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# LOCAL REGULARITY OF SOLUTIONS TO TIME-DEPENDENT SCHRÖDINGER EQUATIONS WITH SMOOTH POTENTIALS

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**Abstract.** Consider solutions to the Schrödinger equation $i\partial u/\partial t = -Pu + Vu$ in a half-space $\{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+\}$ with given boundary values $u = f$ on $\mathbf{R}^n$. Here $P$ is an elliptic constant-coefficient operator in $x$, and $V = V(x)$ is a suitable potential. We prove several results about local regularity and boundary behaviour which were known in the case $V = 0$. In particular, if $f$ belongs to a Sobolev space, then $u$ is locally in a mixed Sobolev space. Moreover, $u$ converges to its boundary values along quasi-every vertical ray, and the corresponding maximal function can be estimated.

## 1. Introduction

We define the Fourier transform in $\mathbf{R}^n$ by setting

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot x} f(x)\, dx$$

and also introduce Sobolev spaces $H_s = H_s(\mathbf{R}^n)$, $s \in \mathbf{R}$, by defining the norm

$$\|f\|_{H_s} = \Big( \int_{\mathbf{R}^n} \big(1 + |\xi|^2\big)^s \big|\hat{f}(\xi)\big|^2 d\xi \Big)^{1/2}.$$

Let $p$ be a polynomial in $\mathbf{R}^n$ which is real and elliptic, i.e. its principal part does not vanish in $\mathbf{R}^n \setminus \{0\}$. We assume that the degree $m$ of $p$ is at least 2. Then set $P = p(D)$, where $D = (D_1, \dots, D_n)$ and $D_k = -i\partial/\partial x_k$.

Let $V$ be a real-valued function in $C^\infty(\mathbf{R}^n)$ with $D^\alpha V$ bounded for every $\alpha$. We define an operator $H = -P + V$ by setting $Hf = -Pf + Vf$ for $f \in H_m$.

Then $H$ is a self-adjoint operator on $L^2(\mathbf{R}^n)$ and $e^{-itH}$ is a unitary operator for $t \in \mathbf{R}$. We set $u(\cdot,t) = e^{-itH} f$, $f \in L^2(\mathbf{R}^n)$. Then $u$ is a measurable function in $\mathbf{R}^{n+1}$, and it is well known that $u$ satifies the Schrödinger equation $i\partial u/\partial t = Hu$ for $f \in H_m$, where the derivative is taken in the $L^2$ sense. Taking the derivatives in the distribution sense, one also has $i\partial u/\partial t = -Pu + Vu$ for $f \in L^2$.

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We shall here study the regularity of $u$ when the initial value $f$ belongs to Sobolev spaces $H_s$. Also the pointwise convergence of $u(\cdot,t)$ to $f$ as $t \to 0$ will be discussed.

To formulate the results, we introduce mixed Sobolev spaces $H_{\varrho,r}$ for $\varrho \geq 0$, $r \geq 0$. We set $H_{\varrho,r} = H_{\varrho,r}(\mathbf{R}^n \times \mathbf{R}) = (G_\varrho \otimes G_r) * L^2(\mathbf{R}^{n+1})$, where $G_\varrho$ and $G_r$ are Bessel kernels in $\mathbf{R}^n$ and $\mathbf{R}$, respectively. The Bessel kernel is given by the formula $\hat{G}_s(\xi) = \left(1 + |\xi|^2\right)^{-s/2}$, and the norm in $H_{\varrho,r}$ is the obvious one.

We introduce the class

$$\mathcal{A} = \Big\{\varphi \in C^\infty(\mathbf{R}^n);\ \text{there exists } \varepsilon > 0 \text{ such that } \left|D^\alpha\varphi(x)\right| \leq C_\alpha\left(1 + |x|\right)^{-1/2-\varepsilon} \text{ for every } \alpha\Big\}$$

and set

$$Sf(x,t) = \varphi(x)\psi(t)u(x,t),$$

where we assume that $\varphi \in \mathcal{A}$ and $\psi \in C_0^\infty(\mathbf{R})$. Let $\mathcal{S}$ denote the Schwartz class.

We then have the following result.

**Theorem 1.** *If $\varrho \geq 0$, $r \geq 0$, then*

$$\|Sf\|_{H_{\varrho,r}} \leq C\,\|f\|_{H_{\varrho+mr-(m-1)/2}}, \qquad f \in \mathcal{S}. \tag{1}$$

*Here the constant $C$ depends on $\varphi$ and $\psi$.*

Theorem 1 expresses a local smoothing property for the Schrödinger equation. For instance, taking $\varrho = r = 0$ in (1) we obtain

$$\|Sf\|_{L^2(\mathbf{R}^{n+1})} \leq C\,\|f\|_{H_{-(m-1)/2}}, \tag{2}$$

and $\varrho = (m-1)/2$, $r = 0$ yields

$$\|Sf\|_{H_{(m-1)/2,0}} \leq C\,\|f\|_{L^2(\mathbf{R}^n)}. \tag{3}$$

From (3) it then follows that $\varphi u(\cdot,t) \in H_{(m-1)/2}(\mathbf{R}^n)$ for almost every $t$ if $f \in L^2(\mathbf{R}^n)$.

Setting

$$u^*(x) = \operatorname{ess\,sup}_{0<t<1}\left|u(x,t)\right|, \qquad x \in \mathbf{R}^n, \quad f \in \mathcal{S},$$

we have the following maximal inequality:

**Theorem 2.** *If $s > \frac{1}{2}$, then for any ball $B \subset \mathbf{R}^n$*

$$\Big(\int_B u^*(x)^2\,dx\Big)^{1/2} \le C_B \|f\|_{H_s}, \qquad f \in \mathcal{S}.$$

To formulate our next result, we shall introduce capacities $C_s$ for $s > 0$. We set

$$C_s(E) = \inf\big\{\, \|g\|_2^2 \,; 0 \le g \in L^2(\mathbf{R}^n), G_s * g \ge 1 \text{ on } E\big\}, \quad E \subset \mathbf{R}^n.$$

By $C_s$-q.e. we mean everywhere except on a set of $C_s$-capacity $0$, and similarly for $C_s$-q.a.

A function $f \in H_s$ can be written as a convolution $f = G_s * g$ with $g \in L^2$. At $C_s$-q.a. points $x$, this convolution is well-defined in the sense that $G_s * |g|(x) < \infty$. These well-defined values of $f$ can be recovered if one knows $f$ almost everywhere. In fact, the means of $f$ in small balls centered at $x$ converge to $G_s * g(x)$ if $G_s * |g|(x) < \infty$.

We shall now describe how to make the solution $u$ to the Schrödinger equation precise by defining it at sufficiently many points. Let $f \in L^2(\mathbf{R}^n)$. The function $u$ is measurable and defined a.e. in $\mathbf{R}^{n+1}$. Let $B_{x,t}(\delta)$ be the ball in $\mathbf{R}^{n+1}$ with center $(x,t)$ and radius $\delta$ and let $B'_{x,t}(\delta) = \big\{(x',t); |x'-x| < \delta\big\}$ be a horizontal disc. We define the value $u(x,t)$ as the limit as $\delta \to 0$ of the mean value of $u$ over either $B_{x,t}(\delta)$ or $B'_{x,t}(\delta)$, at all points $(x,t)$ where this limit exists. We speak of the ball and the disc method.

**Theorem 3.** *Let $s > \frac{1}{2}$ and take $f \in H_s$. Define $u$ as above and make $u$ precise by the ball or the disc method. If $0 < \varrho < s - \frac{1}{2}$, then the following holds for $C_\varrho$-q.a. $x$: The function $u$ is defined at every point of the vertical line $\{x\} \times \mathbf{R}$, its restriction to this line is continuous, and its value at $(x,0)$ is $f(x)$.*

We shall also prove that the local smoothing inequality (2) is best possible in the following sense.

**Theorem 4.** *Assume $P = \Delta$ and $V = 0$. Define $Sf$ as above and assume that*

$$\|Sf\|_{L^2(\mathbf{R}^{n+1})} \le C\,\|f\|_{H_s}, \qquad f \in \mathcal{S}, \tag{4}$$

*for all $\varphi \in C_0^\infty(\mathbf{R}^n)$ and $\psi \in C_0^\infty(\mathbf{R})$. Then $s \ge -\frac{1}{2}$.*

In the cases $n = 1$ and $n = 2$, Theorem 2 can be improved in the following way.

**Theorem 5.** *Assume $n = 1$ or $n = 2$. Let $P = \Delta^k$, where $k = 1,2,3,\ldots$, and define $u^*$ as in Theorem 2. Then for any ball $B \subset \mathbf{R}^n$*

$$\Big(\int_B u^*(x)^2\,dx\Big)^{1/2} \le C_B \|f\|_{H_{n/4}}, \qquad f \in \mathcal{S}.$$

In the case $V = 0$ (and $\varphi \in C_0^\infty$ in Theorem 1) Theorems 1–3 were proved in P. Sjögren and P. Sjölin [5] and Theorem 5 in P. Sjölin [7].

We remark that local smoothness for solutions to Schrödinger equations has also been studied by P. Constantin and J.C. Saut [1], [2].

We also remark that an important tool in the passage from the case $V = 0$ to the case of general $V$ is Duhamel's formula for solutions to Schrödinger equations

$$u(\cdot, t) = e^{itP} f + \int_0^t e^{i(t-\tau)P}\big( - iVu(\cdot, \tau)\big)\, d\tau, \qquad f \in H_m$$

(cf. [1], [2]).

In [6] we proved Theorem 2 in the case when $n \geq 3$ is odd, $P = \Delta$ and $V \in C_0^\infty$ is small, with a different method. The proof in [6] was based on results of A. Melin [3], [4] on intertwining operators. Set $H = -\Delta + V$ and $H_0 = -\Delta$. Melin constructed a bounded linear operator $A$ on $L^2$ such that $HA = AH_0$. One has

$$e^{-itH} = Ae^{-itH_0}A^{-1},$$

and a combination of properties of $A$ and results in [5], [7] for $e^{-itH_0}$ gave Theorem 2 in this special case.

## 2. Proof of Theorem 1

We set

$$S_0 f(x,t) = \varphi(x)\psi(t)e^{itP} f(x), \qquad x \in \mathbf{R}^n,\ t \in \mathbf{R},\ f \in \mathcal{S}(\mathbf{R}^n),$$

where $\varphi \in \mathcal{A}$ and $\psi \in C_0^\infty(\mathbf{R})$. With $\alpha = \frac{1}{2}(m-1)$ one then has

$$\|S_0 f\|_{H_{\varrho,r}} \leq C\,\|f\|_{H_{\varrho+mr-\alpha}}, \qquad \varrho \geq 0,\ r \geq 0. \tag{5}$$

In the case $\varphi \in C_0^\infty(\mathbf{R})$, (5) was proved in [5]. It is also easily seen that the argument in [5] gives (5) with $\varrho = r = 0$ in the case $\varphi = \hat{G}_s$ with $s > \frac{1}{2}$. It is then clear that (5) with $\varrho = r = 0$ holds for all $\varphi \in \mathcal{A}$, since $|\varphi(x)| \leq C\hat{G}_s(x)$ for some $s > \frac{1}{2}$. This result can now be extended as in [5] and gives (5) for all $\varrho \geq 0$, $r \geq 0$.

The conditions on $V$ imply that $V$ is a multiplier on $H_s$, i.e.

$$\|Vf\|_{H_s} \leq C_s\,\|f\|_{H_s}, \qquad s \in \mathbf{R}. \tag{6}$$

Next we prove that

$$\|e^{-itH} f\|_{H_s} \leq C_s\,\|f\|_{H_s}, \qquad t \in \mathbf{R},\ s \in \mathbf{R}. \tag{7}$$

To begin with, it follows from the ellipticity of $P$ that $g \in H_m$ if and only if $g \in L^2$ and $(-P+V)g \in L^2$. More generally,

$$g \in H_{jm} \quad \text{if and only if} \quad g, (-P+V)g, \dots, (-P+V)^j g \in L^2, \tag{8}$$

with a corresponding norm equivalence. Here $j = 0, 1, 2, \dots$. The "only if" part in (8) follows directly, and we shall prove the "if" part by induction.

Assume that $g$, $(-P+V)g$, $\dots$, $(-P+V)^j g \in L^2$. Using an induction assumption, we then conclude that $g$ and $(-P+V)g \in H_{(j-1)m}$. It follows that $Pg \in H_{(j-1)m}$ and $g \in H_{jm}$. Hence (8) is proved.

For $f \in H_{jm}$ we have

$$(-P+V)^k e^{-itH} f = e^{-itH}(-P+V)^k f \in L^2, \qquad 0 \le k \le j,$$

since $(-P+V)^k f \in L^2$ and $e^{-itH}$ is unitary on $L^2$. It follows that

$$\left\| e^{-itH} f \right\|_{H_{jm}} \le C \left\| f \right\|_{H_{jm}}, \qquad j = 0, 1, 2, \dots,$$

and interpolation and duality now give (7).

We shall also verify that the mapping $t \mapsto u(\cdot, t) = e^{-itH} f$ is continuous from $\mathbf{R}$ to $H_s$ if $f \in H_s$. Because of (7), it is sufficient to assume that $f \in \mathcal{S}$. We set $v_t = u(\cdot, t) - u(\cdot, t_0)$ and may assume that $s > 0$. One has

$$\|v_t\|_{H_{s+1}} \le C, \qquad t \in \mathbf{R},$$

and

$$\|v_t\|_2 \to 0, \qquad t \to t_0.$$

We also have

$$\begin{aligned}
\|v_t\|_{H_s}^2 &= \int (1+|\xi|^2)^s \left|\hat{v}_t(\xi)\right|^2 d\xi \\
&\le \int_{|\xi| \le R} (1+R^2)^s \left|\hat{v}_t(\xi)\right|^2 d\xi + \int_{|\xi| > R} (1+|\xi|^2)^{-1} (1+|\xi|^2)^{s+1} \left|\hat{v}_t(\xi)\right|^2 d\xi \\
&\le C(1+R^2)^s \|v_t\|_2^2 + (1+R^2)^{-1} \|v_t\|_{H_{s+1}}^2,
\end{aligned}$$

and it follows that $\|v_t\|_{H_s} \to 0$ as $t \to t_0$. The continuity is proved.

We shall now prove (1). Duhamel's formula gives

$$u(\cdot, t) = e^{itP} f + \int_0^t e^{i(t-\tau)P} \big( -iVu(\cdot, \tau) \big) \, d\tau, \qquad f \in H_m.$$

For $f \in \mathcal{S}$, we multiply by $\varphi \in \mathcal{A}$ and take norms in $H_s$, $s \geq 0$, getting

$$\|\varphi u(\cdot,t)\|_{H_s} \leq \left\|\varphi e^{itP} f\right\|_{H_s} + \int_{I_t} \left\|\varphi e^{i(t-\tau)P}(Vu(\cdot,\tau))\right\|_{H_s} d\tau,$$

where $I_t = [0,t]$, $t \geq 0$, and $I_t = [t,0]$, $t < 0$. Hence,

$$\left(\int_{-T}^{T} \|\varphi u\|_{H_s}^2 \, dt\right)^{1/2} \leq \left(\int_{-T}^{T} \|\varphi e^{itP} f\|_{H_s}^2 \, dt\right)^{1/2} + \left(\int_{-T}^{T} \left(\int_{I_t} \left\|\varphi e^{i(t-\tau)P}(Vu(\cdot,\tau))\right\|_{H_s} d\tau\right)^2 dt\right)^{1/2}.$$

We invoke (5) with $\varrho = s$, $r = 0$ to estimate the first term on the right-hand side.

Using an obvious notation, we have

$$\|\varphi u\|_{L^2(-T,T;H_s)} \leq C\,\|f\|_{H_{s-\alpha}} + \left(\int_{-T}^{T} |t| \left(\int_{I_t} \left\|\varphi e^{i(t-\tau)P}(Vu(\cdot,\tau))\right\|_{H_s}^2 d\tau\right) dt\right)^{1/2},$$

where we have also used the Cauchy–Schwarz inequality in the last term. The square of this term can be majorized by

$$T \int_{-T}^{T}\int_{-T}^{T} \|\varphi e^{it_1 P}(Vu(\cdot,\tau))\|_{H_s}^2 \, dt_1 \, d\tau.$$

Invoking (5) again, one finds that this is less than

$$C \int_{-T}^{T} \|Vu(\cdot,\tau)\|_{H_{s-\alpha}}^2 \, d\tau$$

(where $C$ depends on $T$). Hence,

$$\|\varphi u\|_{L^2(-T,T;H_s)} \leq C\,\|f\|_{H_{s-\alpha}} + C\,\|Vu\|_{L^2(-T,T;H_{s-\alpha})},$$

and from (6) and (7) we obtain

$$\|\varphi u\|_{L^2(-T,T;H_s)} \leq C\,\|f\|_{H_{s-\alpha}}. \tag{9}$$

This estimate yields (1) in the case $r = 0$.

We shall now extend this to the case $r > 0$. An easy consequence of (9) is that

$$\left\|\varphi D_x^\beta u\right\|_{L^2(-T,T;H_s)} \leq C\,\|f\|_{H_{s+|\beta|-\alpha}}. \tag{10}$$

Indeed, Leibniz' rule shows that

$$\varphi D^\beta u = D^\beta(\varphi u) + \sum_{|\gamma|\le|\beta|-1} \varphi_\gamma D^\gamma u,$$

for some $\varphi_\gamma \in \mathcal{A}$. A simple induction argument gives (10).

It follows from the Schrödinger equation that for $\varphi \in \mathcal{A}$

$$\varphi \frac{\partial u}{\partial t} = \sum_{|\gamma|\le m} \varphi_\gamma D^\gamma u,$$

where $\varphi_\gamma \in \mathcal{A}$. Differentiating with respect to $t$ and using the Schrödinger equation again, one obtains more generally

$$\varphi \frac{\partial^k u}{\partial t^k} = \sum_{|\gamma|\le km} \varphi_\gamma D^\gamma u, \qquad k = 0, 1, 2, \ldots,$$

where $\varphi_\gamma \in \mathcal{A}$.

The above equality combined with (10) now gives

$$\left\| \varphi \frac{\partial^k u}{\partial t^k} \right\|_{L^2(-T,T;H_s)} \le C \, \|f\|_{H_{s+mk-\alpha}}, \qquad \varphi \in \mathcal{A}, \ k = 0, 1, 2, \ldots.$$

It follows that

$$\|\varphi\psi u\|_{H_{s,k}} \le C \, \|f\|_{H_{s+mk-\alpha}}, \qquad s \ge 0, \ k = 0, 1, 2, \ldots$$

for $\varphi \in \mathcal{A}$, $\psi \in C_0^\infty(\mathbf{R})$.

Theorem 1 now follows from the above inequality and interpolation.

## 3. The remaining proofs

*Proof of Theorem 2.* Here we use the notation

$$\|Sf\|_{L^2(H_r)} = \left( \int_{\mathbf{R}^n} \|Sf(x,\cdot)\|^2_{H_r(\mathbf{R})} \, dx \right)^{1/2},$$

so that $L^2(H_r) = H_{0,r}$. For $s > \frac{1}{2}$ define $r > \frac{1}{2}$ by $mr - \alpha = s$. Then Theorem 1 implies that

$$\|Sf\|_{L^2(H_r)} \le C \, \|f\|_{H_s}.$$

Since $r > \frac{1}{2}$ the $L^\infty(\mathbf{R})$ norm is majorized by the $H_r(\mathbf{R})$ norm. We obtain

$$\left( \int_B u^*(x)^2 dx \right)^{1/2} \le C_B \, \|f\|_{H_s}$$

for every ball $B$, if we choose $\varphi$ and $\psi$ suitably. Theorem 2 is proved.

We omit the proof of Theorem 3, since it is the same as the last part of the proof of Theorem 2 in [5].

*Proof of Theorem 4.* Set

$$v(x,t) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{it|\xi|^2} \hat{f}(\xi)\, d\xi, \qquad x \in \mathbf{R}^n, t \in \mathbf{R}, f \in \mathcal{S}(\mathbf{R}^n).$$

It is sufficient to prove that if

$$\|\varphi\psi v\|_{L^2(\mathbf{R}^{n+1})} \le C\,\|f\|_{H_s}, \qquad f \in \mathcal{S}, \tag{11}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\psi \in C_0^\infty(\mathbf{R})$, then $s \ge -\frac{1}{2}$.

Choose $g \in C_0^\infty(\mathbf{R})$ so that $0 \notin \operatorname{supp} g$, $g$ is even, and $g(\frac{1}{2}) = 1$. We first assume $n = 1$ and define $f$ by $\hat{f}(\xi) = g(\xi/N)$ where $N > 1$. Then

$$\|f\|_{H_s}^2 = \int_{\mathbf{R}} (1+|\xi|^2)^s |g(\xi/N)|^2 d\xi = \int_{\mathbf{R}} (1+N^2|\eta|^2)^s |g(\eta)|^2 d\eta N \le CN^{2s+1},$$

so that

$$\|f\|_{H_s} \le CN^{s+1/2}. \tag{12}$$

The function $e^{i\xi^2}$ has Fourier transform $ce^{-ix^2/4}$ for some constant $c \ne 0$, and one finds that

$$\begin{aligned} v(x,t) &= ct^{-1/2} \int_{\mathbf{R}} e^{-i(x-y)^2/4t} N\hat{g}(Ny)\, dy \\ &= ct^{-1/2} \int_{\mathbf{R}} e^{-ix^2/4t} e^{2ixy/4t} e^{-iy^2/4t} N\hat{g}(Ny)\, dy \end{aligned}$$

for $t > 0$. Hence,

$$\big|v(x,t)\big| = |c| t^{-1/2} \Big| \int_{\mathbf{R}} e^{i(x/Nt)z/2} e^{-i(1/4N^2t)z^2} \hat{g}(z)\, dz \Big|, \qquad t > 0.$$

The function

$$F(\alpha,\beta) = \int_{\mathbf{R}} e^{i\alpha z/2} e^{-i\beta z^2} \hat{g}(z)\, dz$$

is a continuous function of $(\alpha,\beta)$, and $F(1,0) = 2\pi g(\frac{1}{2}) = 2\pi$. It follows that there exists a $\delta > 0$ so that $\big|F(\alpha,\beta)\big| \ge 1$ if $1 \le \alpha \le 1+\delta$ and $0 \le \beta \le \delta$. We conclude that $\big|v(x,t)\big| \ge ct^{-1/2}$ for $t \ge 1/4N^2\delta$ and $Nt \le x \le Nt + \delta Nt$. When $1/N\delta \le t \le 2/N\delta$, one then has $\big|v(x,t)\big| \ge cN^{1/2}$ for $x$ in an interval of length $\delta Nt \ge 1$. Hence, $\big|v(x,t)\big| \ge cN^{1/2}$ for $(x,t)$ in a set of measure $\ge c/N$.

It follows that

$$\|\varphi\psi v\|_{L^2(\mathbf{R}^{n+1})} \ge cN^{1/2}\Big(\frac{1}{N}\Big)^{1/2} = c,$$

and (11) and (12) yield $N^{s+1/2} \geq c$ for $N > 1$, so that $s \geq -\frac{1}{2}$.

When $n \geq 2$, the argument is similar. One chooses $g$ as above and sets

$$\hat{f}(\xi) = g(\xi_1/N)g(\xi_2/N)\cdots g(\xi_n/N), \qquad \xi = (\xi_1, \ldots, \xi_n).$$

*Proof of Theorem 5.* Set $u_0(x,t) = (e^{itP}f)(x)$ and $u_0^*(x) = \sup_{0<t<1} \left|u_0(x,t)\right|$. It was proved in [7] that

$$\left(\int_B u_0^*(x)^2 dx\right)^{1/2} \leq C_B \|f\|_{H_\gamma}, \qquad f \in \mathcal{S}, \tag{13}$$

where $\gamma = n/4$.

Since $\left(e^{itP}(Vu(\cdot,\tau))\right)(x)$ is a continuous function of $(x,t,\tau)$ if $f \in \mathcal{S}$, it follows from Duhamel's formula that

$$u(x,t) = u_0(x,t) + \int_0^t \left(e^{i(t-\tau)P}\left(-iVu(\cdot,\tau)\right)\right)(x)\, d\tau.$$

Hence for $0 < t < 1$

$$\left|u(x,t)\right| \leq u_0^*(x) + \int_0^1 \sup_{0<t<1} \left|\left(e^{itP}e^{-i\tau P}\left(Vu(\cdot,\tau)\right)\right)(x)\right| d\tau.$$

Using (13) twice we obtain

$$\begin{aligned}\left(\int_B u^*(x)^2 dx\right)^{1/2} &\leq \left(\int_B u^*(x)^2 dx\right)^{1/2} \\ &\quad + \int_0^1 \left(\int_B \left(\sup_{0<t<1} \left|\left(e^{itP}e^{-i\tau P}\left(Vu(\cdot,\tau)\right)\right)(x)\right|\right)^2 dx\right)^{1/2} d\tau \\ &\leq C\|f\|_{H_\gamma} + C\int_0^1 \left\|e^{-i\tau P}\left(Vu(\cdot,\tau)\right)\right\|_{H_\gamma} d\tau \\ &\leq C\|f\|_{H_\gamma} + C\int_0^1 \|u(\cdot,\tau)\|_{H_\gamma}\, d\tau \leq C\|f\|_{H_\gamma},\end{aligned}$$

and the proof is complete.

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