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ON THE PRESERVATION OF DIRECTION-CONVEXITY AND THE GOODMAN–SAFF CONJECTURE

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Abstract. Let $K(\varphi)$ be the set of univalent functions in the unit disk $\mathbf{D}$ which are convex in the direction $e^{i\varphi}$. We determine the set of analytic functions g in $\mathbf{D}$ which preserve $K(\varphi)$ under the Hadamard product, i.e., $g * f \in K(\varphi)$ whenever $f \in K(\varphi)$. This result contains as a special case the proof of a conjecture of Goodman and Saff about $K(\varphi)$ and solves partially a multiplier problem concerning convex univalent harmonic functions in $\mathbf{D}$, posed by Clunie and Sheil-Small.

1. Introduction

A domain $M \subset \mathbf{C}$ is said to be convex in the direction $e^{i\varphi}$ if for every $a \in \mathbf{C}$ the set

$$M \cap \{a + te^{i\varphi} : t \in \mathbf{R}\}$$

is either connected or empty. Let $K(\varphi)$ be the family of univalent analytic functions f in the unit disk $\mathbf{D}$ with $f(\mathbf{D})$ convex in the direction $e^{i\varphi}$ and, similarly, $K_H(\varphi)$ with 'univalent analytic' replaced by 'univalent harmonic'. It is well-known (see W. Hengartner and G. Schober [5], A.W. Goodman and E.B. Saff [4]) that for $r_0 := \sqrt{2} - 1 < r < 1$ generally $f \in K(\varphi)$ does not imply $f(rz) \in K(\varphi)$, but Goodman and Saff conjectured that such an implication may hold for $0 < r \le r_0$. Recently J. Brown [1] proved that

$$f \in K(\varphi) \Rightarrow f(r \circ z) \in K(\psi), \qquad \psi \in I(f),$$

where $I(f) \subset [0, 2\pi)$ is a set of positive measure. It was not shown, however, that $\varphi \in I(f)$ and thus the conjecture remained open. We shall prove the following stronger result:

Theorem 1. *Let $f \in K_H(\varphi)$, $0 < r \le r_0$. Then $f(rz) \in K_H(\varphi)$.*

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This settles the Goodman–Saff conjecture even for univalent harmonic functions. In the analytic case, however, Theorem 1 is a very simple special case of the solution of the following multiplier problem ($*$ denotes the Hadamard product):

Determine the set DCP of all analytic functions g in $\mathbf{D}$ such that $g*f \in K(\varphi)$ for every $\varphi \in \mathbf{R}$ and every $f \in K(\varphi)$.

Theorem 2. *Let g be analytic in $\mathbf{D}$. Then $g \in$ DCP if and only if*

$$\text{for each } \gamma \in \mathbf{R} : g + i\gamma z g' \in K\left(\frac{\pi}{2}\right). \tag{1}$$

Theorem 1, for f analytic, follows from Theorem 2 by choosing $g_r(z) := 1/(1-rz)$ and showing that $g_r \in$ DCP for $0 < r \leq r_0$. If f is harmonic in $\mathbf{D}$, $f = \overline{f_1} + f_2$ with f_1, f_2 analytic in $\mathbf{D}$ and $f_1(0) = 0$, we may define for an analytic g

$$f \tilde{*} g := \overline{(f_1 * g)} + (f_2 * g).$$

It is not true that all functions g satisfying (1) preserve $K_H(\varphi)$ under the operation $\tilde{*}$ (Theorem 1, however, says that this is the case for g_r). For an example see Clunie and Sheil-Small [3, (5.21.1)] where the multiplier happens to satisfy (1). But out result does extend to the class K_H of convex harmonic univalent functions f (where 'convex' indicates that $f(\mathbf{D})$ is convex).

Theorem 3. *Let g be analytic in $\mathbf{D}$. Then $f \tilde{*} g \in K_H$ for all $f \in K_H$ if and only if g satisfies (1), i.e., $g \in$ DCP.*

This theorem solves partially a problem of Clunie and Sheil-Small [3,(7.7)].

The members of $K(\varphi)$ are usually described analytically through a condition due to M.S. Robertson [7] (see also W.C. Royster and M. Ziegler [8]). Unfortunately, this condition is very difficult to deal with when it comes to convolutions (Hadamards products). In the proof of the basic Theorem 2 we shall use a completely different way, namely the concept of periodically monotone functions, introduced by I.J. Schoenberg [11].

Definition. Let u be a real, continuous, 2π-periodic function. It is said to be *periodically monotone* ($u \in$ PM) if there exist numbers $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that u increases on (θ_1, θ_2) and decreases on $(\theta_2, \theta_1 + 2\pi)$.

We shall reduce the discussion of functions in DCP to the characterisation of certain integral kernels which preserve periodic monotonicity. And, as a result of this connection, we also obtain the following very handy criterion for g to be in DCP.

Theorem 4. *Let g be non-constant and analytic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$ with $u(\theta) = \operatorname{Re} g(e^{i\theta})$ three times continuously differentiable. Then $g \in$ DCP if and only if $u \in$ PM with*

$$u'(\theta)u'''(\theta) \leq \left(u''(\theta)\right)^2, \qquad \theta \in \mathbf{R}. \tag{2}$$

2. Proofs

Let $C^k_{2\pi}$ denote the set of real 2π-periodic functions which are k times continuously differentiable. For $u, v \in C^0_{2\pi}$ we define

$$(u * v) := \frac{1}{2\pi}\int_0^{2\pi} u(\psi)v(\theta-\psi)\,d\psi.$$

There will be no confusion in using the same symbol $*$ for different convolutions since from the context it will be always clear which one is meant. In fact, there is a close connection between the two definitions: let g, h be analytic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$, $g(0)=0$, and set

$$u(\theta) := \operatorname{Re} g(e^{i\theta}), \quad v(\theta) := \operatorname{Re} h(e^{i\theta}), \qquad \theta \in \mathbf{R}.$$

Then we have the important relation

$$(u * v)(\theta) = \tfrac{1}{2}\operatorname{Re}(g * h)(e^{i\theta}), \qquad \theta \in \mathbf{R}. \tag{3}$$

(3) is readily verified by writing down the corresponding Fourier expansions.

A function $u \in C^1_{2\pi}$ is said to preserve periodic monotonicity ($u \in \mathrm{PMP}$) if

$$u * v \in \mathrm{PM} \qquad \text{for every } v \in \mathrm{PM}.$$

Let V_n be the de la Vallée-Poussin kernels:

$$V_n(\theta) := \binom{2n}{n}^{-1}(1+\cos\theta)^n, \qquad \theta \in \mathbf{R}, n \in \mathbf{N}. \tag{4}$$

It is known (de la Vallée-Poussin [13]) that for $u \in C^0_{2\pi}$ we have

$$\lim_{n\to\infty}(V_n * u)(\theta) = u(\theta), \qquad \theta \in \mathbf{R}.$$

Furthermore, as has been shown by Pólya and Schoenberg [6], the V_n are variation diminishing. These two properties imply:

Lemma 1. *Let $u \in C^0_{2\pi}$. Then $u \in \mathrm{PM}$ if and only if $V_n * u \in \mathrm{PM}$ for all $n \in \mathbf{N}$.*

Similarly we obtain

Lemma 2. *Let $u \in C^1_{2\pi}$. Then $u \in \mathrm{PMP}$ if and only if $V_n * u \in \mathrm{PMP}$ for all $n \in \mathbf{N}$.*

Indeed, if $u \in \mathrm{PMP}$, $v \in \mathrm{PM}$ then, by Lemma 1, $V_n * u \in \mathrm{PM}$ and hence $(V_n * v) * u = v * (V_n * u) \in \mathrm{PM}$, which implies $V_n * u \in \mathrm{PMP}$. In the other direction, if $v * (V_n * u) \in \mathrm{PM}$ for all $v \in \mathrm{PM}$ then, using dominated convergence,

$$v * u = \lim_{n\to\infty} (V_n * v) * u \in \mathrm{PM}$$

and hence $u \in \mathrm{PMP}$. The crucial part in the proof of Theorem 2 is contained in the following result.

Theorem 5. *Let $u \in C^1_{2\pi}$ be such that*

$$\tilde{u}(\theta) := u(\theta) - iu'(\theta), \qquad 0 \le \theta \le 2\pi, \tag{5}$$

is a (complex) Jordan curve with a convex interior domain. Then $u \in \mathrm{PMP}$.

We remark that a more general definition of the classes PM and PMP has been studied by Schoenberg [11], who also quotes a result of C. Loewner which says that (5) is essentially also a necessary condition for $u \in \mathrm{PMP}$. In another paper [10] we give the complete characterisation of the wider Schoenberg class. For our present purpose, however, this is of no relevance.

We shall reduce the proof of Theorem 5 to the following lemma which is of independent interest.

Lemma 3. *Let u be a trigonometric polynomial satisfying the assumptions of Theorem 5. Let $h \not\equiv$const. be a function in $C^0_{2\pi}$ such that h has at most two sign changes in any interval of length 2π and satisfies*

$$\frac{1}{2\pi}\int_0^{2\pi} h(\psi)\, d\psi = 0. \tag{6}$$

Then $u * h$ has exactly two zeros (which are simple) in $[0, 2\pi)$.

Proof. We first note that $\tilde{u}$ is strongly convex, i.e., there are no three numbers $\theta_1 < \theta_2 \theta_3 < \theta_1 + 2\pi$ such that the points $\tilde{u}(\theta_j)$, $j = 1, 2, 3$, lie on a straight line. In fact, if they were, then by the convexity we conclude that $\tilde{u}(\theta)$ lies on that straight line, $\theta_1 \le \theta \le \theta_3$. This gives a relation

$$au(\theta) + bu'(\theta) + c = 0 \tag{7}$$

on that interval, and since u is a trigonometric polynomial, for all θ. But then $\tilde{u}$ lies completely in that straight line, a contradiction to the assumption. We shall use this information in the following form: let $\psi_1 < \psi_2 < \psi_1 + 2\pi$ and denote by $\mathring{\mathrm{co}}(A)$ the interior of the convex hull of a set $A \subset \mathbf{C}$. Then

$$\mathring{\mathrm{co}}\{\tilde{u}(\psi) : \psi_1 \le \psi \le \psi_2\} \cap \mathring{\mathrm{co}}\{\tilde{u}(\psi) : \psi_2 \le \psi \le \psi_1 + 2\pi\} = \emptyset. \tag{8}$$

Now let $\theta_1 < \theta_2 < \theta_1 + 2\pi$ be such that

$$h(\theta)\begin{cases} \geq 0, & \text{if } \theta_1 \leq \theta \leq \theta_2, \\ \leq 0, & \text{if } \theta_2 \leq \theta \leq \theta_1 + 2\pi. \end{cases}$$

Note that by (6) we can be sure that h has at least two zeros in a period. We define the set $M = M(\theta_1, \theta_2)$ as the set of real 2π-periodic functions g, continuous in $I_1 := (\theta_1, \theta_2)$ and in $I_2 = (\theta_2, \theta_1 + 2\pi)$, such that

$$g(\theta)\begin{cases} \geq 0, & \text{if } \theta \in I_1, \\ \leq 0, & \text{if } \theta \in I_2, \end{cases} \tag{9}$$

and

$$1 = \frac{1}{2\pi}\int_{I_1} g(\psi)\, d\psi = -\frac{1}{2\pi}\int_{I_2} g(\psi)\, d\psi. \tag{10}$$

Clearly $\varrho h \in M$ for some suitable $\varrho > 0$. For $g \in M$ the function $v_g := g * u$ is a trigonometric polynomial and we wish to show that this polynomial cannot have any multiple zero. In fact, for $\varrho \in \mathbf{R}$ we have

$$v_g(\varphi) - iv_g'(\varphi) = \frac{1}{2\pi}\int_{I_1} g(\theta)\tilde{u}(\varphi - \theta)\, d\theta - \frac{1}{2\pi}\int_{I_2} \big(-g(\theta)\big)\tilde{u}(\varphi - \theta)\, d\theta,$$

and from (9), (10) we conclude that

$$\frac{1}{2\pi}\int_{I_1} g(\theta)\tilde{u}(\varphi - \theta)\, d\theta \in \mathring{\text{co}}\big\{\tilde{u}(\psi) : \varphi - \theta_2 \leq \psi \leq \varphi - \theta_1\big\},$$

$$\frac{1}{2\pi}\int_{I_2} \big(-g(\theta)\big)\tilde{u}(\varphi - \theta)\, d\theta \in \mathring{\text{co}}\big\{\tilde{u}(\psi) : \varphi - \theta_1 - 2\pi \leq \psi \leq \varphi - \theta_2\big\},$$

and thus by (8)

$$v_g(\varphi) - iv_g'(\varphi) \neq 0, \qquad \varphi \in \mathbf{R}.$$

Hence v_g and v_g' can never vanish simultaneously and v_g cannot have multiple zeros. Now assume that we can find at least one $g_0 \in M$ such that V_{g_0} has only two zeros (simple, of course) in a period. Then, if $v_{\varrho h}$ has more than two zeros in a period (but, because of the periodicity, an even number), then there exists a $\lambda \in (0,1)$ such that

$$\lambda v_{g_0} + (1 - \lambda)v_{\varrho h} = v_{[\lambda g_0 + (1-\lambda)\varrho h]}$$

has a double zero. But M is a convex set and hence $\lambda g_0 + (1 - \lambda)\varrho h \in M$, a contradiction.

What remains is to construct g_0. We set

$$g_0(\theta) = \begin{cases} 2\pi/(\theta_2-\theta_1), & \text{if } \theta \in I_1, \\ 0, & \text{if } \theta = \theta_1, \theta_2, \\ -2\pi/(\theta_1+2\pi-\theta_2), & \text{if } \theta \in I_2, \end{cases}$$

and extend this definition periodically to $\mathbf{R}$. Then $g_0 \in M$ and

$$v_{g_0}(\varphi) = \frac{1}{\theta_2-\theta_1}\int_{I_1} u(\varphi-\theta)\,d\theta - \frac{1}{\theta_1+2\pi-\theta_2}\int_{I_2} u(\varphi-\theta)\,d\theta,$$

and hence

$$v'_{g_0} = \left(\frac{1}{\theta_2-\theta_1} + \frac{1}{\theta_1+2\pi-\theta_2}\right)\big(u(\varphi-\theta_1) - u(\varphi-\theta_2)\big).$$

The convexity of $\tilde{u}$ implies that $u \in \mathrm{PM}$ and since u is a non-constant trigonometric polynomial v'_{g_0} has only two zeros in a period. The same is therefore true for v_{g_0}. Since $g_0 \in M$ we conclude that v_{g_0} has (exactly) two simple zeros in a period. This completes the proof of Lemma 3.

Proof of Theorem 5. It follows again from the variation diminishing property of the kernels V_n and from

$$V_n * u' = (V_n * u)'$$

that $u_n := V_n * u$ satisfies the assumptions of Theorem 5. Using Lemma 2 we conclude that we have to prove Theorem 5 only for trigonometric polynomials u. Similarly, if $t * u \in \mathrm{PM}$ for all trigonometric polynomials $t \in \mathrm{PM}$, then $u \in \mathrm{PMP}$.

A non-constant trigonometric polynomial t is in PM if and only if t' has exactly two sign changes in any period. Furthermore we obviously have

$$\frac{1}{2\pi}\int_0^{2\pi} t'(\psi)\,d\psi = 0.$$

Hence, if $t \in \mathrm{PM}$, we can apply Lemma 3 to $h := t'$ and obtain that

$$v' = (t * u)' = h * u$$

has (exactly) two sign changes in a period. This proves $v \in \mathrm{PM}$ and hence $u \in \mathrm{PMP}$.

The geometric condition concerning $\tilde{u}$ in Theorem 5 can be replaced by a more analytic one if $u \in C^3_{2\pi}$: we can then describe the convexity by the monotonicity of the tangent rotation at $\tilde{u}$ and by ensuring that the total variation of the argument of the tangent vector is 2π. This leads immediately to:

Lemma 4. *Let $u \in C^3_{2\pi}$ be non-constant and $\tilde{u}$ as in* (5). *Then $\tilde{u}$ fulfills the assumption of Theorem 5 if and only if $u \in \mathrm{PM}$ and*

$$u'(\theta)u'''(\theta) \leq \left(u'(\theta)\right)^2, \qquad \theta \in \mathbf{R}.$$

After these 'real' preliminaries we now turn to the discussion of $K(\varphi)$ and DCP. Also here we need a reduction to polynomial cases. We are working with the analytics version of the de la Vallée-Poussin kernels:

$$W_n(z) := \binom{2n}{n}^{-1} \sum_{k=0}^{n} \binom{2n}{n+k} z^k, \qquad z \in \mathbf{C},\ n \in \mathbf{N}. \tag{11}$$

Note that

$$2\mathrm{Re}\, W_n\left(e^{i\theta}\right) = V_n(\theta) + 1, \qquad \theta \in \mathbf{C},\ n \in \mathbf{N}. \tag{12}$$

Lemma 5. *Let g be analytic in $\mathbf{D}$. Then $g \in K(\varphi)$ if and only if $W_n * g \in K(\varphi)$ for $n \in \mathbf{N}$.*

Proof. Without loss of generality we may assume $g(0) = 0$, $\varphi = \frac{1}{2}\pi$. Let $g \in K(\pi/2)$, $\Gamma = g(\mathbf{D})$. We can construct a sequence of polygonal domains Γ_k with

$$0 \in \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma, \qquad \bigcup_{k \in \mathbf{N}} \Gamma_k = \Gamma,$$

and Γ_k convex in the direction of the imaginary axis. Let g_k be the univalent functions in $\mathbf{D}$ with $g_k(0) = 0$, $\arg g_k'(0) = \arg'(0)$ and $g_k(\mathbf{D}) = \mathbf{\Gamma_k}$. Then $g_k \in K(\pi/2)$ and $g_k \to g$ locally uniformly in $\mathbf{D}$ by Caratheodory's kernel convergence. The functions g_k extend continuously to $\partial\mathbf{D}$ and the direction-convexity is reflected by the property that $u_k(\theta) := \mathrm{Re}\, g_k\left(e^{i\theta}\right)$ is in PM. Hence, since $V_n \in \mathrm{PMP}$, we find using (3), (11), (12):

$$\mathrm{Re}\,(W_n * g_k) = V_n * u_k \in \mathrm{PM}. \tag{13}$$

The elements of $K(\pi/2)$ are, in particular, close-to-convex univalent functions while the polynomials W_n are convex univalent in $\mathbf{D}$ (Pólya and Schoenberg [6]). Hence, by the result of Ruscheweyh and Sheil-Small [9], we conclude that $W_n * g_k$ is close-to-convex univalent in $\mathbf{D}$. This fact together with (13) implies that $W_n * g_k \in K(\pi/2)$. But obviously $W_n * g_k \to W_n * g$ locally uniformly in $\mathbf{D}$ and hence $W_n * g \in K(\pi/2)$ for $n \in \mathbf{N}$.

If, on the other hand, $W_n * g \in K(\pi/2)$ for $n \in \mathbf{N}$ then we have $g \in K(\pi/2)$ since $W_n * g \to g$ locally uniformly in $\mathbf{D}$.

Lemma 6. *Let g be analytic in $\mathbf{D}$. Then $g \in \mathrm{DCP}$ if and only if $W_n * g \in \mathrm{DCP}$ for $n \in \mathbf{N}$.*

Proof. Lemma 5 shows, in particular, that $W_n \in$ DCP and since DCP is obviously closed under convolutions (i.e., $f, g \in$ DCP implies $f * g \in$ DCP) we have $W_n * g \in$ DCP if $g \in$ DCP. If $W_n * g \in$ DCP for $n \in \mathbf{N}$ then for $f \in K(\varphi)$:

$$g * (W_n * f) = (W_n * g) * f \in K(\varphi).$$

With $n \to \infty$ we obtain $g * f \in K(\varphi)$ and thus $g \in$ DCP.

For the proof of Theorem 2 we shall need one further result, due to Clunie and Sheil-Small [3]:

Lemma 7. *Let f_1, f_2 be analytic in $\mathbf{D}$, $f_1(0) = 0$. Then $F = \overline{f_1} + f_2 \in K_H$ if and only if*

$$f_2 - e^{i\varphi} f_1 \in K\left(\frac{\varphi}{2}\right), \qquad \varphi \in \mathbf{R}. \tag{14}$$

Proof of Theorem 2. We show first that (1) is necessary for g to be in DCP. We have $g + i\gamma z g' = g * f_\gamma$ where

$$f_\gamma(z) = \frac{1}{1-z} + i\gamma \frac{z}{(1-z)^2}, \qquad \gamma \in \mathbf{R}.$$

These functions are close-to-convex univalent and map $\mathbf{D}$ onto $\mathbf{C}$ minus a vertical slit. Thus they are in $K(\pi/2)$ and (1) turns out to be a special case of the direction-convexity preservation of g.

Now let g satisfy (1). We observe that this implies that g is convex univalent in $\mathbf{D}$. In fact, since $g * f_\gamma \in K(\pi/2)$ we see that

$$(g * f_\gamma)'(0) = g'(0) \cdot f_\gamma'(0) \neq 0$$

and thus $g'(0) \neq 0$. Furthermore, for $z \in \mathbf{D}$,

$$0 \neq (g * f_\gamma)'(z) = \frac{1}{z}(zg' * f_\gamma) = g' + i\gamma(zg')'$$

and hence

$$\frac{zg''(z)}{g'(z)} + 1 \neq \frac{i}{\gamma}, \qquad \gamma \in \mathbf{R}, \quad z \in \mathbf{D},$$

which gives

$$\operatorname{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) > 0, \qquad z \in \mathbf{D},$$

the convexity condition for g.

The convexity of g implies [9] that $f * g$ is univalent for f close-to-convex, in particular for $f \in K(\varphi)$.

We found already that $W_n \in \mathrm{DCP}$, $n \in \mathbf{N}$, and therefore

$$W_n * (g + i\gamma z g') = (W_n * g) + iz\gamma(W_n * g)' \in K(\tfrac{1}{2}\pi), \qquad \gamma \in \mathbf{R},$$

which shows that $W_n * g$ satisfies (1) as well. In view of Lemma 6 this implies that we have to prove the sufficiency part of Theorem 2 only for polynomials g. Similarly, using Lemma 5, we see that we have to prove $f * g \in K(\varphi)$ only for polynomials $f \in K(\varphi)$. Obviously we may restrict ourselves again to the case $\varphi = \pi/2$, and we may assume $g(0) = 0$. We know already that $f * g$ is univalent in $\mathbf{D}$. Hence to prove $f * g \in K(\pi/2)$ we just have to prove that

$$\mathrm{Re}\left[(f * g)(e^{i\theta})\right] = 2\big(\mathrm{Re}\, f(e^{i\theta})\big) * \big(\mathrm{Re}\, g(e^{i\theta})\big) \in \mathrm{PM}$$

under the assumption that $\mathrm{Re}\, f(e^{i\theta}) \in \mathrm{PM}$. But this is surely true if we can show that $u(\theta) := \mathrm{Re}\, g(e^{i\theta}) \in \mathrm{PMP}$.

We rewrite (1) as follows: let $i\gamma = (1 + e^{i\varphi})/(1 - e^{i\varphi})$, $0 < \varphi < 2\pi$, and note that

$$\arg\left[i(1 - e^{i\varphi})\right] = \tfrac{1}{2}\varphi, \qquad 0 < \varphi 2\pi.$$

Hence

$$(15) \qquad (1 - e^{i\varphi})(g + i\gamma z g') = (g + zg') - e^{i\varphi}(g - zg') \in K(\varphi/2),$$

for $0 < \varphi < 2\pi$. The limiting case $\gamma \to \infty$ can be used to show that (15) holds for $\varphi = 0$ as well. We now apply Lemma 7 and deduce that

$$(16) \qquad F(z) := \overline{g - zg'} + g + zg' = 2\big(\mathrm{Re}\, g(z) + i\mathrm{Im}\, zg'(z)\big) \in K_H.$$

This clearly implies that

$$(17) \qquad \tfrac{1}{2}F(e^{i\theta}) = u(\theta) - iu'(\theta), \qquad 0 \le \theta < 2\pi,$$

is a convex curve in the sense of Theorem 5: u belongs to PMP, and this completes the proof of Theorem 2.

We note that the last steps in this proof are invertible: if the curve (17) is convex in the sense of Theorem 5, then, by a Theorem of Choquet [2], the statement (16) also holds true. Using the other direction of Lemma 7 we conclude that the function g satisfies (1). We have shown:

Lemma 8. *Let g be analytic in $\mathbf{D}$, continuous in $\overline{\mathbf{D}}$ with $u(\theta) = \mathrm{Re}\, g(e^{i\theta}) \in C^1_{2\pi}$. Then $g \in \mathrm{DCP}$ if and only if u fulfills the assumptions of Theorem 5.*

The assertion of Theorem 4 is just a combination of Lemma 4 and Lemma 8.

Proof of Theorem 1. Using $g_r(z) := 1/(1-rz)$ we obtain

$$u_r(\theta) = \operatorname{Re} g_r(e^{i\theta}) = \tfrac{1}{2} + \tfrac{1}{2}\frac{1-r^2}{1+r^2-2r\cos\varphi}.$$

It is a matter of straightforward calculus to show that $u_r(\theta)$ satisfies the conditions of Theorem 4 for $0 < r \leq r_0$. Theorem 1 follows for $f \in K(\varphi)$. A result of Clunie and Sheil-Small [3, Theorem 5.3] extends this immediately to $K_H(\varphi)$.

Proof of Theorem 3. That $f \tilde{*} g \in K_H$ for $f \in K_H$ and $g \in$ DCP follows from Theorem 2 and Lemma 7. On the other hand, Clunie and Sheil-Small [3, (5.5.4)] have shown that

$$f_0(z) = \overline{\frac{1}{1-z} - \frac{z}{(1-z)^2}} + \frac{1}{1-z} + \frac{z}{(1-z)^2} \in K_H.$$

Hence, if g preserves harmonic convexity, we must have $F = f_0 \tilde{*} g \in K_H$ where F is exactly the function (16). As we have seen in the deduction of Lemma 8 this is equivalent to the fact that g satisfies (1) and hence to $g \in$ DCP.

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