Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 257–263

A SUFFICIENT CONDITION FOR THE WOLD–CRAMÉR CONCORDANCE OF BANACH-SPACE-VALUED STATIONARY PROCESSES

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Abstract. A recent result of Makagon and Salehi [7] is applied to obtain a sufficient condition for the concordance of the Wold decomposition and the spectral measure decomposition of Banach-space-valued stationary processes.

A sufficient condition for the Wold-Cramér concordance of Banach-space-valued stationary processes based on the isomorphism theorem (cf. [8, Theorem 3.3]) is presented in [4]. But for a given stationary process no algorithm for the determination of its isomorphism image is known. The concordance theorem for a stationary process with multiplicity one is given (cf. Proposition 7 below) in a recent paper of Makagon and Salehi [7]. If we connect the technics applied in [4] with the result of [7], we get the sufficient condition for the Wold-Cramér concordance, but for initial Banach-space-valued stationary process is formulated below.

In this paper, N, Z and K stand for positive integers, all integers and the unit circle of the complex plane, respectively. By $\mathcal{B}(K)$ we denote the family of Borel subsets of K and by m the normed Lebesgue measure on K. Let B be a complex Banach space with the dual space B^* and H be a complex Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. We denote by $L(B,H)$ the space of all continuous linear operators from B into H and by $\overline{L}^+(B,B^*)$ the space of all continuous antilinear and nonnegative operators from B into B^*.

By a second order stochastic process with values in B we mean a mapping $X\colon Z \to L(B,H)$. X is stationary if its correlation function $R(l,k) = X^*(k)X(l)$ depends only on $l-k$. In that case R has the spectral representation

$$X^*(k)X(l) = R(l-k) = \int_K z^{l-k} F(dz),$$

where F is a weakly countably additive measure on $\mathcal{B}(K)$ with values in $\overline{L}^+(B,B^*)$. F is called the spectral measure of the process X.

We denote $M(X) = \overline{\text{sp}}\{X(l)b\colon l \in Z, b \in B\}$,

$$M_k(X) = \overline{\text{sp}}\{X(l)b\colon l \le k, b \in B\}, \quad M_{-\infty}(X) = \bigcap_{k\in Z} M_k(X).$$

doi:10.5186/aasfm.1989.1408

We say that a stationary process X is separable if $M(X)$ is a separable subspace in H. X is regular if $M_{-\infty}(X) = \{0\}$ and it is singular if $M_{-\infty}(X) = M(X)$.

A second order process X is stationary if and only if X has the unitary shift operator, i.e, if there exists a unitary operator $U\colon M(X) \to M(X)$ such that

$$UX(k) = X(k+1), \qquad k \in Z.$$

If E is the spectral measure of the operator U, then E and F are related by the formula

$$(\alpha) \qquad F(\Delta) = X^*(0)E(\Delta)X(0), \qquad \Delta \in \mathcal{B}(K).$$

Now we recall the Wold decomposition theorem (cf. [2, Theorem 8.6]).

1. Proposition. *Let $X\colon Z \to L(B,H)$ be a stationary process with the shift operator U. Then there exist two processes X^r and X^s with the same shift operator U such that*

(i) $X(k) = X^r(k) + X^s(k)$, $k \in Z$,

(ii) $M(X^r)$ *and* $M(X^s)$ *are orthogonal,*

(iii) *for each* $k \in Z$, $M_k(X^r)$ *and* $M_k(X^s)$ *are contained in* $M_k(X)$,

(iv) X^r *is regular and* X^s *is singular.*

The above decomposition is unique. Namely, $X^s(k) = P_{-\infty}X(k)$ and $X^r(k) = X(k) - X^s(k)$, where $P_{-\infty}$ is the orthogonal projection onto $M_{-\infty}(X)$.

In particular, it follows that $M(X^s) = M_{-\infty}(X)$ and $M(X^r) = M(X) \ominus M_{-\infty}(X)$.

The following definition of multiplicity of a stationary process is given in [7].

2. Definition. Let $X\colon Z \to L(B,H)$ be a separable stationary process. The smallest number $n \in N \cup \{\infty\}$ such that there exists a sequence $\{x_i\}_{i=1}^n \subseteq M(X)$ with the property

$$M(X) = \overline{\mathrm{sp}}\{U^k x_i\colon 1 \le i < n+1,\ k \in Z\}$$

is called the multiplicity of the process X and denoted by $m(X)$. The inequality $i < n+1$ means $i \le n$ if $n < \infty$ and $i < \infty$ if $n = \infty$.

We will say that the spectral measure F of a stationary process X is absolutely continuous (or singular) with respect to a nonnegative scalar measure μ if $\big(F(\cdot)b\big)(b)$ is absolutely continuous (or singular) with respect to μ for all $b \in B$. We will use the notations $F << \mu$, $F \perp \mu$, respectively. Note that, for fixed $\Delta \in \mathcal{B}(K)$, $\big(F(\Delta)b\big)(b) = 0$ for each $b \in B$ if and only if $F(\Delta) = 0$. Hence $F << \mu$ is equivalent to the implication that if $\mu(\Delta) = 0$, then $F(\Delta) = 0$, $\Delta \in \mathcal{B}(K)$.

The following fact is proved in [1, Section 95] and [5, Section 66].

3. Proposition. *Let E be a spectral measure (on $\mathcal{B}(K)$) in a Hilbert space H and $H^a = \{h \in H\colon (E(\cdot)h, h) << m\}$, $H^s = \{h \in H\colon (E(\cdot)h, h) \perp m\}$. Then H^a and H^s are closed linear subspaces of H, $H = H^a \oplus H^s$, H^a and H^s reducing the spectral measure E.*

In the case where E is the spectral measure of the shift operator U of a stationary process X and $H = M(X)$, we shall denote $M^a(X) = H^a$, $M^s(X) = H^s$.

4. Lemma. *Suppose X, $Y\colon Z \to L(B,H)$ are stationary processes with the same shift operator U and $M(Y) \subseteq M(X)$. Denote by F_Y the spectral measure of the process Y. Then*

(i) *$F_Y << m$ if and only if, for each $b \in B$, $Y(0)b \in M^a(X)$ (equivalently $M(Y) \subseteq M^a(X)$);*

(ii) *$F_Y \perp m$ if and only if, for each $b \in B$, $Y(0)b \in M^s(X)$ (equivalently $M(Y) \subseteq M^s(X)$).*

Proof. In (α), $F_Y(\Delta) = Y^*(0)E(\Delta)Y(0)$; hence $\big(F_Y(\Delta)b\big)(b) = (E(\Delta)Y(0)b,\ Y(0)b)$. Thus, by Proposition 3, $\big(F_Y(\cdot)b\big)(b) << m$ if and only if $Y(0)b \in M^a(X)$. Furthermore, $Y(0)b \in M^a(X)$ for each $b \in B$ implies $M(Y) \subseteq M^a(X)$ because $M^a(X)$ reduces the shift operator U and $Y(k)b = U^kY(0)b \in M^a(X)$. We prove (ii) in the same way.

5. Lemma. *Let F be the spectral measure of a stationary process X. Then there exists a unique decomposition $F = F^a + F^s$, where F^a and F^s are measures with values in $\overline{L}^+(B, B^*)$ such that $F^a << m$, $F^s \perp m$.*

Proof. We denote $X^1(k) = P_aX(k)$, $X^2(k) = P_sX(k)$, where P_a, P_s are the orthogonal projections on $M^a(X)$, $M^s(X)$, respectively. Since $M^a(X)$, $M^s(X)$ reduce the shift operator U, both X^1 and X^2 are stationary processes with the same shift operator U. Let F^a be a spectral measure of X^1 and F^s the spectral measure of X^2. By Lemma 4, $F^a << m$, $F^s \perp m$. Moreover,

$$\begin{aligned}\big(F(\Delta)b\big)(b) &= \|E(\Delta)X(0)b\|^2 = \big\|E(\Delta)\big(X^1(0)b + X^2(0)b\big)\big\|^2 \\ &= \big\|E(\Delta)X^1(0)b\big\|^2 + \big\|E(\Delta)X^2(0)b\big\|^2 = \big(F^a(\Delta)b\big)(b) + \big(F^s(\Delta)b\big)(b)\end{aligned}$$

for each $b \in B$; thus $F = F^a + F^s$. Next, for each $b \in B$, $\big(F^a(\cdot)b\big)(b)$ is the absolutely continuous and $\big(F^s(\cdot)b\big)(b)$ the singular part of the measure $\big(F(\cdot)b\big)(b)$ with respect to m. The uniqueness of such decomposition implies the uniqueness of the decomposition of $F = F^a + F^s$.

By [6, Theorem 21.13], it follows that in $M(X)$ there exists a subset $\{x_\gamma\}_{\gamma\in\Gamma}$ (Γ is an index set) such that

$$M(X) = \bigoplus_{\gamma\in\Gamma} \overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\}, \tag{β}$$

where U is the shift operator of a stationary process X. Now fix such decomposition (β) and denote by P_γ the orthogonal projection on $\overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\}$. Let $X_\gamma(k) = P_\gamma X(k)$. We claim that $X_\gamma : Z \to L(B, \overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\})$ is a stationary process with the shift operator U. Indeed, $\overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\})$ reduces U, whence P_γ and U commute. Then

$$X_\gamma(k+1) = P_\gamma X(k+1) = P_\gamma U X(k) = U P_\gamma X(k) = U X_\gamma(k).$$

The following lemma gives a connection between the process X and the family of processes X_γ.

6. Lemma.
(i) $\overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\} = M(X_\gamma)$.
(ii) $\overline{P_\gamma M_k(X)} = M_k(X_\gamma)$.
(iii) $P_\gamma M_{-\infty}(X) \subseteq M_{-\infty}(X_\gamma)$.
(iv) *If X is singular, X_γ is singular for each $\gamma \in \Gamma$.*
(v) *If for each $\gamma \in \Gamma$ X_γ is regular, X is regular.*

Proof. (i) From the definition of P_γ it follows that $\overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\} = P_\gamma M(X)$. We prove that $P_\gamma M(X) = M(X_\gamma)$. In fact,

$$P_\gamma M(X) \subseteq \overline{\mathrm{sp}}\{P_\gamma X(l)b : l \in Z, b \in B\} = \overline{\mathrm{sp}}\{X_\gamma(l)b : l \in Z, b \in B\} = M(X_\gamma).$$

On the other hand, $X_\gamma(l)b = P_\gamma X(l)b \in P_\gamma M(X)$ for each $b \in B$ and $P_\gamma M(X)$ is a closed linear subspace. Hence $M(X_\gamma) \subseteq P_\gamma M(X)$.

(ii) As in (i) we show that $\overline{P_\gamma M_k(X)} = M_k(P_\gamma X) = M_k(X_\gamma)$.

(iii) $P_\gamma M_{-\infty}(X) = P_\gamma\left(\bigcap_{k\in Z} M_k(X)\right) \subseteq \left(\bigcap_{k\in Z} P_\gamma M_k(X)\right)$

$$\subseteq \bigcap_{k\in Z} \overline{P_\gamma M_k(X)} = \bigcap_{k\in Z} M_k(X_\gamma) = M_{-\infty}(X_\gamma).$$

(iv) By the assumption $M(X) = M_{-\infty}(X)$. From (iii)

$$M(X_\gamma) = P_\gamma M(X) = P_\gamma M_{-\infty}(X) \subseteq M_{-\infty}(X_\gamma).$$

Hence $M(X_\gamma) = M_{-\infty}(X_\gamma)$ and each process X_γ is singular.

(v) Since $M(X) = \bigoplus_{\gamma\in\Gamma} P_\gamma M(X)$,

$$M_{-\infty}(X) \subseteq \bigoplus_{\gamma\in\Gamma} P_\gamma M_{-\infty}(X) \subseteq \bigoplus_{\gamma\in\Gamma} M_{-\infty}(X_\gamma) = \{0\},$$

because every process X_γ is regular by assumption.

By Lemma 6(i) it follows that every process X_γ has multiplicity one.

Let now $X = X^r + X^s$ be the Wold decomposition of a process X as in Theorem 1. We denote by F_{X^r} the spectral measure of the process X^r and by F_{X^s} the spectral measure of X^s. Since $M(X^r)$ and $M(X^s)$ are orthogonal subspaces and reduce the shift operator U, we get $F = F_{X^r} + F_{X^s}$ as in the proof of Lemma 5.

The following fact is proved in [7, as Corollary 3.8].

7. Proposition. *If X is a nonsingular stationary process with multiplicity one, F^a is the spectral measure of X^r and F^s is the spectral measure of X^s.*

Now we state the main result of this paper.

8. Theorem. *Let $X\colon Z \to L(B,H)$ be a stationary process with the shift operator U and the spectral measure F. If there exists a decomposition*

$$M(X) = \bigoplus_{\gamma\in\Gamma} \overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\}$$

such that for each $\gamma \in \Gamma$ the orthogonal projection $P_\gamma X$ of the process X on $\overline{\mathrm{sp}}\{U^k x_\gamma : k \in Z\}$ is nonsingular, then $F_{X^r} = F^a$, $F_{X^s} = F^s$.

Proof. By virtue of Lemma 4 it suffices to show that $M(X^r) \subseteq M^a(X)$ and $M(X^s) \subseteq M^s(X)$. The first of these inclusions follows from Lemma 4 because the spectral measure of a regular process is absolutely continuous with respect to m (cf. [2, Theorem 10.2]). We prove that $X^s(0)b \in M^s(X)$ for each $b \in B$. In fact,

$$X^s(0)b = \sum_{\gamma\in\Gamma} P_\gamma X^s(0)b, \qquad b \in B$$

and $P_\gamma X^s(0)b \in P_\gamma M(X^s) = P_\gamma M_{-\infty}(X) \subseteq M_{-\infty}(X_\gamma) = M(X_\gamma^s)$, where the above inclusion follows from Lemma 6(iii). Since, for each $\gamma \in \Gamma$, X_γ has multiplicity one, Proposition 7 implies that the spectral measure of X_γ^s is singular with respect to m. Moreover, U is the shift operator of X_γ and X_γ^s and $M(X_\gamma^s) \subseteq M(X_\gamma) \subseteq M(X)$ for every $\gamma \in \Gamma$. Thus $X^s(0)b \in M^s(X)$ for each $b \in B$, which gives $M(X^s) \subseteq M^s(X)$.

9. Corollary. *If the assumptions of Theorem 8 hold and the spectral measure of X is absolutely continuous with respect to m, then X is regular.*

10. Remarks. (a) If X is a separable stationary process with multiplicity $m(X)$ there exists in $M(X)$ a sequence $\{x_i\}_{i=1}^{m(X)}$ realizing the decomposition (β) (cf. [3, p. 914–918] or [7, Lemma 2.2]). In that case the power of this decomposition is minimal. In a general case the power of the set Γ depends on the choice of a subset realizing (β).

(b) If a Hilbert space H is separable, Proposition 3 obtains a stronger form. Namely, for a spectral measure E in H, there exists in this case the unique decomposition $E = E^a + E^s$, where E^a is a spectral measure in a subspace $H^a \subseteq H$, E^s is a spectral measure in a subspace $H^s \subseteq H$, $H^a \oplus H^s = H$ and $E^a << m$, E^s is concentrated on a set of m-zero measure (then E^a and E^s are supported on disjoint Borel subsets of K). Indeed, if for some orthonormal base $\{e_i\}_{i=1}^{\infty}$ in H we define

$$\mu(\Delta) = \sum_{i=1}^{\infty} (E(\Delta)e_i, e_i) \cdot 2^{-i}, \qquad \Delta \in \mathcal{B}(K),$$

then μ is a nonnegative finite measure and $E << \mu$. Consider the Lebesgue decomposition of μ with respect to m: $\mu = \mu_a + \mu_s$. Let $\Delta_a \in \mathcal{B}(K)$ be a set such that $m(\Delta_a^C) = 0$ and μ_s is concentrated on Δ_a^C. If we put $E^a(\Delta) = E(\Delta_a \cap \Delta)$, $E^s(\Delta) = E(\Delta_a^C \cap \Delta)$, $H^a = E(\Delta_a)H$, $H^s = E(\Delta_a^C)H$, we get the desired decomposition. It is easy to verify that the subspaces H^a and H^s are the same as in Proposition 3.

(c) Suppose that X is a separable stationary process with the shift operator U and the spectral measure F. By (b) it follows that there exists the unique decomposition $F = F^a + F^s$ where $F^a << m$, F^s is concentrated on a set of m-zero measure. In fact, let E be the spectral measure of U. The measures $F^a(\Delta) = X^*(0)E^a(\Delta)X(0)$, $F^s(\Delta) = X^*(0)E^s(\Delta)X(0)$, $\Delta \in \mathcal{B}(K)$, satisfy the above condition.

11. Corollary. *Let $X\colon Z \to L(B,H)$ be a separable stationary process with multiplicity $m(X)$. Let F be the spectral measure of X and U its shift operator. Then*

(a) *There exists a unique decomposition $F = F^a + F^s$, where F^a and F^s are measures with values in $\overline{L}^+(B,B^*)$ such that $F^a << m$, F^s is concentrated on set of m-zero measure;*

(b) *if there exists in $M(X)$ a sequence $\{x_i\}_{i=1}^{m(X)}$ such that*

$$M(X) = \bigoplus_{i=1}^{m(X)} \overline{\mathrm{sp}}\{U^k x_i : k \in Z\}$$

and every process P_iX $(1 \le i < m(X)+1)$ is nonsingular, F^a is the spectral measure of X^r and F^s is the spectral measure of X^s. (P_i denotes the orthogonal projection on $\overline{\mathrm{sp}}\{U^k x_i : k \in Z\}$, $1 \le i < m(X)+1$.)

References

[1] ACHIEZER, N.I., and I.M. GLAZMAN: Theory of linear operators in Hilbert space. - Publications of Kharkov University, Kharkov, 1978 (Russian).

[2] CHOBANYAN, S.A., and A. WERON: Banach-space-valued stationary processes and their linear prediction. - Dissertationes Math. 125, 1975, 1–45.

[3] DUNFORD, N., and J.T. SCHWARTZ: Linear operators, Part II. - Spectral Theory, Interscience, New York, 1963.

[4] HAJDUK-CHMIELEWSKA, G.: The Wold-Cramér concordance problem for Banach-space-valued stationary processes. - Studia Math. 91, 1988, 31–43.

[5] HALMOS, P.: Introduction to Hilbert space and the theory of spectral multiplicity. - Chelsea Publishing Company, New York, 1951.

[6] HEWITT, E., and K.A. ROSS: Abstract harmonic analysis, Part I. - Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.

[7] MAKAGON, A., and H. SALEHI: Infinite dimensional stationary sequence with multiplicity one. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 135–150.

[8] WERON, A.: Second order stochastic processes and the dilatation theory in Banach space. - Ann. Inst. H. Poincaré 16, 1980, 29–38.

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Received 15 January 1988