EMBEDDING OF ORLIZ–SOBOLEV SPACES
IN HÖLDER SPACES

Vesa Lappalainen and Ari Lehtonen

1. Introduction

For a smooth domain $\Omega$ in $\mathbb{R}^n$, e.g. a bounded Lipschitz domain, each function $u$ which belongs to the Sobolev space $W^{1,p}(\Omega)$ is in fact Hölder-continuous in $\Omega$ if $p$ is greater than $n$ (cf. Adams [1], Kufner et al [6] or Necás [14]). A similar embedding property holds also for Orlicz–Sobolev spaces (cf. [1] or [6]).

Typically, the boundary behaviour of $u$ is handled by straightening the boundary to a half space using local coordinate maps and deriving estimates for the Hölder norm of $u$ in terms of the (Orlicz–)Sobolev norm (cf. [14, Chapter 2.3.5]). Instead of using estimates on the boundary we first show that if $p > n$ the Sobolev spaces $W^{1,p}(\Omega)$ can be embedded in a certain local Hölder class $\text{loc Lip}_\alpha(\Omega)$, $\alpha = 1 - n/p$ for any domain $\Omega$. The embedding to $C^\alpha(\Omega)$ is then derived for a large class of domains via the embedding of $\text{loc Lip}_\alpha(\Omega)$ to $C^\alpha(\Omega)$. The following result is obtained as a corollary:

**Theorem.** If $\Omega$ is a bounded uniform domain and $p > n$, then $W^{1,p}(\Omega)$ is continuously embedded in $C^\alpha(\Omega)$.

Note that by a result of P. Jones [5] there exists an extension operator $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ for uniform domains, and the theorem hence follows from the well-known embedding $W^{1,p}(\mathbb{R}^n) \to C^\alpha(\mathbb{R}^n)$. However, for the theorem no extension result is needed, and our approach is based on classical Hölder continuity estimates together with Gehring and Martio’s [3] and Lappalainen’s [7] results on Lip$_\alpha$-extension domains. Therefore our method applies to a larger class of domains than uniform domains.

2. Preliminaries

An *Orlicz function* is any continuous map $A: \mathbb{R} \to \mathbb{R}$ which is strictly increasing, even, convex and satisfies

$$\lim_{\xi \to 0} A(\xi)\xi^{-1} = 0, \quad \lim_{\xi \to \infty} A(\xi)\xi^{-1} = \infty.$$ 

We let $\Omega$ denote a domain in $\mathbb{R}^n$. The *Orlicz class* $K_A(\Omega)$ is the set of all measurable functions $u$ such that

$$\int_{\Omega} A(u(x)) \, dx < \infty.$$
and the Orlicz space $L_A(\Omega)$ is the linear hull of $K_A(\Omega)$. As norm in the Orlicz space we use the Luxemburg norm

$$
\|u\|_{A,\Omega} := \inf \left\{ r > 0 : \int_{\Omega} A(u(x)/r) \, dx \leq 1 \right\}.
$$

The Orlicz–Sobolev space $W^1 L_A(\Omega)$ is the set of functions $u$ such that $u$ and its first order distributional derivatives lie in $L_A(\Omega)$. In the case where $A(\xi) = \xi^p$ we obtain the standard Sobolev space $W^{1,p}(\Omega)$. For a more detailed discussion of Orlicz spaces we refer to [1] and [6].

A domain $\Omega$ in $\mathbb{R}^n$ is called $c$-uniform if each pair of points $x, y \in \Omega$ can be joined by a rectifiable curve $\gamma$ in $\Omega$ such that $l(\gamma) \leq c |x - y|$ and

$$
\text{dist}(\gamma(t), \partial \Omega) \geq c^{-1} \min(t, l(\gamma) - t).
$$

A modulus of continuity is any concave positive increasing function $h : [0, \infty[ \to \mathbb{R}$, $h(0) = 0$. A function $u : \Omega \to \mathbb{R}$ belongs to the local Lipschitz class $\text{loc} \text{Lip}_h(\Omega)$ if there exist constants $b \in ]0,1[$ and $M = m_b$ such that for each $x \in \Omega$ and $y \in B_b(x) := B(x, b \text{dist}(x, \partial \Omega))$

$$
(2.1) \quad |u(x) - u(y)| \leq M h(x, y);
$$

here and hereafter $h(x, y) := h(|x - y|)$. As a matter of fact, it is shown in [7] that it is equivalent to require the condition to hold for $b = 1/2$; the smallest $m_{1/2}$ defines a seminorm of $u$. It should be remarked that this definition differs from the standard definitions of local Hölder spaces. In fact, the class $\text{loc} \text{Lip}_h(\Omega)$ is not a local space but semiglobal in a sense. A function $u$ belongs to the Lipschitz class $\text{Lip}_h(\Omega)$ if there exists a constant $M < \infty$ such that (2.1) holds for all $x, y \in \Omega$. For bounded domains $\text{Lip}_h(\Omega) = C^h(\Omega)$, where $C^h(\Omega)$ is as in [1, 8.37].

Let $h$ and $g$ be two moduli of continuity. A domain $\Omega$ is a $\text{Lip}_{h,g}$-extension domain if $\text{loc} \text{Lip}_h(\Omega)$ is continuously embedded in $\text{Lip}_g(\Omega)$. For short $\text{Lip}_{h,h} =: \text{Lip}_h$ and, for $h(t) = t^\alpha$, $\text{Lip}_h =: \text{Lip}_\alpha$. The following result due to McShane [13] justifies the name extension domain (see also Stein [15], [3] and [7]).

2.1. Theorem. If $\Omega$ is a $\text{Lip}_h$-extension domain and $u \in \text{loc} \text{Lip}_h(\Omega)$, there exists a $\text{Lip}_h$-extension $u^* : \mathbb{R}^n \to \mathbb{R}$.

We can characterize $\text{Lip}_h$-extension domains by using the following metric in $\Omega :$

$$
\text{h}_\Omega(x,y) := \inf_{\gamma(x,y)} \int_{\gamma} \frac{h(\text{dist}(z, \partial \Omega))}{\text{dist}(z, \partial \Omega)} \, d\sigma(z),
$$

where the infimum is taken over all rectifiable curves $\gamma$ in $\Omega$ joining $x$ to $y$. 
2.2. **Theorem.** A domain $\Omega \subset \mathbb{R}^n$ is a $\text{Lip}_{h,g}$-extension domain if and only if there is a constant $1 \leq K(\Omega, h, g) < \infty$ such that

$$h_{\Omega}(x, y) \leq K g(x, y)$$

holds in $\Omega$.

For a proof see e.g. [3] or [7].

3. **Embedding of Orlicz–Sobolev spaces**

Let $A$ denote an Orlicz function. If

$$h(t) := \int_{t^{-n}}^{\infty} \frac{A^{-1}(r)}{r^{1+1/n}} dr$$

is finite at $t = \varepsilon$, then $h$ defines a modulus of continuity on the interval $[0, \varepsilon]$. It is easily seen that the derivative $h'(t) = n A^{-1}(t^{-n})$ is decreasing.

3.1. **Proposition.** If $h(1) < \infty$, then $W^1 L_A(\Omega)$ is continuously embedded in $\text{loc Lip}_h(\Omega)$ for any domain $\Omega \subset \mathbb{R}^n$.

**Proof.** It follows from [1, Theorem 5.35] applied to balls contained in $\Omega$ that each function $u \in W^1 L_A(\Omega)$ is continuous. Now let $B_b(x_0)$ be a ball contained in $\Omega$ and $x_1 \in B_b(x_0)$. Let $t := |x_0 - x_1|$ and choose a ball $B$ of radius $t$ such that $x_0, x_1 \in B \subset B_b(x_0)$. We denote by $|B|$ the Lebesgue measure of $B$ and by

$$u_B := \frac{1}{|B|} \int_B u(z) \, dz$$

the mean value of $u$ in $B$. As in [1] we obtain the following estimate for $x \in B$:

$$|u(x) - u_B| \leq \frac{2t}{|B|} \int_0^1 r^{-n} \int_{B_r} |\nabla u(z)| \, dz,$$

where $B_r$ denotes a ball of radius $rt$ contained in $B$. Since

$$\int_{B_r} |\nabla u(y)| \, dy \leq 2 r^n |B| \|\nabla u\|_{A, B_r} \frac{A^{-1}(r^{-n}/|B|)}{r^{1+1/n}},$$

we obtain

$$|u(x) - u_B| \leq \frac{4}{n^\frac{1}{n}} \|\nabla u\|_{A, \Omega} \int_{1/|B|}^{\infty} \frac{A^{-1}(r)}{r^{1+1/n}} dr,$$

where $\Omega_n := |B(0, 1)|$. Since $h$ is increasing and concave, we have $h(st) \leq h((1+s)t) \leq (1+s)h(t)$ for $s,t > 0$, and therefore

$$|u(x_0) - u(x_1)| \leq \frac{8}{n^\frac{1}{n}} \|\nabla u\|_{A, \Omega} h(t \Omega_n^{1/n})$$

$$\leq \frac{8(1 + \Omega_n^{1/n})}{n^\frac{1}{n}} \|\nabla u\|_{A, \Omega} h(x_0, x_1),$$

which yields the desired result. □

The following theorem is an immediate consequence of Proposition 3.1.
3.2. Theorem. Let $A$ be an Orlicz function and $h$ defined by (3.1). Assume $h(1) < \infty$, $g$ to be a modulus of continuity and $\Omega$ to be a $\text{Lip}_{h,g}$-extension domain. Then $W^1 L_A(\Omega)$ is continuously embedded in $\text{Lip}_g(\Omega)$.

However, $\text{Lip}_{h,g}$-extension domains do not necessarily exist. In order to apply Theorem 3.2 we need to know that they do exist.

3.3. Theorem. Let $h$ be a modulus of continuity. Then the following conditions are equivalent:

1. There are constants $K < \infty$ and $t_K > 0$ such that for every $0 < t \leq t_K$
   \[
   \int_0^t \frac{h(s)}{s} \, ds \leq K h(t).
   \]

2. All bounded uniform domains are $\text{Lip}_h$-extension domains.

3. The unit ball in $\mathbb{R}^n$ is a $\text{Lip}_h$-extension domain.

4. There exists at least one $\text{Lip}_h$-extension domain.

For a proof see [7, p. 27].

Note that if Condition 3.3.(1) holds for all $t > 0$, then all uniform domains are $\text{Lip}_h$-extension domains.

3.4. Corollary. Assume $A$ to be an Orlicz function with

\[
A'(\xi) \geq \frac{p}{\xi} \quad \text{for a.e. } \xi \geq \xi_0.
\]

for some $p > n$ and $\xi_0 > 0$ and $\Omega$ to be a bounded uniform domain.

Then $W^1 L_A(\Omega)$ is continuously embedded in $C^h(\Omega)$, where $h$ is defined by (3.1).

Proof. We just combine Theorem 3.2 with $g := h$ and Theorem 3.3 with the following lemma.

3.5. Lemma. Let $t_K := A(\xi_0)^{-1/n}$ and $K := p/(p - n)$. Then, for $0 < t \leq t_K$, $h(t)$ is finite and

\[
\int_0^t \frac{h(s)}{s} \, ds \leq K h(t).
\]

Proof. Integrating the inequality (3.2) we obtain $A(\xi) \geq (A(\eta)/\eta^p) \xi^p$ for $\xi \geq \eta \geq \xi_0$ by the absolute continuity of $A$, and hence $A^{-1}(r) \leq (\eta A(\eta)^{-1/p}) r^{1/p}$ for $r \geq A(\eta)$. Now for $\eta = A^{-1}(t^{-n})$ the definition (3.1) of $h$ yields

\[
h(t) \leq \frac{\eta}{A(\eta)^{1/p}} \int_{t^{-n}}^\infty \frac{r^{1/p}}{r^{1+1/n}} \, dr = \frac{\eta}{A(\eta)^{1/p}} K n t^{1-n/p}.
\]

Since $h'(t) = n A^{-1}(t^{-n})$ and $\eta/A(\eta)^{1/p} = A^{-1}(t^{-n}) t^{n/p}$, we have $h(t) \leq K h'(t) t$. 

4. Examples

Let $\Omega$ be a bounded uniform domain in $\mathbb{R}^n$.

4.1. Let $p > n$ and $\alpha := 1 - n/p$. Then $W^{1,p}(\Omega)$ is continuously embedded in $C^{\alpha}(\overline{\Omega})$. This follows immediately from Corollary 3.4 since for $A(\xi) := \xi^p$ we have $A'(\xi)/A(\xi) = p/\xi$.

4.2. Let $A(\xi) := e^\xi$. Then the modulus of continuity defined by (3.1) is given by $h(t) = n^2 \left( \ln(1/t) + 1 \right) t$. For any $\alpha \in ]0,1[$ the Orlicz–Sobolev space $W^{1,L_A}(\Omega)$ is compactly embedded in $C^{\alpha}(\overline{\Omega})$. By Corollary 3.4, $W^{1,L_A}(\Omega)$ is continuously embedded in $C^h(\overline{\Omega})$. Since $h(t)/t^\alpha \to 0$ as $t \to 0$, the result follows from the Ascoli–Arzela theorem.

4.3. Let $A(\xi) := \xi^n \left( \ln(\xi) \right)^q$ and assume $q > n$. Then $h(t) = n \left( \ln(\eta) \right)^{-q/n} \times (n/(q - n) \ln(\eta) + 1)$, where $\eta := A^{-1}(t^{-n})$. Then, if $\Omega$ has the strong local Lipschitz property in $\mathbb{R}^n$, the Orlicz–Sobolev space $W^{1,L_A}(\Omega)$ is continuously embedded in $C^h(\overline{\Omega})$. This follows from [1, Theorem 8.36]. However, the modulus of continuity $h$ does not satisfy Condition 3.3.(1) and therefore there does not exist any Lip$_h$-extension domains.

The snowflake or the Koch curve described in Mandelbrot [10, p. 42] bounds a uniform domain whose boundary is very irregular. Examples of domains which are Lip$_\alpha$-extension domains but not uniform can be found in [7] and [3]. Also, in [7] there are examples of Lip$_\beta$-extension domains which are not Lip$_\alpha$-extension domains for any $\alpha < \beta$. 

References