BMO AND TEICHMÜLLER SPACE

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1. Introduction

Let us consider the class $Q_0$ of quasiconformal maps $F$ of the Riemann sphere fixing $\infty$ and conformal at $\infty$ with normalization $f(z) \sim z$ at $\infty$. M. Reimann [18] proved that $\log \det(DF) \in \text{BMO}(\mathbb{R}^2)$. We show the connection of this with a fundamental result of Teichmüller theory. Let $M_0$ be the class of $L^\infty$ functions $\mu$ with compact support and $\|\mu\| < 1$. Ahlfors and Bers [3] show that the solution $F \in Q_0$ of the Beltrami equation

$$\partial F = \mu \partial F$$

depends holomorphically on $\mu$. One formulation of Teichmüller space $T$ (see Becker [4], Gehring [11]) on a fixed domain $\Omega$ containing $\infty$ is to set

$$T = \{ \log F'(z) : F \in Q_0 \text{ conformal on } \Omega \text{ with quasiconformal extension} \}.$$  

Then it is proved that, if $\Omega$ is a quasidisk, $T$ is an open set in the space $B(\Omega)$ of functions $g$ analytic on $\Omega$ with

$$\|g\|_B = \sup_{\Omega} |g'(z)| \text{ dist}(z, \partial \Omega) < \infty, \quad |zg(z)| \to 0 \quad \text{as} \quad |z| \to \infty.$$  

Now $B(\Omega)$ is a closed subspace of the Bloch space. Coifman, Rochberg and Weiss [8] prove that $B(\Omega)$ is the class of analytic functions BMO with respect to $\Omega$. Furthermore, by Jones [12], if $\Omega$ is a quasidisk every $g \in B(\Omega)$ has an extension to a function of BMO$(\mathbb{R}^2)$.

We remove the assumption that $\Omega$ is a quasidisk, and the restriction to considering $\log \partial F$ where $F$ is conformal. The formula $\log \partial F$ will be defined first for smooth $F$. Now we do not take the principle branch of $\arg \partial F$ but a continuous value so that $\log \partial F$ depends holomorphically on $\mu$ and is essentially unique.

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Theorem 1. Suppose for $\mu \in M_0$ we let

$$\Pi(\mu) = \log(\partial F'),$$

where $F = F^\mu \in Q_0$. Then the map $\Pi$ is a well defined holomorphic map of $M_0$ into $\text{BMO}(\mathbb{R}^2)$.

This has a number of implications.

Corollary 1. For any open set $\Omega$ with $\infty \in \mu$ and for any $F$ conformal on $\Omega$ with quasiconformal extension to $\mathbb{C}$ the function $\log F'(z)$ ($z \in \Omega$) has an extension to $\text{BMO}(\mathbb{R}^2)$.

We shall use the "$L^2$ norm" $\|\cdot\|_*$ for $\text{BMO}$, see Section 2.

Corollary 2. For any $\mu, \nu \in M_0$ with corresponding $F, G \in Q_0$

$$\|\log \partial F - \log \partial G\|_* \leq K \|\mu - \nu\|_\infty + O(\|\mu - \nu\|)^2$$

with $K = K(\|\nu\|)$. In particular at $G = z$

$$\|\log \partial F\|_* \leq 3 \|\mu\|_\infty + O(\|\mu\|_\infty)^2,$$

and $3$ is best possible.

Remark. Reimann [19] obtained an upper bound for $\log \det(DF)$ which tends to $0$ as $\|\mu\|_\infty \to 0$. The $3$ comes from Iwaniec’s bound for the Hilbert transform [13].

Let $\Omega$ be a domain containing $\infty$, and $M(\Omega)$ the subset $\mu \in M_0$ which are supported by $\mathbb{C} \setminus \Omega$. The map

$$\Pi: M(\Omega) \rightarrow \log(F'(z))$$

is a holomorphic map into $B(\mathbb{C})$, see Ahlfors and Bers [3].

Bers [5] and Gehring [11] ask for a characterization of domains with a univalence criteria, i.e., domains for which there exists $a = a(\Omega)$ so that for any $g \in B(\Omega)$ with $\|g\|_B < a$ then there is $F$ conformal on $\Omega$ with $\log F' = g$.

Now let $\mathcal{H}$ be the Hilbert transform

$$\mathcal{H}f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2} \, dm(\xi).$$

A classical result of Fefferman and Stein [9] implies for $\delta \in L^\infty(\Omega^C)$ that $g = \mathcal{H}\delta \in \text{BMO}(\mathbb{R}^2)$ and thus $g|_\Omega \in B(\Omega)$. Bers [5] essentially proves the converse, i.e.,

$$B(\Omega) = \mathcal{H}L^\infty(\Omega^C),$$
provided $\Omega$ is a quasicircle. We show that this integral representation for $B(\Omega)$ (which is stronger than BMO extension property for elements in $B(\Omega)$) holds if and only if $\Omega$ has a univalence criteria.

The equivalence occurs by the application of the $\lambda$-lemma of Sullivan, Thurston [22] and Bers, Royden [7] to the structure of the "pseudo" Teichmüller space

$$S = \{\log F': F \text{ conformal on } \Omega \text{ and } F = z + O(1) \text{ near } \infty\}.$$ 

A complex curve $L \subset S$ is a holomorphic map $\Lambda: \{\vert \lambda \vert < 1\} \to S$ with tangent $\tau$ at $\lambda = 0$ given by $\tau = (\partial \Lambda / \partial \lambda)_{|\lambda=0}$. The set of tangents at $g = \Lambda(0)$ is the tangent space $V_g$ at $g$. Clearly $V_g$ is a subset of $B(\Omega)$ and if $0 \in \text{Int} S$ then $V_0 = B(\Omega)$. We answer a question of Bers [6] who asked when $V_0 = B(\Omega)$. Shiga used [7], [22] to characterize $V_g$.

**Theorem 2.** The following are equivalent for an open set $\Omega$ with $\infty \in \Omega$:

(i) $\Omega$ has a univalence criteria;
(ii) the tangent space $V_0 = B(\Omega)$;
(iii) $B(\Omega)$ has integral representation, i.e.,

$$g(z) = \mathcal{H}\delta, \quad \delta \in L^\infty(\Omega^C).$$


2) It is interesting that the linear condition (iii) implies the nonlinear condition (i).

Finally we note that (iii) implies that (i) is actually a local condition.

**Corollary 3.** Let $\Omega$ be a domain so that $\Omega^C$ is the union of disjoint compact sets $A_1, A_2$. Then $\Omega$ has a univalence criteria if and only if $A_1^C, A_2^C$ do.

Instead of assuming $\mu$ has compact support we could just as well consider

$$M = \{\mu \in L^\infty(\hat{C}): \|\mu\|_\infty < 1\}$$

with normalization for solutions $F$,

$$F(0) = 0, \quad F(1) = 1, \quad F(\infty) = \infty.$$ 

It is not so clear how to define $\log(\partial F)$ even in the smooth case. Nevertheless we prove

**Theorem 3.** The map $\Pi: M \to \log(\partial F)$, for quasiconformal $F = F^\mu$ solving $\partial F = \mu \partial F$ and fixing $0$, $1$ and $\infty$, is a well defined holomorphic map of $M$ into $\text{BMO}(\mathbb{R}^2)$.
Finally we shall observe that this theory actually gives a new result for singular integrals.

**Corollary 4.** For any $\delta \in L^\infty$ (supported on a compact set) \[ \mathcal{H}(\delta \mathcal{H} \delta) - \frac{1}{2} (\mathcal{H} \delta)^2 \in \text{BMO}(\mathbb{R}^2). \]

**Remark.** In general $(\mathcal{H} \delta)^2$ or $\mathcal{H}(\delta \mathcal{H} \delta)$ do not belong to $\text{BMO}(\mathbb{R}^2)$.

This quadratic formulae in $\mu$ easily yields a commutator type result. For $\mu \in L^\infty$ define the operator $Q = \langle \mu, \mathcal{H} \rangle$ by
\[ Qf = \mathcal{H}(\mu \mathcal{H}f) + \mathcal{H}(f \mathcal{H} \mu) - (\mathcal{H} \mu) (\mathcal{H}f). \]

Corollary 4 proves that $Q$ is a bounded operator from $L^\infty$ to BMO. Coifman, Rochberg and Weiss [8] prove that the operator
\[ Wf = \mathcal{H}(f \mathcal{H} \mu) - (\mathcal{H} \mu) (\mathcal{H}f) \]
is a bounded operator of $L^p$, $1 < p < \infty$. Their result does not hold for $p = \infty$ so that the $Q$ operator is the correct generalization, note also that the first term of the $Q$ operator
\[ \mathcal{H}(\mu \mathcal{H}f) \]
is a bounded operator on $L^p$, $1 < p < \infty$, for $\mu \in L^\infty$.

2. Preliminary results

In this section we collect some tools necessary for the proof of the main theorem. We shall be using $\text{BMO}(\mathbb{R}^2)$ with “$L^2$ norm”
\[ \|f\|_* = \left( \sup_Q \int_Q \left| f - \int_Q f \, dx \, dy \right| \, dx \, dy \right)^{1/2}, \]
taken over all disks $Q$, where $\int_Q$ is the mean value. The basic theory of BMO is given by Fefferman and Stein [9], where it is proved that BMO is the dual of the Hardy space $H^1$. In particular, the unit ball of BMO is weakly compact and since for $1 < p < \infty$ the set
\[ E = \left\{ u \in L^p(\mathbb{R}^2) : \text{supp} \, u \text{ compact, } \int_{\mathbb{R}^2} u \, dm = 0 \right\} \]
is dense in $H^1$, for each sequence $\{f_n\} \subset \text{BMO}$ with $\|f_n\| \leq K$ \footnote{$K$ denotes a constant which may vary from line to line, any dependence on parameters is made explicit, e.g.: $K = K(m)$ with $m = \|\mu\|_\infty$.} there is a subsequence $\{f_n\}$ and an $f \in \text{BMO}$ so that for any $h \in E$
A map $\chi: M_0 \to \text{BMO}$ is holomorphic if for each $\mu_0$ and $\mu_1$ in $L^\infty$, with $\mu_0 + \lambda \mu_1 \in M_0$ for $|\lambda| < 1$, the map $\chi(\mu_0 + \lambda \mu_1)$ is holomorphic in $\lambda$. This is well known to be equivalent to the stronger condition that $\chi$ has continuous (Gateaux) differential.

Now Montel’s theorem does not hold for Banach spaces but there is the following weak form. This is based on the fact that a map $\chi: \{|\lambda| < 1\} \to \text{BMO}$ will be holomorphic if and only if $\int h(z)\chi(\lambda)\,dx\,dy$ is analytic for all $h \in E$. Thus given a sequence $\chi_n$ of holomorphic maps from $\{|\lambda| < 1\}$ into BMO so that $\|\chi_n\|_* \leq K$ then there is a limit $\chi$ so that $\chi_n \to \chi$ weakly, i.e., for any $h \in E$ and compact $U \subset \{|\lambda| < 1\}$, $\int h\chi_n(\lambda)\,dx\,dy \to \int h\chi(\lambda)\,dx\,dy$ uniformly for $\lambda \in U$.

The theory of quasiconformal solutions of

$$\bar{\partial}F = \mu\partial F, \quad \mu \in M_0,$$

may be seen in Ahlfors and Bers [3]. Thus $F(z) = z + o(1)$ near $\infty$ and we have the canonical solution $F = F^\mu$:

$$F(z) = z + C\mu + C\mu H\mu + \cdots,$$

where $C$ and $H$ denote the Cauchy and Hilbert transforms. It is important that $F - z$ depends holomorphically on $\mu$ (in the above simple minded way) as a map into the space of bounded continuous functions. Now let $DF$ be the differential of $F$, then $\mu \in C_0^\infty$ implies $DF$ and $DF^{-1}$ are both $C^\infty$, see [16], p. 233. M. Reimann considered the function

$$g(z) = \log \det |DF|$$

and proved $g \in \text{BMO}$ with norm only dependent on $\|\mu\|_\infty$. As a simple observation we have

**Lemma 1.** For canonical $F = F^\mu$, $\log |\partial F|$ is BMO and

$$\|\log |\partial F||_* \leq K(\|\mu\|_\infty).$$

We have only to write

$$\log \det |DF| = \log |\partial F|^2 (1 - |\mu|^2) = 2 \log |\partial F| + \log(1 - |\mu|)$$

and observe

$$\log(1 - |\mu|^2) \in L^\infty \subset \text{BMO}.$$
We now consider the definition of \( \log(\partial F) \). This is no problem when \( \mu \in C_0^\infty \cap M_0 \) and may be accomplished in two different ways. One way is to take \( \log(\partial F) = 0 \) at \( \infty \) and use a path \( \gamma \) from \( \infty \) to \( z \), along which \( \log(\partial F) \) is continuously defined. It is clear that this \( \log(\partial F) \) is uniquely defined independently of \( \gamma \). A second way is to consider a continuous deformation of \( \mu \) to 0,

\[
\mu(t) = t\mu, \quad 0 \leq t \leq 1,
\]

and an associated family \( F^t \)

\[
\partial F^t = t\mu \partial F^t
\]

of canonical maps. Beginning with \( F^0 = z \) and \( \log \partial F = 0 \) we may continuously define \( \log \partial F^t(z) \) at fixed \( z \). It is clear that this definition agrees with the first (by the usual homotopy argument) and in fact any \( C^\infty \) deformation of \( \mu \) into 0 may be used. Thus \( \log(\partial F) \) is uniquely defined for the \( C^\infty \) case.

Now for the general case, as \( F \) may be totally singular on a circle, the first method fails. It is necessary to show that the second method may be generalized and gives \( \log \partial F \) uniquely, whatever the representation. Thus we begin with the \( C^\infty \) case and establish \( BMO \) bounds. There are various difficulties caused by the fact the \( C_0^\infty \cap M_0 \) is not \( L^\infty \) dense in \( M_0 \), and a type of weak convergence for sequences of quasiconformal \( F_n \) is needed. The usual good convergence, see Lehto and Virtanen [15], p. 185, ensures \( \partial F_n \to \partial F \) (a.e.).

Assume \( ||\mu_n||_\infty, ||\mu||_\infty \leq k \leq 1 \) fixed. We assume that

(i) \( F_n \to F \) uniformly on compact sets.
(ii) \( \mu_n(z) \to \mu(z) \) (a.e.) on compact sets.

**Lemma 2** (Lehto, Virtanen [15], p. 216). For \( k < 1 \), there are \( p = p(k) > 2 \) so that the above conditions imply \( ||\partial F^n - \partial F||_p \to 0 \) on compact subsets of \( R^2 \).

3. **Proof of Theorem 1 (part 1)**

We have to obtain a well defined \( \log \partial F \) for \( F = F^\mu \) and \( \mu \in M_0 \). First we assume \( \mu \) smooth and obtain a uniform estimate.

**Lemma 3.** For canonical \( F = F^\mu, \mu \in C_0^\infty \cap M_0 \),

\[
||\log \partial F||_\ast \leq K(||\mu||_\infty).
\]

For fixed \( k, ||\mu||_\infty \leq k \leq 1 \) and \( F = F^\mu \), consider the following one parameter family \( F^\lambda \) defined for \( |\lambda| < 1/k \). Let \( F^\lambda \) be the canonical map satisfying

\[
\partial F^\lambda = \lambda \mu \partial F^\lambda.
\]

Observe by the Ahlfors–Bers theory that \((F^\lambda), (F^\lambda)^{-1}\) are \( C^\infty \) and the function

\[
f^\lambda \equiv \log \partial F^\lambda
\]
depends holomorphically on \( \lambda \). Thus we may write

\[
  f^\lambda = \lambda \alpha_1 + \lambda^2 \alpha_2 + \cdots
\]

where each \( \alpha_j(z) \in C^\infty \), and the series converges for \( |\lambda| < 1/k \).

Note that without loss of generality we may assume \( k \geq \frac{1}{2} \) (otherwise the bounds still apply for smaller dilatations). Choose a number \( \varrho \) depending continuously on \( k \) so that

\[
  1 < \varrho < \frac{1}{k}.
\]

Let \( \Gamma \) be the positively-oriented circle \( \{ |\lambda| = \varrho \} \). Now we apply Lemma 1 to \( F^\lambda \) for \( \lambda \in \Gamma \), noting that the dilatation \( \lambda \mu \), satisfies \( \|\lambda \mu\|_\infty = K(k) < 1 \). Thus for \( \lambda \in \Gamma \)

\[
  \|\text{Re} f^\lambda\|_\ast \leq K(k).
\]

However

\[
  \text{Re} f^\lambda = \frac{1}{2}(\lambda \alpha_1 + \overline{\lambda \alpha_1}) + \frac{1}{2}(\lambda^2 a_2 + \overline{\lambda^2 a_2}) + \cdots
\]

and therefore

\[
  \alpha_j = \frac{1}{2\pi i} \int_\Gamma (\text{Re} f^\lambda) \frac{d\lambda}{\lambda j + 1}
\]

and consequently

\[
  \|\alpha_j\|_\ast \leq \frac{K(k)}{\varrho^j}.
\]

Thus, evaluating \( f^1 = \log \partial F \)

\[
  \|\log \partial F\|_\ast = \|\alpha_1 + a_2 + \cdots\|_\ast \leq \frac{K(k)}{\varrho - 1} \leq K'(k).
\]

In order to define \( \log(\partial F) \) for general \( \mu \) we proceed as follows. For any fixed \( k > \|\mu\|_\infty \) choose a sequence \( \mu_n \in C_0^\infty \) so that the conditions (i), (ii) of Lemma 2 hold. Let \( F \) be the canonical solutions of

\[
  \partial F_n = \mu \partial F_n
\]

and

\[
  f_n = \log(\partial F_n).
\]

Now we apply Lemma 3 and the weak compactness of the unit ball of BMO. Thus there is an \( f \in \text{BMO} \) so that some subsequence \( f_{n_k} \to f \) weakly. However by Lemma 2

\[
  |\partial F_{n_k}| \to |\partial F| \quad \text{(a.e)}.
\]
In particular,

\[ \log |\partial F_{n_k}| \to \log |\partial F| \quad \text{(a.e.)} \]

Now Lemma 2 implies

\[ \int (\log |F_n|) u \, dx \, dy \to \int (\log |F|) u \, dx \, dy \]

for all \( u \in E \). Thus

\[ \Re f = \log |\partial F| \]

and we may define

\[ \Pi(\mu) = f. \]

Let us show \( f \) is well defined. Now suppose \( \nu_n \) is another sequence in \( C_0^\infty \cap M \) converging to \( \mu \) so that (i), (ii) hold. Let \( g \) be the corresponding weak limit. For the sequence \( \nu_n \) we associate an analytic family \( \lambda \mu_n \), \( |\lambda| < 1/k' \), \( k' > k \) and corresponding quasiconformal \( F^\lambda_n \)

\[ \partial F^\lambda_n = \lambda \mu_n \partial F^\lambda_n \]

with

\[ f^\lambda_n = \log(\partial F^\lambda_n) \]

which is a uniformly bounded holomorphic family in \( \lambda \). Thus (at least for a subsequence) there is a weak limit \( f^\lambda \), holomorphic in \( \lambda \). By the above analysis

\[ \Re f^\lambda = \log |\partial F^\lambda|, \quad |\lambda| < \frac{1}{k'}. \]

Similarly for \( \nu_n \) we define \( \lambda \nu_n \) and corresponding \( G^\lambda_n \)

\[ \partial G^\lambda_n = \lambda \nu_n \partial G^\lambda_n \]

with

\[ g^\lambda_n = \log(\partial G^\lambda_n). \]

Thus there is a weak limit \( g^\lambda \) holomorphic in \( |\lambda| < 1/k' \). But as \( \nu_n \to \mu \) a.e. \( \lambda \nu_n \to \lambda \mu \) a.e. weakly and \( |\partial G^\lambda_n| \to |\partial F^\lambda| \) (a.e.). Therefore

\[ \Re g^\lambda = \log |\partial F^\lambda|. \]

Now \( g^\lambda \), \( f^\lambda \) are holomorphic functions with the same real part and value (0) for \( \lambda = 0 \). Thus \( g^\lambda \equiv f^\lambda \) for \( |\lambda| < 1/k' \) and in particular (for \( \lambda = 1 \)) \( g \equiv f \).
4. Proof of Theorem 1 (part 2)

In this section we prove holomorphic dependence of \( \log F^{\nu(\lambda)} \) for \( \nu(\lambda) \) holomorphic in \( \lambda \). By the Ahlfors–Bers theory \( \log \partial F^{\nu(\lambda)} \) is holomorphic in \( \lambda \) if \( \nu(\lambda) \in C^\infty_0 \cap M_0 \).

For general holomorphic \( \Lambda : \{ |\lambda| < 1 \} \to M_0 \) there exist sequences of holomorphic mappings \( \Lambda_n \in C^\infty_0 \cap M_0 \) so that \( \Lambda_n(\lambda) \to \Lambda(\lambda) \) a.e., in fact for any \( r < \infty \)

\[
\| \Lambda_n(\lambda) - \Lambda(\lambda) \|_r \to 0, \quad n \to \infty,
\]

and for \( \lambda \) on a compact subset of the unit disk. To see this, take \( \varphi_n \in C^\infty_0 \) to be an approximation of the identity. Define

\[
\Lambda_n(\lambda) = \int \varphi_n(z - x - iy) \lambda(\lambda) \, dx \, dy.
\]

Now as \( \varphi_n > 0 \), \( \int \varphi_n \, dx \, dy = 1 \),

\[
\| \Lambda_n(\lambda) \| \leq k < 1.
\]

Observe that \( \Lambda_n \in M_0 \cap C^\infty \) and that by the standard estimate, for all \( r < \infty \),

\[
\| \Lambda_n(\lambda) - \Lambda(\lambda) \|_r \to 0
\]

for \( \lambda \) on any (fixed) compact set of the unit disk.

Consequently the corresponding sequences

\[
g_n^\lambda = \log(\partial G_n^\lambda)
\]

will converge weakly to a holomorphic function \( g^\lambda \). However, as before, this is the unique definition of \( \log(\partial G^\lambda) \) which is therefore holomorphic in \( \lambda \).

5. Proof of Theorem 2

As preliminary results we need the following form of the \( \lambda \)-lemma (see [7], [22]).

**Lemma 5.** Let \( L \) be a holomorphic curve in \( S \), i.e., \( L = \Lambda(\{ |\lambda| < 1 \}) \),

\[
\Lambda : \{ |\lambda| < 1 \} \to S
\]

is holomorphic with \( \Lambda(0) = 0 \). Then for \( |\lambda| < \frac{1}{3} \) there is a holomorphic map

\[
\Gamma : \{ |\lambda| < \frac{1}{3} \} \to M(\Omega)
\]

so that

\[
\log \partial F(\lambda) = \Lambda(\lambda).
\]

This is because if \( g = g^\lambda \in L \) depends holomorphically on \( L \) then so does

\[
F^\lambda = \int z \, e^{g^\lambda} \, dz
\]

which is univalent on \( \Omega \). By the \( \lambda \)-lemma \( F^\lambda \) extends to a quasiconformal mapping which, for \( |\lambda| < \frac{1}{3} \), has dilatation \( \mu^\lambda \) depending holomorphically on \( \lambda \).
Remark. The maximal complex manifold containing \( g = 0 \) is essentially the class of all quasiconformal deformations with \( \mu \in \mathcal{M}(\Omega) \).

Since \( \log \partial F^\mu = \log(1 + \mathcal{H}\mu + \cdots) \), \( D\Pi(0) = \mathcal{H} \) and the above lemma gives

**Lemma 6.** The tangent space at 0 is

\[ V = \mathcal{H}L^\infty(\Omega^C). \]

Remark. In particular there is a nontrivial complex manifold through 0 if and only if \( \text{Area}(\Omega^C) > 0 \).

Finally we prove the theorem. Our previous comments give (i) \( \rightarrow \) (ii) \( \rightarrow \) (iii). We use Theorem 1 to complete the chain of equivalences. Now as \( B(\Omega) = \mathcal{H}L^\infty(\Omega^C) \), \( \mathcal{H} \) is an open map of \( L^\infty(\Omega^C) \) onto \( B(\Omega) \). However as \( \Pi|_\mathcal{M}(\Omega) \) is holomorphic, by the implicit function theorem there is a \( b > 0 \) so that \( \Pi \) maps \( \{ \mu \in \mathcal{M}(\Omega) : \|\mu\|_\infty < b \} \) onto an open neighborhood of 0. Thus \( \Omega \) has univalence criteria.

### 6. Proof of corollaries

Observe that Corollaries 1, 2 are just special cases of Theorem 1. We next have to complete the proof of Corollary 3. For each \( g \in B(\Omega) \) we use the Cauchy integral to write

\[ g = g_1 + g_2 \]

where \( g_j \in B(A_j^C) \). Thus if \( B(A_j^C) \) has integral representations so does \( B(\Omega) \). So by Theorem 2 we only have to prove the converse. Suppose now that \( g \in B(A_1^C) \) say. However, \( g \in B(\Omega) \) and there is a \( t > 0 \) so that for \( |\lambda| < t \)

\[ \lambda g = \log F'_\lambda \]

for \( F_\lambda \) univalent (normalized) on \( \Omega \). But \( g \) is analytic on \( A \) so that \( F_\lambda \) has an analytic extension to \( A_1^C \). Also \( F_\lambda \) is 1 : 1 on a set of Jordan curves \( \gamma_k \) separating \( A_1 \) from \( A_2 \). Thus by the argument principle \( F_\lambda \) is univalent inside \( \gamma_k \). Consequently, \( B(A_1^C) \) has univalence criteria.

Finally we prove Corollary 4 by simply observing that the second derivative of \( \Pi \) with respect to \( \mu \) must be in BMO:

\[ \log(\partial F) = \log(1 + \mathcal{H}\mu + \mathcal{H}\mu\mathcal{H}\mu + \cdots) = \mathcal{H}\mu + (\mathcal{H}\mu\mathcal{H}\mu - \frac{1}{2}(\mathcal{H}\mu)^2) + \cdots. \]
7. Proof of Theorem 3

The method is almost exactly the same as the proof of Theorem 1, so we only point out the differences. The first problem is defining $\log(\partial F')$ for $F, F^{-1} \in C^\infty$. Now if $F$ has dilatation $\mu$, we use holomorphic family $F^\lambda$ defined by quasiconformal solutions of

$$\bar{\partial}F^\lambda = \lambda \mu \partial F^\lambda, \quad |\lambda| < \frac{1}{\|\mu\|_\infty}$$

fixing $0, 1, \infty$. Now $F$ as holomorphically dependent on $\lambda$ we obtain a holomorphic function

$$f^\lambda = \log(\partial F^\lambda)$$

which letting $\log(1) = 0$ defines $\log(\partial F')$. A homotopy argument shows the uniqueness of this definition. The rest of the proof proceeds analogously.

8. VMO and quasiconformal mapping

A function $f \in \text{VMO}$ if

$$\int_Q |f - \int f \, dx \, dy| \, dx \, dy \to 0$$

uniformly as area of all disks $Q$ satisfies $|Q| \to 0$. We consider the canonical map from the set of continuous dilatations to quasiconformal $F$ (fixing $0, 1, \infty$).

**Corollary 6.** The canonical map

$$\Pi(\mu) = \log(\partial F^\mu)$$

is a well defined holomorphic map from $M \cap C$ into $\text{VMO}_{\text{loc}}$. In particular for continuous $\mu$, $\log \det(DF^\mu) \in \text{VMO}_{\text{loc}}$.

The proof is a simple deduction from Theorem 1. The canonical map is the restriction of a holomorphic map to a convex submanifold. We have only to show $\log(\partial F) \in \text{VMO}_{\text{loc}}$. Now there exist $\mu_n \in M \cap C^\infty$ so that

$$\|\mu - \mu_n\|_\infty \to 0.$$ 

Thus, by Theorem 3, if $F = F^\mu$ and $F_n = F^{\mu_n}$

$$\|\log(\partial F_n) - \log(\partial F)\|_* \to 0.$$ 

Thus $\log(\partial F)$ is in the BMO closure of $C$ and is therefore $\text{VMO}_{\text{loc}}$.

**Remark.** One should note that for continuous $\mu \partial F$ is not necessarily even $L^\infty_{\text{loc}}$.
References


