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CHARACTERISTIC PROPERTIES OF THE NEVANLINNA CLASS N AND THE HARDY CLASSES H^p AND H_h^p

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For a measurable function $u(z) \geq 0$ defined in the unit disc D: $|z| < 1$, we introduce a characteristic function which in the case of a meromorphic function becomes its Nevanlinna characteristic function in the Ahlfors–Shimizu form. In terms of new characteristic functions we prove necessary and sufficient conditions for a function to belong to the Nevanlinna class N, to the Hardy classes H^p, $0 < p < +\infty$, and to the hyperbolic Hardy classes H_h^p, $0 < p < +\infty$.

1. For a measurable function $u(z) \geq 0$ defined in the unit disc D: $|z| < 1$ on the complex z-plane, we introduce the characteristic function $\mathbf{P}(r, u)$ in the form

$$\mathbf{P}(r,u) = \int_0^r \frac{S(t,u)}{t}\,dt, \qquad 0 < r < 1,$$

where

$$S(t,u) = \frac{1}{\bar{u}} \iint\limits_{|z|<t} \big(u(z)\big)^2\,dx\,dy, \qquad z = x + iy,\ 0 < t < 1,$$

and put $\mathbf{P}(1,u) = \lim_{r\to 1} \mathbf{P}(r,u)$.

If $f(z)$ is a meromorphic function in D and

$$f_p^{\#}(z) = \tfrac{1}{2}p\big|f'(z)\big|^{p/2-1}\big|f'(z)\big|\,\big(1 + \big|f(z)\big|^p\big)^{-1},\ 0 < p < +\infty,$$

then $\mathbf{P}(r, f_p^{\#}) = \frac{1}{2}pT_p(r,f)$, where $T_p(r,f)$ is the characteristic for the meromorphic function $f(z)$ introduced by S. Yamashita [4]; for $p = 2$ we get $T_2(r,f) = T(r,f)$, the Nevanlinna characteristic function of $f(z)$ in the Ahlfors–Shimizu form.

Lemma 1. *Let $S(r,u) < +\infty$ for any r, $0 < r < 1$. Then*

$$\mathbf{P}(r,u) = \frac{1}{\bar{u}} \iint\limits_{|z|<r} \big(u(z)\big)^2 \ln\frac{r}{|z|}\,dx\,dy, \qquad z = x + iy,$$

for any r, $0 < r \leq 1$.

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Proof. First suppose that $0<r<1$. Since $S(r,u)<+\infty$ for $0<r<1$, we get

$$S(r,u)=\int_0^r tS_1(t,u)\,dt,$$

where

(1) $$S_1(t,u)=\frac{1}{\bar u}\int_0^{2\bar u}\left(u(te^{i\theta})\right)^2 d\theta.$$

For any $f\in L(0,a)$, $a>0$,

(2) $$\int_0^a\left(\frac{1}{x}\int_0^x f(t)\,dt\right)dx=\int_0^a f(t)\ln\frac{a}{t}\,dt$$

(see, for instance, [8, p. 59]). Putting $f(t)=tS_1(t,u)$ in (2) and using (1), we get

$$\begin{aligned}\mathbf{P}(r,u)&=\int_0^r\frac{S(x,u)}{x}\,dx=\int_0^r tS_1(t,u)\ln\frac{r}{t}\,dt\\ &=\frac{1}{\bar u}\int_0^r t\ln\frac{r}{t}\left(\int_0^{2\bar u}\left(u(te^{i\theta})\right)^2 d\theta\right)dt=\frac{1}{\bar u}\int_0^r\int_0^{2\bar u}\left(u(te^{i\theta})\right)^2\ln\frac{r}{t}t\,dt\,d\theta\\ &=\frac{1}{\bar u}\iint\limits_{|z|<r}\left(u(z)\right)^2\ln\frac{r}{|z|}\,dx\,dy,\qquad z=te^{i\theta}=x+iy.\end{aligned}$$

In the case $r=1$, consider the characteristic function $\chi_r(z)$ of the disc D_r: $|z|<r<1$, i.e.,

$$\chi_r(z)=\begin{cases}1, & \text{if } |z|<r,\\ 0, & \text{if } r\le|z|<1.\end{cases}$$

Then

$$\mathbf{P}(r,u)=\frac{1}{\bar u}\iint\limits_{|z|<1}\left(u(z)\right)^2\ln\frac{r}{|z|}\,dx\,dy,\quad 0<r<1.$$

Since $0\le\chi_r(z)\ln(r/|z|)\uparrow\ln 1/|z|$ as $r\to 1-0$, the conclusion of Lemma 1 holds in the case $r=1$ by a well-known Fatou theorem.

Remark. In the case $u(z)=f^{\#}(z)=|f'(z)|\left(1+|f(z)|^2\right)^{-1}$ for a meromorphic function $f(z)$ in D, Lemma 1 is proved by S. Yamashita ([3, Lemma 2.2]).

2. For the Green potential

$$\omega(w)=\iint\limits_{|z|<1}\ln\left|\frac{1-\bar w z}{z-w}\right|dx\,dy,\quad z=x+iy,\quad w\in D,$$

we proved in [2] the following lemma.

Lemma 2. *If $\omega(w)$ is a Green potential in D, then*

$$\omega(w) = \tfrac{1}{2}\bar{u}(1-|w|^2), \qquad w \in D.$$

3. Let $\varphi_w(z) = (z+w)/(1+\bar{w}z)$, $w \in D$ is fixed, and $u_w(z) = u(\varphi_w(z))|\varphi'_w(z)|$, obviously, $u_0(z) = u(z)$.

For a point $\xi = e^{i\theta} \in \Gamma$: $|z| = 1$ and any δ, $0 < \delta < 1$, we consider two tangents drawn at the point $\xi = e^{i\theta}$ to the circle Γ_δ: $|z| = \delta$, and denote by $\Delta(\theta, \delta)$ the domain in D whose boundary consists of these two tangents and the largest subarc on Γ_δ. Suppose that for any δ, $0 < \delta < 1$, there exists a set $M(\delta)$ on Γ such that the linear measure $M(\delta) = 2\bar{u}$ and

$$A(\theta,\delta,u) = \iint\limits_{\Delta(\theta,\delta)} \big(u(z)\big)^2\,dx\,dy, \qquad z = x+iy,$$

is finite or infinite for each θ, $e^{i\theta} \in M(\delta)$ (cf. [1]).

Lemma 3. *Let $u(z) \geq 0$ be a measurable function in D. The following assertions are equivalent:*

(i) *For any fixed δ, $0 < \delta < 1$, the function $A(\theta,\delta,u)$ is a summable function of the argument θ on $[0, 2\bar{u}]$;*

(ii) $\iint\limits_{|z|<1} (1-|z|)\big(u(z)\big)^2\,dx\,dy < +\infty$, $z = x+iy$;

(iii) $\iint\limits_{|w|<1} \mathbf{P}(1,u_w)\,d\xi\,d\eta < +\infty$, $w = \xi + i\eta$.

Proof. The equivalence of (i) and (ii) is proved by V.I. Gavrilov ([1, Theorem 1, in which one must replace $u(z)$ with $\big(u(z)\big)^2$].

To prove the equivalence of (ii) and (iii), we note that

$$1-|z| = \frac{2}{\bar{u}(1+|z|)} \iint\limits_{|w|<1} \ln\left|\frac{1-w\bar{z}}{w-z}\right| d\xi\,d\eta, \quad w = \xi + i\eta,$$

holds by Lemma 2. Hence, (ii) holds if and only if

$$\iint\limits_{|z|<1} \frac{2\big(u(z)\big)^2}{\bar{u}(1+|z|)} \left(\iint\limits_{|w|<1} \ln\left|\frac{1-w\bar{z}}{w-z}\right| d\xi\,d\eta \right) dx\,dy < +\infty. \tag{3}$$

Changing the order of integration in (3), we see that (ii) holds if and only if

$$\frac{1}{\bar{u}} \iint\limits_{|w|<1} \left(\iint\limits_{|z|<1} \big(u(z)\big)^2 \ln\left|\frac{1-\bar{z}w}{w-z}\right| dx\,dy \right) d\xi\,d\eta < +\infty. \tag{4}$$

Since $\ln\left|(1-\bar{z}w)/(w-z)\right| = \ln\left|(1-\bar{w}z)/(z-w)\right|$ for any z, $w \in D$,

$$\mathbf{P}(1,u_w) = \frac{1}{\bar{u}} \iint\limits_{|z|<1} \big(u(z)\big)^2 \ln\left|\frac{1-\bar{w}z}{z-w}\right| dx\,dy,$$

(4) holds if and only if

$$\iint\limits_{|w|<1} \mathbf{P}(1,u_w)\,d\xi\,d\eta < +\infty.$$

4. For a meromorphic function $f(z)$ defined in D, we put

$$f_p^{\#}(z) = \tfrac{1}{2}p\left|f(z)\right|^{p/2-1}\left|f'(z)\right|\left(1+\left|f(z)\right|^p\right)^{-1}, \quad 0<p<+\infty.$$

Then $0 \le f_p^{\#}(z) \le +\infty$ and $f_p^{\#}(z) = +\infty$ at the zeros and the poles of $f(z)$.

Theorem 1. *For a meromorphic function $f(z)$ in D and for any p, $0<p<+\infty$, the following assertions are equivalent:*

(i) *For any fixed δ, $0<\delta<1$, the function $A(\theta,\delta,f_p^{\#})$ is a summable function of the argument θ on $[0,2\bar{u}]$;*

(ii) $\iint\limits_{|z|<1}(1-|z|)\big(f_p^{\#}(z)\big)^2\,dx\,dy < +\infty$, $z=x+iy$;

(iii) $\iint\limits_{|w|<1}\mathbf{P}\big(1,(f_p^{\#})_w\big)\,d\xi\,d\eta < +\infty$, $w=\xi+i\eta$;

(iv) $\mathbf{P}(1,f_p^{\#}) < +\infty$;

(v) *$f(z)$ is a function of bounded type; i.e., the Nevanlinna characteristic $T(r,f)$ is bounded as $r\to 1$.*

Proof. According to ([4, Lemma 1]), the function $f_p^{\#}(z)$, $0<p<+\infty$, is locally summable in D. Letting $u(z)=f_p^{\#}(z)$ in our Lemma 3, we obtain the equivalence of (i), (ii) and (iii). The equivalence of (ii), (iv) and (v) is proved by S. Yamashita ([4, Theorem 1]).

Remark. In the case $p=2$ the equivalence of (i), (ii), (iv) and (v) in Theorem 1 is proved by V.I. Gavrilov ([1, Theorem 2]).

5. If $f(z)$ is a holomorphic function in D, we put $f_p^*(z) = \frac{1}{2}p|f(z)|^{p/2-1}|f'(z)|$, $0<p<+\infty$. Then $0\le f_p^*(z)\le+\infty$ and $f_p^*(z)=+\infty$ at the zeros of $f(z)$. If $p=2$, then $f_p^*(z)=\left|f'(z)\right|$ (cf. [5]).

Theorem 2. *For a holomorphic function $f(z)$ in D and for any p, $0<p<+\infty$, the following assertions are equivalent:*

(i) *For any fixed δ, $0<\delta<1$, the function $A(\theta,\delta,f_p^*)$ is a summable function of the argument θ on $[0,2\bar{u}]$;*

(ii) $\int\limits_{|z|<1}\int (1-|z|)\big(f_p^*(z)\big)^2\,dx\,dy < +\infty$, $z = x+iy$;

(iii) $\int\limits_{|w|<1}\int \mathbf{P}\big(1,(f_p^*)_w\big)\,d\xi\,d\eta < +\infty$, $w = \xi + i\eta$;

(iv) $\mathbf{P}(1, f_p^*) < +\infty$;

(v) *$f(z)$ belongs to the Hardy class H^p.*

Proof. Since f_p^*, $0 < p < +\infty$, is a locally summable function in D (see [5]), putting $u(z) = f_p^*(z)$ in Lemma 3 we obtain the equivalence of (i), (ii) and (iii) in Theorem 2. The equivalence of (i), (ii), (iv) and (v) is proved by S. Yamashita ([5, Theorems 1 and 2]).

Remark. In the case $p = 2$ the equivalence of (i), (ii), (iv) and (v) is proved by V.I. Gavrilov ([1, Theorem 3]).

6. Let B denote the class of holomorphic functions $f(z)$ in D for which $|f(z)| < 1$ in D. For a function $f(z) \in B$, let $f^h(z)$ denote the hyperbolic derivative of $f(z)$, i.e., $f^h(z) = |f'(z)|\big(1-|f(z)|^2\big)^{-1}$. Consider $\lambda(f(z)) = \lambda(f) = -\ln\big(1-|f(z)|\big)$.

Following S. Yamashita [6], we say that a function $f(z) \in B$ belongs to the hyperbolic Hardy class H_h^p, $0 < p < +\infty$, if

$$\sup_{0<r<1} \frac{1}{2\bar{u}} \int_0^{2\bar{u}} \big(\sigma(f(z))\big)^p\,d\theta < +\infty, \qquad z = re^{i\theta},$$

where $\sigma(f(z)) = \frac{1}{2}\ln\big(1+|f(z)|\big)/\big(1-|f(z)|\big)$.

Theorem 3. *For any function $f(z) \in B$ and for any p, $0 < p < +\infty$, the following assertions are equivalent:*

(i) *For any fixed δ, $0 < \delta < 1$, the function $A(\theta,\delta,\lambda(f)^{(p-1)/2}f^h)$ is a summable function of the argument θ on $[0, 2\bar{u}]$;*

(ii) $\int\limits_{|z|<1}\int (1-|z|)\lambda\big(f(z)\big)^{p-1}\big(f^h(z)\big)^2\,dx\,dy < +\infty$, $z = x+iy$;

(iii) $\int\limits_{|w|<1}\int \mathbf{P}\left(1, \big(\lambda(f)^{(p-1)/2}f^h\big)_w\right)\,d\xi\,d\eta < +\infty$, $w = \xi+i\eta$;

(iv) $\mathbf{P}\big(1, \lambda(f)^{(p-1)/2}f^h\big) < +\infty$;

(v) $f(z) \in H_h^p$.

Proof. Since $\big(f(z)\big)^{(p-1)/2}f^h(z)$ is a locally summable function in D for any p, $0 < p < +\infty$, putting $u(z) = \lambda\big(f(z)\big)^{(p-1)/2}f^h(z)$, $0 < p < +\infty$, in Lemma 3, we obtain the equivalence of (i), (ii) and (iii) in Theorem 3. The equivalence of (i), (ii), (iv) and (v) in Theorem 3 is proved by S. Yamashita ([6, Theorems 1 and 4]).

Lemma 4. *Let* $\mathbf{P}(1,u) < +\infty$. *Then*

(5) $$\iint\limits_{|z|<r} \big(u(z)\big)^2 dx\,dy = o\left(\frac{1}{\ln r}\right), \qquad r \to 1,\ z = x + iy.$$

Proof. By Lemma 1 and the condition $\mathbf{P}(1,u) < +\infty$ we have

$$\mathbf{P}(1,u) = \lim_{r\to 1} \mathbf{P}(r,u) = \frac{1}{\bar{u}} \lim_{r\to 1} \iint\limits_{|z|<r} \big(u(z)\big)^2 \ln \frac{r}{|z|} dx\,dy$$

$$= \frac{1}{\bar{u}} \lim_{r\to 1} \ln r \iint\limits_{|z|<r} \big(u(z)\big)^2 dx\,dy + \frac{1}{\bar{u}} \lim_{r\to 1} \iint\limits_{|z|<r} \big(u(z)\big)^2 \ln \frac{1}{|z|} dx\,dy$$

$$= \frac{1}{\bar{u}} \lim_{r\to 1} \ln r \iint\limits_{|z|<r} \big(u(z)\big)^2 dx\,dy + \mathbf{P}(1,u),$$

from which the assertion of Lemma 4 follows.

Remark. Since $\ln r \sim (r-1)$ as $r \to 1$, the assertion (5) may be rewritten in the form

$$\lim_{r\to 1} (r-1) \iint\limits_{|z|<r} \big(u(z)\big)^2 dx\,dy = 0$$

when $\mathbf{P}(1,u) < +\infty$.

8. Conclusion. (i) Putting $u(z) = f_p^{\#}(z)$, $0 < p < +\infty$, in Lemma 4 we obtain the following result: If $f(z)$ is meromorphic in D and $T(r,f) = O(1)$, $r \to 1$, then

$$\lim_{r\to 1} (r-1) \iint\limits_{|z|<r} \big(f_p^{\#}(z)\big)^2 dx\,dy = 0, \qquad z = x + iy.$$

In the case $p = 2$ this result is mentioned in [7].

(ii) Putting $u(z) = f_p^*(z)$, $0 < p < +\infty$, in Lemma 4, we obtain a result of S. Yamashita ([7, Theorem 3]): If $f(z)$ belongs to the Hardy class H^p, $0 < p < +\infty$, then

$$\iint\limits_{|z|<r} \big(f_p^*(z)\big)^2 dx\,dy = o\left(\frac{1}{1-r}\right), \qquad r \to 1.$$

(iii) Putting $u(z) = \big(\lambda(f(z))\big)^{p-1} f^h(z)$ in Lemma 4, we obtain the following result: If $f(z)$ belongs to the hyperbolic Hardy class H_h^p, $0 < p < +\infty$, then

$$\iint\limits_{|z|<r} \big(\lambda(f)\big)^{p-1} \big(f^h(z)\big)^2 dx\,dy = o\left(\frac{1}{1-r}\right), \qquad r \to 1.$$

In the case $p = 1$ this result is mentioned in [7].

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