THE ESSENTIAL STATE DIAGRAM
OF A LINEAR OPERATOR

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Given an operator $T: D(T) \subset X \to Y$ where X and Y are normed spaces, we call T on F_+-operator [C2] if there exists a subspace E of finite codimension in X for which $(T/E)^{-1}$ exists and is continuous. We investigate properties of T related to the quantities $\alpha(T)$, $\beta(T)$, $\bar{\beta}(T)$ and the property $T \in F_+$ and construct a state diagram, called the *essential state diagram*, analogous to the Taylor–Halbert model [TH] (see also [G1], [G2] and [G3]); the latter will be referred to as the THG state diagram. A consequence is the following observation: T is an F_+-operator if and only if its adjoint T' is a φ_--operator (Corollary 1.17). Various other state diagrams have appeared in the literature. The reader may consult the survey monograph of V.M. Onieva [O] for further references.

Let X, Y, Z, ... denote normed linear spaces. The completion of X will be denoted by $\tilde{X}$. The class of linear transformations (henceforth called "operators") T defined on a linear subspace $D(T)$ of X with range contained in Y is denoted by $L(X, Y)$. The range and null space of T are denoted by $R(T)$ and $N(T)$ respectively. The restriction of T to a linear subspace M of X is denoted by T/M; note that $T/M = T/M \cap D(T)$. The operator T is called *bounded* if T is continuous and $D(T) = X$. T is called *closed* if its graph $\{(x, Tx): x \in D(T)\}$ is a closed subset of $X \times Y$. Let X_T be the space $D(T)$ normed by $\|x\|_T = \|x\| + \|Tx\|$. The graph operator G_T of T is the operator in $L(X_T, X)$ defined by $G_T x = x$ ($x \in X_T$). We write $G = G_T$. Clearly TG is a bounded operator in $L(X_T, Y)$. Let E be a linear subspace of X. Following Pietsch [P] we denote by J_E^X and Q_E^X respectively the natural injection of E into X and the natural quotient map of X onto X/E. The *adjoint* T' of T is defined by $T' = (TJ_{D(T)}^X)'$ where the righthand side is the conjugate defined in the usual sense [G3; 50]. Note that $T' \in L(Y', D(T)')$. We clearly have $(J_E^X)' = Q_{E^\perp}^{X'}$, and if E is closed then $(Q_E^X)' = J_{E^\perp}^{X'}$. It is evident that the state diagrams II.3.14 and II.4.11 of [G3] are valid in the general case. Given two linear subspaces M, N of X auch that $M \cap N = 0$ we write $M \oplus N$ for $M + N$. Let $\alpha(T) = \dim N(T)$, $\beta(T) = \mathrm{codim}\, R(T)$ and $\bar{\beta}(T) = \mathrm{codim}\, \overline{R(T)}$. T is called a φ_+-operator if $\alpha(T) < \infty$ and $R(T)$ is closed, and a φ_--operator if $R(T)$ is closed and $\beta(T) < \infty$. If T is closed and X and Y are Banach spaces then $T \in F_+$ if and only if $T \in \varphi_+$ (Proposition 1.5). The F_+-operators retain the usual properties associated with

the φ_+-operators of Gohberg–Krein [GohK] (see [C3]). Other properties of F_+-operators relate to stability of complemented ranges and generalised inverses [C4], and to the Tauberian property [C5].

We classify the operators T in $L(X,Y)$ as follows:

Ie: $R(T)$ is closed and $\beta(T) < \infty$;

IIe: $R(T)$ is not closed but $\bar{\beta}(T) < \infty$;

$IIIe$: $\bar{\beta}(T) = \infty$;

  $1e$: T is an F_+-operator;

  $2e$: $T \notin F_+$ but $\alpha(T) < \infty$;

  $3e$: $\alpha(T) = \infty$.

If $\beta(T) < \infty$ and $R(T)$ is closed we say that T is in the state Ie and write $T \in Ie$. If T is both in state $IIIe$ and state $2e$ then we write $T \in III_2e$. The other cases are treated similarly. It is clear that each of the sets $\{Ie, IIe, IIIe\}$ and $\{1e, 2e, 3e\}$ are partitions of $L(X,Y)$. There are thus nine disjoint classes $I_1, \ldots, III_3e$ whose union is $L(X,Y)$. The classes should be compared to the classes $I, II, III, 1, 2, 3$ in Goldberg [G3]. It is obvious that $I \subset Ie$, $II \subset IIe$, $III \supset IIIe$ and $1 \subset 1e$, $2 \subset 1e \cup 2e$ and $3 \supset 3e$.

A subspace M is called a *principal subspace* if M is closed and has finite codimension. In such a case T/M is called a *principal restriction* of T.

1.1. Lemma. *If T has no principal restriction having a continuous inverse then there exists an infinite dimensional subspace M of $D(T)$ for which T/M is precompact ([Ka], [G3]; see [G3; 80]).*

1.2. Lemma. *If $X = M \oplus N$ where M is a principal subspace then the projection of X onto M with null space N is bounded.*

1.3. Corollary. *If M is a principal subspace of X and if T/M is continuous, then T is continuous.*

1.4. Lemma. *Let $T \in F_+$. Then there exists a principal subspace M of X for which T/M has a continuous inverse.*

Proof. There exists a finite codimensional subspace E of X such that T/E has a continuous inverse. Suppose T has no principal restriction having a continuous inverse. Then by Lemma 1.1 there exists an infinite dimensional subspace N of $D(T)$ such that T/N is precompact. But then $E \cap N$ is infinite dimensional and $T/E \cap N$ is precompact with a continuous inverse, contradicting the non-precompactness of the unit ball in an infinite dimensional normed space.

1.5. Proposition. *Let T be closed. Then*
(a) *if X is complete, we have $T \in F_+ \implies T \in \varphi_+$,*
(b) *if X and Y are complete, we have $T \in \varphi_+ \implies T \in F_+$.*

Proof. (a) Let $T \in F_+$. Then by Lemma 1.4 there exists a principal subspace M and a finite dimensional subspace F of $D(T)$ such that $D(T) = M \oplus F \oplus N(T)$,

and such that $(T/M)^{-1}$ is continuous. Let $Tm_n \to y(M_n \in M)$. Then (m_n) is Cauchy, and so $m_n \to x$ (for an $x \in X$) since X is complete. Thus $(m_n, Tm_n) \to (x, y)$ in $X \times Y$. But T is closed; hence $x \in D(T)$ and $y = Tx$. Since M is closed in $D(T)$, we have $x \in M$. Therefore TM is closed and so $R(T) = TF + TM$ is closed (see e.g. [G3; 16]). Hence $T \in \varphi_+$.

(b) Let $T \in \varphi_+$. Then $D(T) = M \oplus N(T)$ where M is principal in $D(T)$ and TM is closed. By the closed graph theorem $(T/M)^{-1}$ is continuous. Therefore $T \in F_+$.

1.6. Corollary. *We have $T' \in F_+$ if and only if $T' \in \varphi_+$.*

1.7. Lemma. *Let M be a principal subspace of $D(T)$. Then $\left(J_M^X\right)' T' = \left(T J_M^X\right)'$.*

Proof. Write $J = J_M^X$. We clearly have $D(T') \subset D\left((TJ)'\right)$ and $D(J'T') = D(T')$ since J' is bounded. Let $y' \in D\left((TJ)'\right)$. Then $y'TJ$ is continuous on M, and hence on $D(T)$ since M is a principal subspace of $D(T)$ by Corollary 1.3. Hence $y' \in D(T')$ and the equality follows. □

1.8. Lemma. *Let $S \in L(X, Z)$ be a restriction of T where $R(T) \subset Z \subset Y$. Then $R(S') = R(T')$.*

Proof. We have $T = J_Z^Y S$, whence $T' = S' Q_{Z\perp}^{Y'}$ (since J_Z^Y is bounded). Since $D(S') \subset R\left(Q_{Z\perp}^{Y'}\right) = Y'/Z^\perp$ it follows immediately that $R(T') = R(S')$. □

The normed space X is called an *operator range* [C1] if it is the range of a bounded operator defined on a Banach space.

1.9. Lemma [1]. *If R_1 and R_2 are disjoint complementary operator ranges in a Banach space X then R_1 and R_2 are closed.*

1.10. Lemma. *If $T \in F_+$ then $T' \in \varphi_-$.*

Proof. Let $T \in F_+$. By Lemma 1.4 there exists a principal subspace E of $D(T)$ such that $T J_E^X \in 1$. Write $J = J_E^X$. Then $(TJ)' \in I$ by the THG state diagram [G3; 61]. Hence $J'T' \in I$ by Lemma 1.7. Now let $x' \in X'$ and choose $y' \in Y'$ such that $x' + E^\perp = T'y' + E^\perp$. Select $w \in E^\perp$ so that $x' = T'y' + w$. This shows that $X' = R(T') + E^\perp$. But $R(T')$ is an operator range (see e.g. [C1; 228]). Hence by Lemma 1.9, $R(T')$ is closed (alternatively, apply [G3; IV.1.12]). Therefore $T' \in \varphi_-$.

1.11. Proposition.
 (i) *If $T \in Ie$ we have $T' \in 1e \cup 2e$,*
 (ii) *If $T \in IIe$ we have $T' \in 1e \cup 2e$,*
 (iii) *If $T \in IIIe$ we have $T' \in 3e$,*
 (iv) *If $T \in 1e$ we have $T' \in Ie$,*
 (v) *If $T \in 2e$ we have $T' \in IIe \cup IIIe$,*
 (vi) *If $T \in 3e$ we have $T' \in IIIe$.*

Proof. If $\dim D(T) < \infty$ all the statements (i)–(vi) are elementary or trivial. We shall therefore suppose that $\dim D(T) = \infty$. The statements (i), (ii) and (iii) follow immediately from the equality $N(T') = R(T)^\perp$ [G3; II.3.7], while (iv) is a restatement of Lemma 1.10. To prove (v), assume $T \in 2e$. Then there exists an infinite dimensional subspace E of D(T) for which TJ_E^X is precompact by Lemma 1.1. Write $J = J_E^X$. Then $(TJ)'$ is compact [G3; III.1.11]. Since $(TJ)'$ is an extension of $J'T'$, it follows that $J'T'$ is compact. Suppose (if possible) that $T' \in Ie$. Let $X' = R(T') \oplus W$ where $\dim W < \infty$. Then

$$J'(X') = J'\big(R(T') \oplus W\big) \subset R(J'T') + J'W = J'(X')$$

since $J' \in I$. Thus $R(J'T') + J'W = J'(X') = X'/E^\perp$. But $R(T')$ is an operator range and so therefore is its continuous linear image $R(J'T')$. Consequently $R(J'T')$ is closed by Lemma 1.9. Since $J'T'$ is compact it follows that $R(J'T')$ is finite dimensional. Therefore $\dim X'/E^\perp < \infty$. Hence $\dim E = \dim E' = \dim X'/E^\perp < \infty$, a contradiction. Therefore $T' \notin Ie$ and (v) follows. Finally to prove (vi) we have $N(T) \subset {}^\perp R(T')$ whence $N(T)^\perp \supset \overline{R(T')}$. Hence if $T' \in Ie \cup IIe$ then $\operatorname{codim} N(T)^\perp < \infty$, which implies $\dim N(T) < \infty$ and (vi) follows. □

1.12. Proposition. *If $T \in I_1 e \cup II_1 e$ we have $T' \in I_1 e$.*

Proof. Let $T \in I_1 e \cup II_1 e$. Then from Proposition 1.11, $T' \in Ie \cap (1e \cup 2e)$. Thus $\alpha(T') < \infty$ and $R(T')$ is closed, i.e., T' is a closed φ_+-operator. Therefore $T' \in F_+$ by Corollary 1.6 and the result follows. □

1.13. Lemma. *Let $\overline{R(T)} \oplus M = Y$ where $\dim M < \infty$. Then $R\big((Q_M^Y T)'\big) = R(T')$.*

Proof. Write $Q = Q_M^Y$. Since $\dim M < \infty$ there exists a bounded projection P of Y onto $\overline{R(T)}$ with null space M by Lemma 1.2. Now let $y' \in D(T')$. Then

$$T'y' = \overline{y'T} = \overline{y'PT} = \overline{y'PQT} = (QT)'(y'P) \in R\big((QT)'\big).$$

Thus $R(T') \subset R\big((QT)'\big)$. On the other hand since Q is bounded, $(QT)' = T'Q'$. Hence $R\big((QT)'\big) \subset R(T')$. □

1.14. Lemma. *Let $S \in L(Y, Z)$ be a bounded operator. Then $ST \in F_+$ implies $T \in F_+$.*

Proof. Let $ST \in F_+$ and let M be a finite dimensional subspace of $D(T) = D(ST)$ for which ST/M has a continuous inverse. Then $\|S\|\, \|Tm\| \geq \|STm\| \geq c\, \|m\|$ for some $c > 0$ and all $m \in M$. Hence $T \in F_+$. □

1.15. Proposition.
(a) *If $T \in I_2 e \cup II_2 e$ we have $T' \notin II_1 e$.*
(b) *If Y is complete then $T \in I_2 e$ implies $T' \in III_1 e$.*

Proof. (a) Let $T \in I_2 e \cup II_2 e$ and suppose $T' \in II_1 e$. Since $T' \in 1e$, T' is a φ_+-operator by Corollary 1.6, and hence $R(T')$ is closed contradicting $T' \in IIe$. Therefore $T' \notin II_1 e$.

(b) Let Y be complete and let $T \in I_2 e$. Then there exists a finite dimensional subspace N of Y and a closed finite codimensional subspace E of $D(T)$ such that $R(T) \oplus M = Y$ and $E \oplus N(T) = D(T)$. Write $Q = Q_M^Y$ and $J = J_E^X$. It is clear that $\alpha(TJ) = 0$ and since J is an isomorphism it follows that $TJ \in 2$. Moreover $R(TJ) = R(T)$. Hence $QTJ \in I_2$. The state diagram [G3; 61] now gives $(QTJ)' \in III_1$ where by Lemma 1.7, $(QTJ)' = (TJ)'Q' = J'T'Q'$. Since $Q' = J_{M^\perp}^{Y'}$ we clearly have $R(J'T'Q') = R(J'T')$ by Lemma 1.13 and in particular $R(J'T')$ is closed. But since $TJ \in I_2 e$, Proposition 1.11 gives

$$J'T' \in II_1 e \cup III_1 e \cup II_2 e \cup III_2 e.$$

It follows that $J'T' \in III_1 e$. But $J'T' \in 1e$ implying that $T' \in 1e$ by Lemma 1.14, and the finite dimensionality of $E^\perp$ implies immediately that $T' \in IIIe$. Therefore $T' \in III_1 e$. □

1.16. Proposition. *Let Y be complete. Then $T \in I_3 e$ implies $T' \in III_1 e$.*

Proof. Let $T \in I_3 e$. Then $Y = R(T) \oplus M$ where $\dim M < \infty$. Write $Q = Q_M^Y$. Then $QT \in I \cap 3e \subset I_3$. Since Y is complete the *THG* state diagram [G3; 61] gives $T'Q' = (QT)' \in III_1$. But $Q' = J_{M^\perp}^{Y'}$ where $\operatorname{codim} M^\perp < \infty$. Therefore $T' \in 1e$. Hence by Lemma 1.11, $T' \in III_1 e$. □

The results of this section are summarized in the state diagram 2.7 of the next section. Note that the resulting configuration is the same as II.3.14 of [G3].

1.17. Corollary. *We have $T \in F_+$ if and only if $T' \in \varphi_-$.*

2. The essential state diagram for a closed operator

2.1. Lemma. *Let $T \in L(X, Y)$ be closed and let $Q \in L(X, Y)$ be a quotient map with finite dimensional null space. Then QT is closed.*

Proof. Let $(x_n, QTx_n) \to (x, y + N)$ where $N = N(Q)$. Then there exists a sequence $w_n \in N$ such that $Tx_n - y + w_n \to 0$. The finite dimensionality of N now implies that (w_n) is bounded: indeed if (w_n) is unbounded then for some subsequence w_n, we have $\|w_{n'}\| \to \infty$ and $(Tx_{n'} + w_{n'})/\|w_{n'}\| \to 0$. Since $(w_{n'}/\|w_{n'}\|)$ is bounded in the finite dimensional space N there exists a subsequence $(w_{n''})$ of $(w_{n'})$ and $k \in N$ such that $w_{n''}/\|w_{n''}\| \to k$. But then $Tx_{n''}/\|w_{n''}\| \to -k$ and $x_{n''}/\|w_{n''}\| \to 0$. Since T is closed, this implies that $k = 0$, contradicting $\|k\| = 1$. Therefore (w_n) is bounded. Passing to a subsequence if necessary we may suppose that w_n is convergent. Let $w = \lim w_n$. Then $Tx_n \to y - w$. Since T is closed we now have $x \in D(T)$ and $Tx = y - w$. Therefore $QTx = y$, showing that QT is closed. □

2.2. Theorem [2]. *Let $T \in F_+(X,Y)$ and $S \in F_+(Y,Z)$. Then $ST \in F_+$ whenever $ST \neq 0$.*

Proof. See [C2; 2.12 and 2.13]. □

2.3. Lemma. *Let T be closed and X complete and let $R(T')$ be closed. Then $R(T)$ is closed.*

Proof. Let X be complete and let $T_0 \in L\big(X, \overline{R(T)}\big)$ be an astriction of T. Then T_0 is closed, and $R(T_0') = R(T')$ by Lemma 1.8. Thus T_0' has a continuous inverse by the closed graph theorem. From the THG state diagram for closed operators [G3; 66] we see that $T_0 \in I_1 \cup I_3$. Thus T_0 is surjective, i.e. $R(T) = R(T_0) = \overline{R(T)}$. □

2.4. Corollary. *Let T be closed and let X be complete. Then $T \in IIe$ implies $T' \notin 1e$.*

Proof. Let $T \in IIe$. Then $R(T)$ is not closed and hence $R(T')$ is not closed by Lemma 2.3. Consequently $T' \notin 1e$. □

2.5. Proposition. *Let T be closed and let X be complete. Then $T \in I_2e$ implies $T' \notin 1e$.*
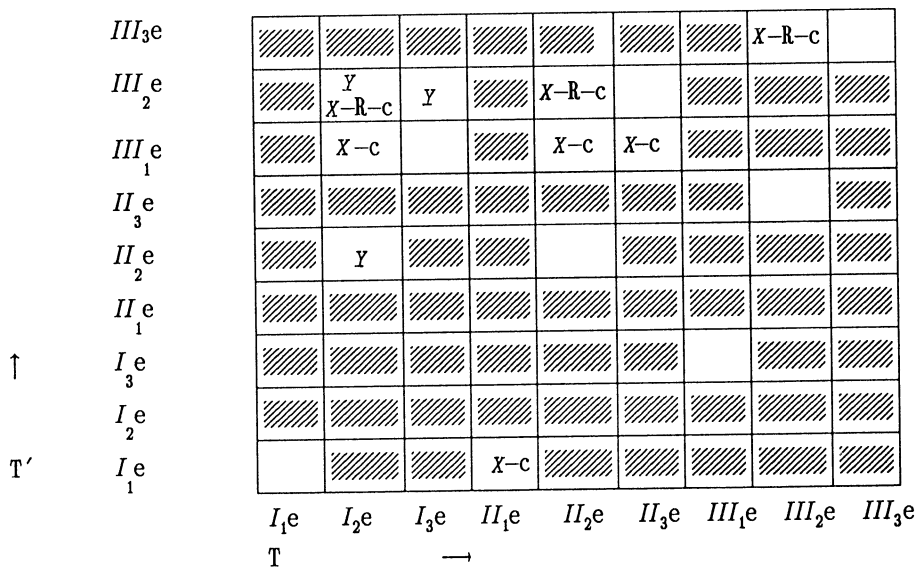
Proof. Let $T \in I_2e$ and let the operators Q and J be as in the proof of Proposition 1.15. Then $QTJ \in I_2$ and $QTJ \in L(E, Y/M)$ where E is complete. Moreover QTJ is closed by Lemma 2.1. By the THG state diagram for closed operators [G3; 66], $J'T'Q' = (QTJ)' \in II_2 \cup III_2$. But $R(J'T'Q') = R(J'T')$ by Lemma 1.13 and therefore $R(J'T')$ is not closed; in particular, $J'T' \notin 1e$ which implies by Theorem 2.2 that $T' \notin 1e$ (since $J' \in 1e$ and $J'T' = (TJ)' \neq 0$). □

2.6. Proposition. *Let T be closed and let X be reflexive. Then $T \in 2e$ implies $T' \notin IIIe$.*

Proof. Let $T \in 2e$. There exists a closed finite codimensional subspace E such that $E \oplus N(T) = D(T)$. Write $J = J_E^X$. Then $TJ \in 2$ and hence from the THG state diagram for closed operators and Lemma 1.7, $J'T' \in I \cup II$. It follows immediately that $\bar{\beta}(T') < \infty$, i.e. $T' \notin IIIe$. □

The results of this section are summarized in the diagram below. This configuration is identical to that of II.4.11 of [G3].

2.7. Essential state diagram for closed operators.

	I_1e	I_2e	I_3e	II_1e	II_2e	II_3e	III_1e	III_2e	III_3e
III_3e	▨	▨	▨	▨	▨	▨	▨	X–R–c	
III_2e	▨	Y / X–R–c	Y	▨	X–R–c		▨	▨	▨
III_1e	▨	X–c		▨	X–c	X–c	▨	▨	▨
II_3e	▨	▨	▨	▨	▨	▨	▨		▨
II_2e	▨	Y	▨	▨		▨	▨	▨	▨
II_1e	▨	▨	▨	▨	▨	▨	▨	▨	▨
I_3e	▨	▨	▨	▨	▨	▨		▨	▨
I_2e	▨	▨	▨	▨	▨	▨	▨	▨	▨
I_1e		▨	▨	X–c	▨	▨	▨	▨	▨

$T' \uparrow$ $T \longrightarrow$

Y: cannot occur if Y is complete,

$X - c$: cannot occur if X is complete and T is closed,

$X - R - c$: cannot occur if X is reflexive and T is closed.

3. Completeness of the essential state diagram

In this section we give examples of essential states which can occur and thus show that the blank squares appearing in the diagram 2.7 all eventuate. We do this by suitably modifying the examples in Section II.5 of [G3]. To simplify the notation we shall for example write $(T, T') \in (II_3, III_1)e$ if $T \in II_3e$ and $T' \in III_1e$. Note that in all the examples of this section T is a bounded operator.

3.1. *T bounded, with* $X = Y = l_2$. $(I_1, I_1)e$: Let T be the identity operator on X. $(I_3, III_1)e$: Let T be defined by $T((x_k)) = (x_{2k})$. Then $T \in I_3e$ and $T' \in III_1e$ by 1.17. $(III_1, I_3)e$: Define T by $T((x_k)) = (0, x_1, 0, x_2, \ldots)$. Then $T \in III_1e$ and $T' \in I_3e$ by 1.17. (II_2, II_2): The same example as for (II_2, II_2) in [G3], i.e., $T((x_k)) = (x_k/k)$, using 2.7. $(II_3, III_2)e$: The operator $T: (x_k) \to (x_{2k}/k)$ is compact and has dense range. Therefore $T \in II_3$, and then $T' \in III_2$ by 2.7. $(III_2, II_3)e$: Let T be the adjoint of the operator in the previous example. Then $T' \in II_3e$ by 2.7. $(III_3, III_3)e$: Let T be the zero operator. A nontrivial example similar to the corresponding one in [G3] is also easily constructed.

3.2. *T compact, with* $X = l_2$ *and* Y *not complete.* $(I_2, II_2)e$: The same examples as that given for (I_2, II_2) in [G3] serves. $(I_3, III_2)e$: Let T be defined by $T((x_k)) = (x_{2k}/k)$ and let $Y = R(T) \subset l_2$. Then $T \in I_3$, and since T' is compact it is clear that $T' \notin 1e$. Hence $T' \in III_2e$ by 1.17.

3.3. T bounded, with X not complete and $Y = l_2$. $(I_2, III_1)e$: Let $\{x_\alpha\}$ be a Hamel basis for l_2 and let X be the linear space l_2 renormed by $\|\sum \lambda_\alpha x_\alpha\| = \sum |\lambda_\alpha|$. Let T be the identity map from X onto l_2. Then $T \in Ie$ and is bounded. Suppose $T \in F_+$. Let M be a principal subspace of X for which $T/M \in 1$. Then $\| \|$ and the Hilbert norm $\| \|_2$ are equivalent on M and hence also on $l_2 = M \oplus M^\perp$ by the fact that any two norms are equivalent on the finite dimensional space $M^\perp$, which is clearly impossible. Therefore $T \notin F_+$. Consequently $T \in I_2e$ and the essential state diagram for closed operators 2.7 now gives $T' \in III_1e$.

$(II_1, I_1)e$: see [G3, 68]. $(II_2, III_1)e$: Similar to the example for (II_2, III_1) in [G3; 68]. $(II_3, III_1)e$: Similar to the example for (II_3, III_1) in [G3] using the operator $(x_k) \to (x_k) \to (x_{2k})$ in place of the left shift operator.

3.4. T compact, X complete but not reflexive, $Y = l_2$. $(II_2, III_2)e$: The same example in [G3] for (II_2, III_2) serves. $(III_2, III_3)e$: Similar to the example for (III_2, III_3) in [G3] but using the operator $(x_n) \to (0, x_1, 0, x_2, \ldots)$ (on l_1) in place of the right shift operator.

3.5. T compact, X complete but not reflexive, Y not complete. $(I_2, III_2)e$: The same example as for (I_2, III_2) of [G3].

A construction of S. Goldberg [G2] reducees the THG diagram for closed operators to the bounded case. A similar reduction using [G2] carries through for the "essential" classification. Goldberg's construction has recently been extended to arbitrary operators by L.E. Labuschagne [L].

4. Uniqueness character of the essential state diagram

The essential state diagram was achieved by partitioning the class $\{T: \alpha(T) < \infty\}$ into two subclasses, namely the F_+ and non F_+-operators. We do not claim that our classification is the only one resulting in the same configuration as the THG model. However, the following alternative classification seems worthy of comment.

1a: The class of operators T for which $(T/M)^{-1}$ exists and is continuous on some principle subspace M complementary to $N(T)$.

2a: The complement of 1a in the class of $\{T: \alpha(T) < \infty\}$.

The states we now consider are Ie, IIe, $IIIe$, $1a$, $2a$, $3a$. The example below shows that this classification results in a different configuration to that of the THG model.

4.1. Example. There exists $T \in Ie \cap 2a$ such that $T' \in Ie \cap 1a$.

Let Y be an infinite dimensional Banach space, let f be a discontinuous linear functional on Y and let $T: Y_f \to Y$ be the graph operator associated with f. Set $X = Y_f$. Then T^{-1} is discontinuous. However $T^{-1}/N(f)$ is an isometry and hence $T \in F_+$. Thus T is an injective and surjective F_+-operator, and $T \in I \cap 2a$. The essential state diagram 2.7 now gives $T' \in Ie$ while THG gives $T' \in 1$. Therefore $T' \in Ie \cap 1a$.

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