A REGURAL CURVE THAT CANNOT BE COVERED WITH FINITELY MANY CHORD-ARC CURVES

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1. Introduction

Generalizing a theorem by Hayman and Wu [HW], Fernandez and Hamilton in [FH] prove that every chord-arc curve in the plane has finite conformal length. Finite conformal length is a property inherited by subcurves and by finite unions of curves. In particular every finite union of chord-arc curves will have finite conformal length. A partial converse to this is the trivial observation that finite conformal length of a curve implies regularity in the sense of Alhfors [A1] and David [D2]. We give an example of a curve that is regular, yet not a subset of any finite union of chord-arc curves, not even of quasicircles.

Let γ be a rectifiable, closed curve in the plane $\mathbf{R}^2 = \mathbf{C}$. We will use the letter γ to denote both an arc length parametrization $\gamma\colon [a,b] \to \mathbf{C}$, and the image set $\gamma = \gamma([a,b])$. The arc length distance between two points on γ is understood as the shorter distance between them along γ, i.e. $l(\gamma(t),\gamma(s)) = \min\{|t-s|, b-a-|t-s|\}$. The corresponding arc length measure is also called l.

A curve γ satisfies the *chord-arc* condition with constant L, if the arc length distance between points on γ never exceeds L times their ordinary distance:

$$l(z_1, z_2) \leq L|z_1 - z_2| \qquad \text{for all} \quad z_1, z_2 \text{ on } \gamma.$$

Chord-arc curves are also called *Lavrentiev* curves. A chord-arc curve is Jordan. Evidently it also satisfies the Ahlfors [A2] *three points condition*: "at least one of the arcs between any two points z_1 and z_2 on the curve has diameter at most a constant C times $|z_1 - z_2|$", characterizing *quasicircles* among (not necessarily rectifiable) Jordan loops. For details see [G].

A *regular* curve is a curve whose total length inside any circle $B(z_0, r) = \{z \in \mathbf{C} \mid |z - z_0| \leq r\}$ never exceeds a constant D times the radius r of the circle:

$$l(B(z_0, r) \cap \gamma) \leq Dr \qquad \text{for all} \quad z_0 \in \mathbf{C}, \ r > 0.$$

The chord-arc curves are precisely the regular quasicircles, but not all regular curves are such. A regular curve may well contain cusps or multiple points. The following example is intended to refute the conjecture that every regular curve is
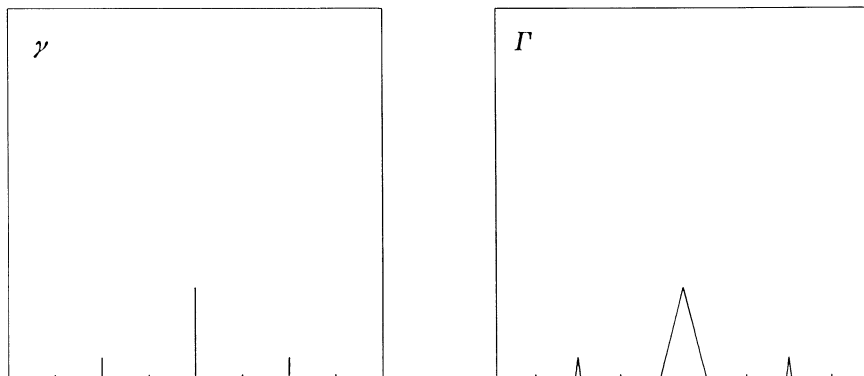
Figure 1.

at least contained in the union of finitely many quasicircles. Two variants of the example will be presented, the first being easier to handle, the second having the additional property of being a Jordan loop.

Instead of writing down a formal definition of the curves in question we prefer to describe them using Figure 1, and a few explanatory words.

2. The simpler example

The set γ consists of the boundary of the unit square and of vertical lines, one for each dyadic rational point on the base interval of our square. These should become shorter with increasing order of the corresponding dyadic rational. We choose height 2^{-2m} at $p/2m$, p odd, as indicated in Figure 1. There is a natural parametrization of γ as a closed rectifiable curve passing twice through every point on these lines and once through the remaining points on the boundary of the unit square. The interesting part of γ is the arc consisting of the base interval of the square and of the vertical lines, call it the *basic arc*. Any subarc of it between two distinct real points $(x,0)$, and $(x',0)$ contains a miniaturized and dilated copy of the basic arc. This is now used to demonstrate that γ, obviously regular, cannot be covered using finitely many chord-arc curves. Assume the contrary. Without loss of generality we may think of γ as being covered with as few as possible such curves γ_1, γ_2, ..., γ_n. As the same is true with respect to the miniaturized and dilated copies of the basic arc no subarc of γ between two distinct real points can be covered using less than n of the curves γ_1, γ_2, ..., γ_n, i.e. all of them. In particular every γ_k is dense on the unit interval of the real line and so by compactness each γ_k contains the unit interval. On the other hand the union of the curves γ_k does also cover the vertical lines used in the construction of γ. Therefore, by finiteness at least one of the curves γ_k contains a sequence of points outside the real line converging to $(1/2,0)$, say. But it is easy to see and will follow from the lemma in Section 3 that no quasicircle can contain both an interval and

a sequence of points outside that interval converging to any internal point of the interval.

3. The Jordan loop example

We start the construction with a Cantor set

$$F = [0,1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{k=1}^{2^m} E_{m,k},$$

where the open intervals $E_{m,k}$ deleted at the mth stage have length $l_m = l(E_{m,k}) = 2^{-3m-3}$. The curve Γ will now consist of the boundary of the unit square with the real unit interval replaced by the graph of the continuous function

$$f: [0,1] \to [0,1/4]: x \to \begin{cases} 2^{-2m} \operatorname{dist}(x,F)/\frac{1}{2}l_m, & \text{if } x \in E_{m,k}, \\ 0, & \text{if } x \in F. \end{cases}$$

Obviously Γ is regular. To prove that Γ cannot be covered with finitely many quasicircles we have to modify the proof given for the first version of the example, since now Γ does not contain any subinterval of the real line the real points of Γ forming a Cantor set.

Lemma. *If σ is a quasicircle in the plane $\mathbf{C}$ passing through an imaginary point bi, $b > 0$, and through some real points $a_1 < a_1 < \cdots < a_n$, then the three points constant C of σ is at least equal to $\min\{b/d, -a_0/2b, a_n/2b\}$, where d is the norm of the partition $a_0, \ldots, a_n$), i.e. $d = \max\{a_j - a_{j-1} \mid j = 1, \ldots, n\}$.*

Proof. By the three points property for every z on one of the two arcs between a_{j-1} and a_j:

$$|\operatorname{Im} z| \leq C|a_{j-1} - a_j| \leq Cd.$$

Hence either $b \leq Cd$—in which case we are done—or σ cannot pass through bi, not even leave the strip $|\operatorname{Im} z| \leq b$ on one of its arcs between any points a_{j-1} and a_j, $j = 1, \ldots, n$. The union of such arcs is an arc of σ joining a_0 and a_n without ever leaving the strip $|\operatorname{Im} z| \leq Cd$. We assume $a_0 < 0 < a_n$, since the alternative case is trivial. Thus our arc passes through some imaginary point ci with $|c| \leq Cd \leq b$. On the other hand the complementary arc of σ passes through bi. Now one of the arcs between bi and ci contains a_0 and the other a_n, so they have diameters at least $-a_0$ and a_n, respectively. But $|bi - ci| \leq b + |c| \leq 2b$, so

$$\min\{-a_0, a_n\} \leq 2Cb,$$

which completes the proof of the lemma.

Let us now continue our examination of the Jordan loop Γ. Assume that Γ is covered with minimally many chord-arc curves $\gamma_1, \gamma_2, \ldots, \gamma_n$. By minimality

of n and compactness of γ_k every γ_k contains the Cantor set F, in particular every γ_k contains the endpoints of all the deleted intervals $E_{m,j}$. Choose some $E_{m,j}$, call its midpoint $x_{m,j}$ and assume m is large enough, $m \geq 10$ say. The distance $\mathrm{dist}(x_{m,j}, E_{m',j'})$ for $m' \leq m$, $(m,j) \neq (m',j')$, will then be larger than $2^{-m}/5$. Thus the set $\left\{ x \in [0,1] \;\middle|\; x \notin F, \mathrm{dist}(x, x_{m,j} \leq 2^{-m}/10 \right\}$ does not contain any interval of length more than $l_m < 2^{-3m}$. At least one γ_k passes through the point $(x_{m,j}, 2^{-2m})$. Translating $x_{m,j}$ to the origin we may apply the lemma to γ_k choosing $b = 2^{-2m}$, $d \leq 2^{-3m}$ and both $-a_0$ and $a_m \geq 2^{-m}/20$. We find $C \geq 2^m/40$ which may be made arbitrarily large in contradiction to the finite number of possible choices of γ_k. $\square$

We finally point out that replacing successively all intervals on γ with copies of the basic arc will in the limit lead to a curve, no proper subarc of which can be embedded in the union of finitely many quasicircles.

4. Another property of the curves γ and Γ

In a preprint to appear ([L]) V. Lappalainen proves that the bounded domain D with boundary γ (or Γ) is a John domain which cannot be expressed as the union of finitely many quasiconvex domains. (In fact, it was through this work of Lappalainen's that the author's interest was directed to these curves.)

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