1}) + \cdots + (1/\nu_{im_i}).$$

The area of $\Omega(G)/G$ is the sum $A_1 + \cdots + A_r$.

The *first area inequality* asserts that if G is generated by N elements, then

$$\text{Area}\,(\Omega(G)/G) \leq 4\pi(N-1).$$

This is sharp; equality is achieved for Schottky groups.

A sharp estimate of r, the number of components of $\Omega(G)/G$, in terms of N, is not known.

Since no A_i can be less than $\pi/21$ (cf. (4.1) and (2.2)), the area inequality yields the trivial estimate $r \leq 84(N-1)$. Ahlfors improved this to $r \leq 18(N-1)$.

Assume now that Δ_1 is an *invariant component* of $\Omega(G)$, so that $g(\Delta_1) = \Delta_1$ for all $g \in G$. The *second area inequality* asserts that in this case

$$\text{Area}\,(\Omega(G)/G) \leq 2\text{Area}\,(\Delta_1/G).$$

Corollary: If G has two invariant components, Δ_1 and Δ_2, then

$$\Omega(G) = \Delta_1 \cup \Delta_2.$$

In this case both Δ_1 and Δ_2 are simply connected. We will give a more precise description of groups with two invariant components later, see Section 8.

5. The measure of the limit set

In his paper on the finiteness theorem Ahlfors asked whether the area (= 2-dimensional Lebesgue measure) of the limit set $\Lambda(G)$ of a finitely generated Kleinian group is zero. Later he proved that this is so if G, considered as a group of 3-dimensional non-Euclidean motions, has a fundamental (finitely many sided) non-Euclidean polyhedron.

Kleinian groups which have a fundamental polyhedron are now called *geometrically finite*; the difference between geometrically finite and geometrically infinite groups turns out to be very important.

A Kleinian group G is called a *function group* if it has an invariant component, i.e., if $\Omega(G)$ has a component Δ such that $g(\Delta) = \Delta$ for all $g \in G$. Maskit gave a *topological description* of all geometrically finite function groups.

Of course, geometric finiteness is not a necessary condition for a group G to have a limit set of vanishing area. There are several known sufficient conditions, for instance, Bonahan proved that $\text{mes}\,\Lambda(G) = 0$ if G is *not a free product*. Yet we do not know whether there is a finitely generated G with $\text{mes}\,\Lambda(G) > 0$.

Occasionally the question can be bypassed using a result of Sullivan which we proceed to describe.

A bounded measurable function $\mu(z)$ defined on $\mathbf{C}$ is called *compatible with respect to a Kleinian group G* if

$$(5.1) \qquad \mu\big(g(z)\big)\overline{g'(z)}/g'(z) = \mu(z) \qquad \text{a.e.} \quad \text{for } g \in G.$$

Sullivan proved that (5.1) implies that

$$(5.2) \qquad\qquad \mu(z) = 0 \qquad \text{a.e.} \quad \text{on } \Lambda(G).$$

Of course, this result is not vacuous only for a group G with mes $\Lambda(G) > 0$.

(We remark, though this is of no importance for the present paper, that Sullivan's result remains valid for non-Kleinian groups, i.e., for the case when G is a discrete group of Möbius transformations with $\Lambda(G) = \hat{\mathbf{C}}$.)

6. Beltrami equations

Let D be a domain in $\mathbf{C}$. An element μ of the Banach space $L_\infty(D)$ is (or, more precisely, is defined by) a bounded measurable complex valued function $\mu(z)$, $z \in D$; the norm

$$\|\mu\|_\infty = \|\mu\|_{L_\infty(D)}$$

is the smallest number k such that $|\mu(z)| \le k$ a.e. in D. If $\|\mu\|_\infty < 1$, μ is called a *Beltrami coefficient* in D.

A Beltrami coefficient in D defines a (not necessarily smooth) *Riemannian metric* on D,

$$(6.1) \qquad\qquad ds^2 = \sigma^2 |dz + \mu d\bar{z}|^2$$

where σ is a positive function: every Riemannian metric can be so represented.

A *μ-conformal map* w of D onto a subdomain $D' \subset \hat{\mathbf{C}}$ is a homeomorphic bijection $w\colon D \to D'$ such that the distributional derivatives

$$(6.2) \qquad\qquad w_z = \frac{w_x - iw_y}{2}, \qquad w_{\bar{z}} = \frac{w_x + iw_y}{2}$$

are locally square-integrable measurable functions satisfying a.e. the *Beltrami equation*

$$(6.3) \qquad\qquad w_{\bar{z}} = \mu w_z.$$

The geometric meaning of this equation is that w is a *conformal map with respect to the Riemannian metric* (6.1).

The existence of local homeomorphic solutions of (6.3), and the main properties of these solutions, for the case when μ is merely bounded measurable, rather than Hölder continuous, have been proved by Morrey in 1938. (Later Bojarski showed that w_z and $w_{\bar{z}}$ are locally L_p, for some $p > 2$ depending on $\|\mu\|_\infty$.)

Morrey's result, together with Koebe's general uniformization principle, imply that for $D = \mathbf{C}$ and for any Beltrami coefficient μ in $\mathbf{C}$ there exist a unique μ-*conformal automorphism* $z \to w^\mu(z)$ of $\mathbf{C}$ onto itself which fixes the points $0, 1$ (and ∞).

A direct proof of this *generalized Riemann mapping theorem*, as well as the main local properties of quasiconformal maps, are contained in the 1960 paper by Ahlfors and Bers, together with the technically important additional result that, for every $z \in \mathbf{C}$, the number of $w^\mu(z)$ depends *holomorphically* on $\mu \in L_\infty(\mathbf{C})$.

An analogue, and a *corollary* of the generalized Riemann mapping theorem is the statement Φ: if μ is a Beltrami coefficient defined in the upper half-plane U, then there exists a unique μ-conformal homeomorphism $z \to w_\mu(z)$ of $\hat{U} = U \cup \mathbf{R} \cup \{\infty\}$ onto itself, which fixes the real axis $\mathbf{R}$, and the points 0, 1, ∞. For every fixed $z = x + iy$, $y \geq 0$, $w_\mu(w)$ depends *real analytically* on μ.

A *Beltrami differential* m on Riemann surface S is a rule which assigns to every local parameter ζ, defined on a plane domain $D \subset S$, a Beltrami coefficient $\mu(\zeta)$ such that the $(-1,1)$ differential $\mu \, d\bar{\zeta}/d\zeta$ is invariant under parameter changes. It is also required that the absolute value $|m| = |\mu|$, which is a globally defined scalar measurable function on S, an element of $L_\infty(S)$, satisfy a.e. the inequality $\|\mu\|_{L_\infty} < 1$. In these circumstances we say that $m = \mu \, d\bar{\zeta}/d\zeta$, *locally*.

Let F be a bijective homeomorphism of a Riemann surface S onto a Riemann surface S_1. We say that a Beltrami differential m defined on S is the Beltrami differential of F if, for every point P on S, for every local parameter ζ defined near and vanishing at P, and for every local parameter ω defined near and vanishing at $F(P)$, we have that near 0 the map $\omega \circ F \circ \zeta^{-1}$ has a Beltrami coefficient $\mu(\zeta)$, and $m = \mu(\zeta) \, d\bar{\zeta}/d\zeta$ locally, near 0.

A map which has a Beltrami differential is called *quasiconformal*.

7. Quasiconformal deformations of Kleinian groups

Let G be as before a Kleinian group and μ a Beltrami coefficient on $\hat{\mathbf{C}}$ compatible with G, i.e., satisfying relation (5.1) for every $g \in G$. Sullivan's theorem stated in Section 5 implies that it suffices to require that (5.1) hold a.e. in $\Omega(G)$.

A direct calculation shows that, for every g in G, the mapping $\omega^\mu \circ g$ satisfies (a.e.) the same Beltrami equation as w^μ. But two μ-conformal maps differ only by a conformal one, so that

$$w^\mu \circ g = g^\mu \circ w^\mu$$

where g^μ is a holomorphic function. But

$$g^\mu = w^\mu \circ g \circ (w^\mu)^{-1}$$

is clearly a topological self-map of $\hat{\mathbf{C}}$ and thus an element of $\mathbf{M}$.

One sees at once that the group

$$G^\mu = w^\mu G (w^\mu)^{-1}$$

is a Kleinian group, that $w^\mu\big(\Omega(G)\big) = \Omega(G^\mu)$ and $w^\mu\big(\Lambda(G)\big) = \Lambda(G^\mu)$. The isomorphism

$$g \mapsto w^\mu \circ g \circ (w^\mu)^{-1} = g^\mu$$

is called a *quasiconformal deformation* of G onto G^μ, induced by μ.

This construction can be reversed, and this suggests a method for proving uniformization theorems, called *variation of conformal structure* or, less formally, the *μ-trick*.

As an example we prove the classical *retrosection theorem*: every closed Riemann surface S of genus $p > 1$ can be uniformized by a Schottky group, that is, there is a Schottky group G with $\Omega(G)/G$ conformal to S.

Repeating the construction of a Schottky group in Section 3, we choose $2p$ disjoint compact circular discs on $\mathbf{C}$, call them D_{01}, D'_{01}, ..., D_{0p}, D'_{0p} and p Möbius transformations A_{01}, ..., A_{0p} such that (i) the complement in $\hat{\mathbf{C}}$ of $D_{01} + D'_{01} + \cdots + D'_{0p}$ is a plane domain of connectivity $2p$, and (ii) A_{0j} maps the interior of D_{0j} onto the exterior of D'_{0j}, $j = 1, \ldots, p$. The group G_0 generated by A_{01}, ..., A_{0p} is a Schottky group. It is not difficult to construct a smooth (say of class C^2) quasiconformal map of $S_0 = \Omega(G_0)/G_0$ onto the given Riemann surface S. This map has a Beltrami differential μ defined first over some fundamental domain for G_0 and then extended over all of $\Omega(G_0)$ by the condition

$$\mu\big(g(z)\big)\overline{g'(z)} = \mu(z)g'(z)$$

for $z \in \Omega(G_0)$ and $g \in G_0$. Since μ is compatible with the group G_0, the group

$$G = w^\mu G_0 \circ (w^\mu)^{-1}$$

is a Kleinian group, more precisely a Schottky group, and $\Omega(G)/G$ is conformally equivalent to the given Riemann surface S.

We do not discuss the additional conditions which must be imposed in order to make the uniformizing Schottky group G unique.

8. Quasi-Fuchsian groups

A *quasi-Fuchsian* group (of the first kind) G is a Kleinian group whose limit set $\Lambda(G)$ is a Jordan curve on $\hat{\mathbf{C}}$. We shall consider only finitely generated quasi-Fuchsian groups G, and shall assume, for the sake of simplicity, that G has no elliptic or parabolic elements.

It is often convenient to orient the Jordan curve $\Lambda(G)$ and to assume that it passes through the points 0, 1, ∞ (in this order). The two simply connected components of $\hat{\mathbf{C}} \setminus \Lambda(G)$ will be denoted by $\tilde{U}(G)$ and $\tilde{L}(G)$, $\tilde{U}(G)$ being to the left of $\Lambda(G)$. A Fuchsian group is also quasi-Fuchsian, with $\Lambda(G) = \mathbf{R} \cup \{\infty\}$ and $\tilde{U}(G) = U$. A quasi-Fuchsian group G represents two closed Riemann surfaces,

$$S_1 = \tilde{U}(G)/G \qquad \text{and} \qquad S_2 = \tilde{L}(G)/G.$$

The *theorem on simultaneous uniformization* (or, rather a special case of this theorem) asserts that S_1 and S_2 may be prescribed arbitrarily, except that they must have the same genus $p > 1$.

The only proof I know involves the μ-trick (whereas all uniformization theorems mentioned before have also "classical" proofs which do not depend on quasiconformal mappings).

Sketch of proof. Let G_0 be a Fuchsian group such that U/G_0 is (conformal to) the given surface S_1 (the existence of G_0 can be proved by the method in Section 6). Let L denote the lower half-plane. Then L/G_0 is conformal to the mirror image $\bar{S}_1$ of S_1. (The points and open sets of $\bar{S}_1$ are those of S_1, the local parameters on $\bar{S}_1$ are the complex conjugates of the local parameters on S_1.) Considered as a quasi-Fuchsian group G_0 has the components U and L. The quotient U/G_0 is already what we want it to be, the complex structure of L/G_0 is to be changed into that of S_2 by the μ-trick.

There is a quasiconformal homeomorphism f of $\bar{S}_1 = L/G$ onto S_2. It can be lifted to a homeomorphism $\tilde{f}$ of $\bar{S}_1$ onto the universal covering surface of S_2. Let m be the Beltrami differential of the map which coincides with the identity on U and with the map $\tilde{f}$ on L. This m is compatible with G_0. One verifies, as in Section 6, that

$$G = w^{\mu} G_0 (w^{\mu})^{-1}$$

is a quasi-Fuchsian group with $\tilde{U}(G)/G$ conformally equivalent to S_1 and $\tilde{L}(G)/G$ to S_2.

Remark 1. We can now answer a question hinted at the end of Section 4. A finitely generated Kleinian group with two invariant components is quasi-Fuchsian.

Remark 2. Bowen proved that the Hausdorff dimension of the limit set of a quasi-Fuchsian not Fuchsian group is strictly between 1 and 2.

9. Teichmüller spaces

The *Teichmüller space* $T(G)$ of a Fuchsian group G is the space of normalized quasiconformal deformations of G. For the sake of simplicity we assume, as before, that G is finitely generated, purely hyperbolic and of the first kind; then $S = U/G$ is a closed Riemann surface of some genus $p > 1$.

We denote by Q the group of all quasiconformal self-mappings of the closed upper half-plane $\hat{U} = U \cup \mathbf{R} \cup \{\infty\}$, by Q_n the subgroup which fixes the points 0, 1, ∞, and by Q_0 the normal subgroup which fixes every $x \in \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. We say that $u \in Q(G)$ if $u \in Q$ and uGu^{-1} is again a Fuchsian group. Every element u of $Q(G)$ induces a *quasiconformal deformation*

$$G \ni g \mapsto u \circ g \circ u^{-1} = \hat{g} \in uGu^{-1};$$

this deformation is called *normalized* if $u \in Q_n$. Two deformations induced by two elements u_1 and u_2 are identical if and only if u_1 and u_2 coincide on $\hat{\mathbf{R}}$. The deformation induced by u will be denoted by $[u]$; thus $[u]$ determines and is determined by $u \,|\, \hat{\mathbf{R}}$. The space $T(G)$ is the space of all $[u]$, $u \in Q_n(G)$.

Every $u \in Q(G)$ induces an *allowable bijection* of $T(ugu^{-1})$ onto $T(G)$ which takes

$$[w] \in T(ugu^{-1})$$

onto

$$[\alpha \circ w \circ u^{-1}] \in T(G),$$

where α is a real Möbius transformation chosen so that $\alpha \circ w \circ u^{-1}$ fixes 0, 1 and ∞. The allowable self-mappings of $T(G)$ form the *modular group* $\mathrm{Mod}\,(G)$ of G and

$$X(G) = T(G)/\mathrm{Mod}\,(G)$$

is the set of $\mathrm{PSL}(2\mathbf{R})$ conjugacy classes of quasiconformal images of G.

There is a canonical way of representing $T(G)$ as a bounded domain in a complex vector space, namely the space $B_2(L, G)$ of bounded automorphic forms of weight (-4) for the group G, acting on the lower half-plane. The elements of this vector space are holomorphic functions $\varphi(z)$, $z \in L$, satisfying the functional equation

$$(9.1) \qquad \varphi\big(g(z)\big)g'(z)^2 = \varphi(z), \qquad (g \in G, \; z \in L).$$

Thus $\varphi(z)\,dz^2$ is a *holomorphic quadratic differential* on the Riemann surface L/G. The Riemann–Roch theorem implies that

$$\dim B_2(L, G) = 3p - 3.$$

One defines in this space the norm

$$\|\varphi\| = \sup\big|y^2\varphi(x + iy)\big|$$

where $-\infty < x < +\infty$, $-\infty < y < 0$. (Note that $\big|y^2\varphi(z)\big|$ is G-invariant.)

We now proceed to define a mapping of $T(G)$ into $B_2(L, G)$. Let τ be an element of $T(G)$. There is a Beltrami coefficient μ for G, defined in U, and compatible with G, such that $\tau = w_\mu \,|\, \hat{\mathbf{R}} = [w_\mu]$. We continue μ over all of $\mathbf{C}$ by setting

$$\mu_0(z) = \begin{cases} \mu(z) & \text{for } z \in U, \\ 0 & \text{for } z \in L. \end{cases}$$

The Beltrami coefficient μ_0 is also G-compatible, and one can verify that for any two Beltrami coefficients μ and ν, defined in U and compatible with G the *following statements are equivalent.*

(i) $[w_\mu] = [w_\nu]$,

(ii) $w^{\mu_0} \,|\, L = w^{\nu_0} \,|\, L$,

(iii) $w^{\mu_0} \,|\, \hat{\mathbf{R}} = w^{\nu_0} \,|\, \hat{\mathbf{R}}$.

Now $w^{\mu_0} \,|\, L$ is a schlicht holomorphic function in L, so that it is natural (at least for a complex analyst) to form the *Schwarzian derivative* of $w^{\mu_0} \,|\, L$ which we denote by φ^μ; thus

$$(9.2) \qquad \varphi^\mu = \{w, z\} = (w''/w')' - \tfrac{1}{2}(w''/w')^2,$$

where $w = w^{\mu_0} \,|\, L$.

One can verify that $\varphi^\mu(z)$ satisfies relation (9.1) and thus belongs to $B_2(L, G)$. Also, since φ^μ is the Schwarzian derivative of a schlicht function in L, we conclude from the *Kraus–Nehari inequality* that

$$\|\varphi\| \le \frac{3}{2}.$$

It can be shown that the map

$$(9.3) \qquad T(G) \to B_2(L, G)$$

which takes $w_\mu \,|\, \mathbf{R}$ into φ^μ is a *bijection* of $T(G)$ onto a subdomain of the ball $\|\varphi\| < 3/2$ in $B_2(L, G)$, and there is a unique natural way of defining a complex structure in $T(G)$ such that (9.3) becomes a *holomorphic homeomorphism*. The same is then true of every element of the modular group and of every allowable mapping.

It is usually convenient to identify $T(G)$ with its image under the mapping (9.3). The image contains the open ball $\|\varphi\| < 1/2$ in $B_2(L, G)$. This statement is a corollary of the elegant *Ahlfors–Weil lemma* which can be restated as follows.

Let $\psi(z)$ be a holomorphic quadratic differential in L, for the group G, and assume that $\|\psi\| < 1/2$, i.e., that $2|\psi(z)| < y^{-2}$ for $z \in L$. Set

$$\begin{cases} \mu(z) = -2y^2 \psi(\bar{z}) & \text{for } z \in L, \\ \mu(z) = 0 & \text{for } z \in U. \end{cases}$$

Then $\mu(z)$ is a Beltrami coefficient compatible with G in $\mathbf{C}$, and the Schwarzian derivative of $w^\mu \,|\, L$ is $\psi = \varphi^\mu$.

We conclude this section by noting that $T(G)$ is a cell (this follows, for instance from Teichmüller's mapping theorem) and that $T(G)$ is holomorphically convex (there are, by now, several proofs of this fact).

Remark 1. Practically all results of this and the preceding section remain valid when the Fuchsian group G is not finitely generated, or when it contains elliptic or parabolic elements, or when it is of the second kind. This may require some modifications of definitions (for instance, the finite dimensional space $B_2(L, G)$ may have to be replaced by an infinitely dimensional Banach space). Also, some proofs may become significantly harder.

Remark 2. We did not yet mention Teichmüller spaces of Riemann surfaces. Assuming their definition known, and assuming G to be torsion free and of the first kind, we can state a basic theorem.

There exists a *canonical isomorphism* between the Teichmüller space $T(G)$ and the Teichmüller space of the Riemann surface U/G.

A modified isomorphism theorem can be proved if G is of the second kind and/or has torsion.

10. Boundary groups

Let G, $B_2(L, G)$, $T(G)$ have the same meaning as in the preceding section, and let Ψ denote the bijection (9.3). We identify $T(G)$ with its image under Ψ and denote by $\mathrm{Cl}(T(G))$ the closure of $T(G) = \Psi(T(G))$ in $B_2(L, G)$; then $\partial T(G) = \mathrm{Cl}(T(G)) \setminus T(G)$ is the set theoretical boundary of $T(G)$ in $B_2(L, G)$. We will associate a Kleinian group to every point of $\partial(T(G))$.

If $\psi \in B_2(L, G)$ is given, we denote by $\eta_j(z)$, $j = 1, 2$, the two solutions of the ordinary differential equation

$$2\eta''(z) + \eta(z)\psi(z) = 0,$$

subject to the initial conditions

$$\eta_1 = \eta_2' = 1, \qquad \eta_1' = \eta_2 = 0, \qquad \text{at} \quad z = -i.$$

These two functions are holomorphic in L, and depend continuously on $\psi \in B_2(L, G)$. The quotient

$$V^\psi(z) = \eta_1(z)/\eta_2(z)$$

is a locally single-valued meromorphic solution of the Schwarzian differential equation

$$\{V^\psi(z), z\} = \psi(z).$$

Comparing this with (9.2) we conclude that there exists a Möbius transformation α_ψ (dependent on ψ) such that

$$w^{\mu_0} = \alpha_\psi \circ V^\psi.$$

This is valid for $\psi \in T(G)$ or, more precisely, for $\Psi(\psi) \in T(G)$. Indeed w^{μ_0} was defined assuming that $[w_\mu] \in T(G)$. Also, if ψ converges to a point $\hat\psi$ on $\partial T(G)$, there is no reason to assume that α_ψ converges.

On the other hand, V^ψ is defined for all ψ in $B_2(G)$ and it makes sense to study the group

$$G^\psi = V^\psi G(V^\psi)^{-1}, \qquad \psi \in \partial T(G).$$

Such groups are called *boundary groups* (of G). The facts are as follows.

The group G^ψ is Kleinian and has a *single simply connected invariant component*. (Groups with the latter property are called *b-groups*. It is not known whether every *b*-group is a boundary group.)

In general (in the sense of measure or in the sense of dimension) G^ψ has no other components. Such groups are called *(totally) degenerated*. Totally degenerated groups are *geometrically infinite* (cf. Section 5). As a matter of fact, this is how geometrically infinite groups were discovered.

A function group which is not totally degenerated has non-invariant components. Assume that $D_1, \ldots, D_n$ is a complete list of non-conjugate non-invariant components, and let Γ_j be the stabilizer of D_j. Also set $S_j = D_j/\Gamma_j$. Then each S_j has *at least one puncture*, and if the non-Euclidean area of S_j is A_j, we have that

$$(10.1) \qquad\qquad A_1 + \cdots + A_r \leq A$$

where A is the area of L/G. (This is simply the second area inequality, cf. Section 4.) The group G^ψ is called *partially degenerate* (or *regular*) according to whether (10.1) is valid with the strict inequality sign (or with the equality sign).

The terms "regular", "partially degenerated" and "totally degenerated" make sense for any *b*-group. Abikoff proved that regular *b*-groups are boundary groups.

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