Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 41–46

EMBEDDING OF ORLIZ–SOBOLEV SPACES IN HÖLDER SPACES

Vesa Lappalainen and Ari Lehtonen

1. Introduction

For a smooth domain Ω in $\mathbf{R}^n$, e.g. a bounded Lipschitz domain, each function u which belongs to the Sobolev space $W^{1,p}(\Omega)$ is in fact Hölder-continuous in $\overline{\Omega}$ if p is greater than n (cf. Adams [1], Kufner et al [6] or Necăs [14]). A similar embedding property holds also for Orlicz–Sobolev spaces (cf. [1] or [6]).

Typically, the boundary behaviour of u is handled by straightening the boundary to a half space using local coordinate maps and deriving estimates for the Hölder norm of u in terms of the (Orlicz–) Sobolev norm (cf. [14, Chapter 2.3.5.]). Instead of using estimates on the boundary we first show that if $p > n$ the Sobolev spaces $W^{1,p}(\Omega)$ can be embedded in a certain local Hölder class $\operatorname{loc}\operatorname{Lip}_\alpha(\Omega)$, $\alpha = 1 - n/p$ for any domain Ω. The embedding to $C^\alpha(\overline{\Omega})$ is then derived for a large class of domains via the embedding of $\operatorname{loc}\operatorname{Lip}_\alpha(\Omega)$ to $C^\alpha(\overline{\Omega})$. The following result is obtained as a corollary:

Theorem. *If Ω is a bounded uniform domain and $p > n$, then $W^{1,p}(\Omega)$ is continuously embedded in $C^\alpha(\overline{\Omega})$.*

Note that by a result of P. Jones [5] there exists an extension operator $W^{1,p}(\Omega) \to W^{1,p}(\mathbf{R}^n)$ for uniform domains, and the theorem hence follows from the well-known embedding $W^{1,p}(\mathbf{R}^n) \to C^\alpha(\mathbf{R}^n)$. However, for the theorem no extension result is needed, and our approach is based on classical Hölder continuity estimates together with Gehring and Martio's [3] and Lappalainen's [7] results on Lip_h-extension domains. Therefore our method applies to a larger class of domains than uniform domains.

2. Preliminaries

An *Orlicz function* is any continuous map $A\colon \mathbf{R} \to \mathbf{R}$ which is strictly increasing, even, convex and satisfies

$$\lim_{\xi\to 0} A(\xi)\xi^{-1} = 0, \quad \lim_{\xi\to\infty} A(\xi)\xi^{-1} = \infty.$$

We let Ω denote a domain in $\mathbf{R}^n$. The *Orlicz class* $K_A(\Omega)$ is the set of all measurable functions u such that

$$\int_\Omega A\big(u(x)\big)\,dx < \infty,$$

doi:10.5186/aasfm.1989.1417

and the *Orlicz space* $L_A(\Omega)$ is the linear hull of $K_A(\Omega)$. As norm in the Orlicz space we use the Luxemburg norm

$$\|u\|_{A,\Omega} := \inf\Big\{r > 0 : \int_\Omega A\big(u(x)/r\big)\,dx \le 1\Big\}.$$

The *Orlicz–Sobolev space* $W^1L_A(\Omega)$ is the set of functions u such that u and its first order distributional derivatives lie in $L_A(\Omega)$. In the case where $A(\xi) = \xi^p$ we obtain the standard Sobolev space $W^{1,p}(\Omega)$. For a more detailed discussion of Orlicz spaces we refer to [1] and [6].

A domain Ω in $\mathbf{R}^n$ is called *c-uniform* if each pair of points $x, y \in \Omega$ can be joined by a rectifiable curve γ in Ω such that $l(\gamma) \le c\,|x-y|$ and

$$\operatorname{dist}\big(\gamma(t), \partial\Omega\big) \ge c^{-1} \min\big(t, l(\gamma) - t\big).$$

A *modulus of continuity* is any concave positive increasing function $h\colon [0,\infty[\to \mathbf{R}$, $h(0) = 0$. A function $u\colon \Omega \to \mathbf{R}$ belongs to the *local Lipschitz class* $\operatorname{loc}\operatorname{Lip}_h(\Omega)$ if there exist constants $b \in\,]0,1[$ and $M = m_b$ such that for each $x \in \Omega$ and $y \in B_b(x) := B\big(x, b\operatorname{dist}(x,\partial\Omega)\big)$

$$|u(x) - u(y)| \le M h(x,y); \tag{2.1}$$

here and hereafter $h(x,y) := h\big(|x-y|\big)$. As a matter of fact, it is shown in [7] that it is equivalent to require the condition to hold for $b = 1/2$; the smallest $m_{1/2}$ defines a seminorm of u. It should be remarked that this definition differs from the standard definitions of local Hölder spaces. In fact, the class $\operatorname{loc}\operatorname{Lip}_h(\Omega)$ is not a local space but semiglobal in a sense. A function u belongs to the *Lipschitz class* $\operatorname{Lip}_h(\Omega)$ if there exists a constant $M < \infty$ such that (2.1) holds for all $x, y \in \Omega$. For bounded domains $\operatorname{Lip}_h(\Omega) = C^h(\overline{\Omega})$, where $C^h(\overline{\Omega})$ is as in [1, 8.37].

Let h and g be two moduli of continuity. A domain Ω is a $\operatorname{Lip}_{h,g}$*-extension domain* if $\operatorname{loc}\operatorname{Lip}_h(\Omega)$ is continuously embedded in $\operatorname{Lip}_g(\Omega)$. For short $\operatorname{Lip}_{h,h} =: \operatorname{Lip}_h$ and, for $h(t) = t^\alpha$, $\operatorname{Lip}_h =: \operatorname{Lip}_\alpha$. The following result due to McShane [13] justifies the name extension domain (see also Stein [15], [3] and [7]).

2.1. Theorem. *If Ω is a Lip_h-extension domain and $u \in \operatorname{loc}\operatorname{Lip}_h(\Omega)$, there exists a Lip_h-extension $u^*\colon \mathbf{R}^n \to \mathbf{R}$.*

We can characterize Lip_h-extension domains by using the following metric in Ω:

$$h_\Omega(x,y) := \inf_{\gamma(x,y)} \int_\gamma \frac{h\big(\operatorname{dist}(z,\partial\Omega)\big)}{\operatorname{dist}(z,\partial\Omega)}\,ds(z),$$

where the infimum is taken over all rectifiable curves γ in Ω joining x to y.

2.2. Theorem. *A domain $\Omega \subset \mathbf{R}^n$ is a $\mathrm{Lip}_{h,g}$-extension domain if and only if there is a constant $1 \le K(\Omega, h, g) < \infty$ such that*

$$h_\Omega(x,y) \le K\, g(x,y) \tag{2.2}$$

holds in Ω.

For a proof see e.g. [3] or [7].

3. Embedding of Orlicz–Sobolev spaces

Let A denote an Orlicz function. If

$$h(t) := \int_{t^{-n}}^{\infty} \frac{A^{-1}(r)}{r^{1+1/n}}\, dr \tag{3.1}$$

is finite at $t = \varepsilon$, then h defines a modulus of continuity on the interval $[0, \varepsilon]$. It is easily seen that the derivative $h'(t) = n\, A^{-1}(t^{-n})$ is decreasing.

3.1. Proposition. *If $h(1) < \infty$, then $W^1L_A(\Omega)$ is continuously embedded in $\operatorname{loc}\mathrm{Lip}_h(\Omega)$ for any domain $\Omega \subset \mathbf{R}^n$.*

Proof. It follows from [1, Theorem 5.35] applied to balls contained in Ω that each function $u \in W^1L_A(\Omega)$ is continuous. Now let $B_b(x_0)$ be a ball contained in Ω and $x_1 \in B_b(x_0)$. Let $t := |x_0 - x_1|$ and choose a ball B of radius t such that $x_0, x_1 \in B \subset B_b(x_0)$. We denote by $|B|$ the Lebesgue measure of B and by

$$u_B := \frac{1}{|B|} \int_B u(z)\, dz$$

the mean value of u in B. As in [1] we obtain the following estimate for $x \in B$:

$$\bigl|u(x) - u_B\bigr| \le \frac{2t}{|B|} \int_0^1 r^{-n} \int_{B_r} \bigl|\nabla u(z)\bigr|\, dz,$$

where B_r denotes a ball of radius rt contained in B. Since

$$\int_{B_r} \bigl|\nabla u(y)\bigr|\, dy \le 2\, r^n\, |B|\, \|\nabla u\|_{A,B_r}\, A^{-1}\bigl(r^{-n}/|B|\bigr),$$

we obtain

$$\bigl|u(x) - u_B\bigr| \le \frac{4}{n\, \Omega_n^{1/n}} \|\nabla u\|_{A,\Omega} \int_{1/|B|}^{\infty} \frac{A^{-1}(r)}{r^{1+1/n}}\, dr,$$

where $\Omega_n := \bigl|B(0,1)\bigr|$. Since h is increasing and concave, we have $h(s\,t) \le h\bigl((1+s)t\bigr) \le (1+s)h(t)$ for $s, t > 0$, and therefore

$$\begin{aligned}
\bigl|u(x_0) - u(x_1)\bigr| &\le \frac{8}{n\, \Omega_n^{1/n}} \|\nabla u\|_{A,\Omega}\, h(t\, \Omega_n^{1/n}) \\
&\le \frac{8\,(1 + \Omega_n^{1/n})}{n\, \Omega_n^{1/n}} \|\nabla u\|_{A,\Omega}\, h(x_0, x_1),
\end{aligned}$$

which yields the desired result. □

The following theorem is an immediate consequence of Proposition 3.1.

3.2. Theorem. *Let A be an Orlicz function and h defined by (3.1). Assume $h(1) < \infty$, g to be a modulus of continuity and Ω to be a* $\mathrm{Lip}_{h,g}$*-extension domain. Then $W^1L_A(\Omega)$ is continuously embedded in* $\mathrm{Lip}_g(\Omega)$.

However, $\mathrm{Lip}_{h,g}$-extension domains do not necessarily exist. In order to apply Theorem 3.2 we need to know that they do exist.

3.3. Theorem. *Let h be a modulus of continuity. Then the following conditions are equivalent:*

(1) *There are constants $K < \infty$ and $t_K > 0$ such that for every $0 < t \le t_K$*

$$\int_0^t \frac{h(s)}{s}\,ds \le K\,h(t).$$

(2) *All bounded uniform domains are* Lip_h*-extension domains.*
(3) *The unit ball in* $\mathbf{R}^n$ *is a* Lip_h*-extension domain.*
(4) *There exists at least one* Lip_h*-extension domain.*

For a proof see [7, p. 27].

Note that if Condition 3.3.(1) holds for all $t > 0$, then all uniform domains are Lip_h-extension domains.

3.4. Corollary. *Assume A to be an Orlicz function with*

$$\frac{A'(\xi)}{A(\xi)} \ge \frac{p}{\xi} \qquad \text{for a.e. } \xi \ge \xi_0. \tag{3.2}$$

for some $p > n$ and $\xi_0 > 0$ and Ω to be a bounded uniform domain.

Then $W^1L_A(\Omega)$ is continuously embedded in $C^h(\overline{\Omega})$, where h is defined by (3.1).

Proof. We just combine Theorem 3.2 with $g := h$ and Theorem 3.3 with the following lemma.

3.5. Lemma. *Let $t_K := A(\xi_0)^{-1/n}$ and $K := p/(p-n)$. Then, for $0 < t \le t_K$, $h(t)$ is finite and*

$$\int_0^t \frac{h(s)}{s}\,ds \le Kh(t).$$

Proof. Integrating the inequality (3.2) we obtain $A(\xi) \ge \big(A(\eta)/\eta^p\big)\,\xi^p$ for $\xi \ge \eta \ge \xi_0$ by the absolute continuity of A, and hence $A^{-1}(r) \le \big(\eta\,A(\eta)^{-1/p}\big)\,r^{1/p}$ for $r \ge A(\eta)$. Now for $\eta = A^{-1}(t^{-n})$ the definition (3.1) of h yields

$$h(t) \le \frac{\eta}{A(\eta)^{1/p}} \int_{t^{-n}}^{\infty} \frac{r^{1/p}}{r^{1+1/n}}\,dr = \frac{\eta}{A(\eta)^{1/p}}\,Knt^{1-n/p}.$$

Since $h'(t) = nA^{-1}(t^{-n})$ and $\eta/A(\eta)^{1/p} = A^{-1}(t^{-n})t^{n/p}$, we have $h(t) \le Kh'(t)t$.

4. Examples

Let Ω be a bounded uniform domain in $\mathbf{R}^n$.

4.1. Let $p > n$ and $\alpha := 1 - n/p$. Then $W^{1,p}(\Omega)$ is continuously embedded in $C^\alpha(\overline{\Omega})$. This follows immediately from Corollary 3.4 since for $A(\xi) := \xi^p$ we have $A'(\xi)/A(\xi) = p/\xi$.

4.2. Let $A(\xi) := e^\xi$. Then the modulus of continuity defined by (3.1) is given by $h(t) = n^2 \big(\ln(1/t)+1\big)\, t$. For any $\alpha \in \,]0,1[$ the Orlicz–Sobolev space $W^1L_A(\Omega)$ is compactly embedded in $C^\alpha(\overline{\Omega})$. By Corollary 3.4, $W^1L_A(\Omega)$ is continuously embedded in $C^h(\overline{\Omega})$. Since $h(t)/t^\alpha \to 0$ as $t \to 0$, the result follows from the Ascoli–Arzela theorem.

4.3. Let $A(\xi) := \xi^n \big(\ln(\xi)\big)^q$ and assume $q > n$. Then $h(t) = n\big(\ln(\eta)\big)^{-q/n} \times \big(n/(q-n)\ln(\eta) + 1\big)$, where $\eta := A^{-1}(t^{-n})$. Then, if Ω has the strong local Lipschitz property in $\mathbf{R}^n$, the Orlicz–Sobolev space $W^1L_A(\Omega)$ is continuously embedded in $C^h(\overline{\Omega})$. This follows from [1, Theorem 8.36]. However, the modulus of continuity h does not satisfy Condition 3.3.(1) and therefore there does not exist any Lip_h-extension domains.

The snowflake or the Koch curve described in Mandelbrot [10, p. 42] bounds a uniform domain whose boundary is very irregular. Examples of domains which are Lip_α-extension domains but not uniform can be found in [7] and [3]. Also, in [7] there are examples of Lip_β-extension domains which are not Lip_α-extension domains for any $\alpha < \beta$.

References

[1] ADAMS, R.A.: Sobolev spaces. - Pure and Applied Mathematics 65. Academic Press, New York–San Francisco–London, 1975.

[2] GEHRING, F.W., and O. MARTIO: Quasidisks and the Hardy–Littlewood property. - Complex Variables Theory Appl. 2, 1983, 67–78.

[3] GEHRING, F.W., and O. MARTIO: Lipschitz classes and quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 203–219.

[4] GEHRING, F.W., and B.S. OSGOOD: Uniform domains and the quasihyperbolic metric. - J. Analyse Math. 36, 1979, 50–74.

[5] JONES, P.: Quasiconformal mappings and extendability of functions in Sobolev spaces. - Acta Math. 147, 1981, 71–88.

[6] KUFNER, A., O. JOHN and S. FUČIK: Function spaces. - Noordhoff International Publishing Leyden; Academia, Prague, 1977.

[7] LAPPALAINEN, V.: Lip_h-extension domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 56, 1985.

[8] LAPPALAINEN, V.: Local and global Lipschitz classes. - Seminar on Deformations, Łódź-Lublin. To appear.

[9] LEHTONEN, A.: Embedding of Sobolev spaces into Lipschitz spaces. - Seminar on Deformations, Łódź-Lublin. To appear.

[10] MANDELBROT, B.: The fractal geometry of nature. - W.H. Freeman and Company, San Francisco, 1982.

[11] MARTIO, O.: Definitions for uniform domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 179–205.

[12] MARTIO, O., and J. SARVAS: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978/79, 383–401.

[13] MCSHANE, E.J.: Extensions of range of functions. - Bull. Amer. Math. Soc. 40, 1934, 837–842.

[14] NEČAS, J.: Les méthodes directes en théorie des équations elliptiques. - Masson et C^ie Editeurs, Paris; Academia, Editeurs, Prague, 1967.

[15] STEIN, E.M.: Singular integrals and differentiability properties of functions. - Princeton University Press, Princeton, New Jersey, 1970.

University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF–40100 Jyväskylä
Finland

Received 7 October 1987