Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 16, 1991, 47–69

COMPACTNESS PROPERTIES OF μ-HOMEOMORPHISMS

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1. Introduction

In the theory of 2-dimensional quasiconformal mappings one considers embeddings of a plane domain D which are ACL and satisfy the Beltrami equation

$$f_{\bar{z}} = \mu f_z \tag{1a}$$

where μ is a complex function of D such that $\|\mu\|_\infty < 1$. Lehto [L1, L2] and David [D] have considered the more general situation where $|\mu(z)| < 1$ almost everywhere but it may be that $\|\mu\|_\infty = 1$. These conditions do not guarantee the solvability of (1a), and both Lehto and David had to make some additional assumptions on μ. David's condition was basically that the areal measure of the set $\{z \in D : |\mu(z)| > c\}$, $c < 1$, has a majorant which is an exponential function of $(1-c)^{-1}$ (see (2a) for the exact formula); he called this kind of solutions of (1a) μ-homeomorphisms.

We prefer to use the dilatation K_f of f at x instead of the complex dilatation and define that f is a *μ-homeomorphism* if it is ACL and if there are $\alpha > 0$, $C > 0$ and $K_0 \geq 1$ such that, when m is the areal measure,

$$m\Big(\{z \in D : K_f(x) \text{ defined and } K_f(z) > K\}\Big) \leq Ce^{-\alpha K} \tag{1b}$$

when $K \geq K_0$. Here $K_f(z)$ is defined if the differential $Df(z)$ exists and is non-zero; if $Df(z)$ exists and is singular but non-zero, $K_f(z)$ is defined with $K_f(z) = \infty$. The numbers α, C, and K_0 are the parameters for μ-homeomorphisms (such as the number K for K-quasiconformal mappings) and if (1b) is true we say that f is an (α, C, K_0)-homeomorphism.

We will extend the compactness properties of K-quasiconformal mappings to μ-homeomorphisms. Let $f_1, f_2, \ldots$ be μ-homeomorphisms of a domain D of $\bar{\mathbf{C}}$ into $\bar{\mathbf{C}}$ with fixed parameters. If they converge towards a map, then the limit map is either a μ-homeomorphism or a map of D onto one or two points. If the maps f_j fix three given points, then there is a subsequence which converges towards a μ-homeomorphism. These results follow from our theorems in Sections 3 and 5.

In addition to the compactness properties, we will also prove the so-called good approximation theorem for μ-homeomorphisms (Corollary 6C).

doi:10.5186/aasfm.1991.1628

Notations and conventions. For the definition of terms like ACL (absolute continuity on lines) we refer to standard treatises on quasiconformal maps like [A] or [LV]. As usual in the theory of quasiconformal mappings, the functions we consider will often be only almost everywhere defined and so when we say that a function is defined on a domain D we often mean that it is defined in D with the exception of a set of measure zero.

We use the following notations:

We identify $\mathbf{C}$ and $\mathbf{R}^2$, and $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We use the euclidean metric on $\mathbf{C}$ and on $\bar{\mathbf{C}}$ we may use the spherical metric denoted by k (obtained by means of the stereographic projection).

m = the areal measure on $\mathbf{C}$;
σ = the areal measure of $\bar{\mathbf{C}}$ with respect to the spherical metric;
λ = the linear measure of an arc = the Hausdorff 1-measure;
$B(x,r)$ = the open euclidean ball with center x and radius r;
∂A is the topological boundary of A and $\bar{A}$ its closure, both taken in $\bar{\mathbf{C}}$.

2. μ-homeomorphisms

We will now give more detailed definitions for the present situation. Let f be an orientation preserving embedding of a domain D of $\bar{\mathbf{C}}$ into $\bar{\mathbf{C}}$. If f has a non-singular differential at x, the complex dilatation of f at x is

$$\mu_f(x) = \frac{f_{\bar{z}}}{f_z}.$$

The dilatation K_f of f at x is the ratio of the long and short axis of the dilatation ellipsoid of f at x; that is, $K_f(x)$ is defined whenever f has a non-singular derivative at x and then $K_f(x)$ is the number

$$K_f(x) = \frac{1 + |\mu_f(x)|}{1 - |\mu_f(x)|}$$

where $K_f(x) = \infty$ if $|\mu_f(x)| = 1$.

David considered mappings $f\colon D \to \mathbf{C}$, $D \subset \mathbf{C}$, which are ACL and hence a.e. differentiable [LV, III.3.2] and which satisfy (1a) a.e. in D for some μ such that $\|\mu\|_\infty \le 1$ and such that for some $\alpha > 0$, $C_0 > 0$ and $\varepsilon \in (0,1]$,

$$m\Big(\big\{z \in D : |\mu(z)| > 1 - \varepsilon\big\}\Big) \le C_0 e^{\alpha} e^{-\alpha/\varepsilon} \tag{2a}$$

whenever $\varepsilon \le \varepsilon_0$; such a map was called a μ-homeomorphism. It is easy to see that David's definition and the definition (1b) by means of K_f give the same class of mappings. The exponents in (1b) and (2a) are related in the following manner:

If (2a) is true for α, C_0 and ε_0, then (1b) is true for $\alpha/2$ and certain C and K_0. Conversely, if (1b) is true for α, C and K_0, then (2a) is true for 2α and certain C_0 and ε_0.

David showed that given μ satisfying (2a), there is an ACL embedding f satisfying (1a) a.e. and furthermore, f is uniquely determined in the sense that if g is another such map, then it is of the form hf where h is conformal. He also showed that such a map f has a.e. a non-singular differential and hence the complex dilatation μ_f of f is a.e. defined and it coincides a.e. with μ. One important property is that f and f^{-1} are absolutely continuous.

David's Theorem 1 in [D], containing the existence theorem and some distortion estimates, is our starting point. We will not make use of David's method of proof which was a very complicated extension of the method in Ahlfors' book [A], a real computational *tour de force.* Our approach is more geometric, with much inspiration drawn from the monograph of Lehto–Virtanen [LV]. Some proofs are modelled on the proof of the good approximation theorem for n-dimensional quasiconformal mappings in [T].

We reserve the word quasiconformal for maps which are quasiconformal in the ordinary, non-extended sense, and the word μ-homeomorphism refers to the extended class we consider here. The notion corresponding to K-quasiconformality is that f is an (α, C, K_0)-*homeomorphism* which means that it satisfies (1b) with these constants. The number α is the most important of the three numbers α, C and K_0, and a μ-homeomorphism f is said to be an (α)-*homeomorphism*, and α an *exponent* of f, if there are C and K_0 such that f is an (α, C, K_0)-homeomorphism. (Note that the exponent of a μ-homeomorphism is not well-defined but we call any such α that f is an (α)-homeomorphism an exponent of f).

The problem with the preceding definition is that the point ∞ is in a special position since every neighbourhood of ∞ has infinite measure. Thus, if $\infty \in D$ or $\infty \in \partial D$, it might be advisable to change the definition. (Note that no problems arise if $\infty \in fD$.) The easiest way to handle this situation is to replace the euclidean measure by the spherical measure in the definition of a μ-homeomorphism and we call such maps *spherical μ-homeomorphisms,* and the words spherical (α)-homeomorphism and spherical (α, C, K_0)-homeomorphism will be used similarly. Note that if $\infty \in \bar{D}$, then a euclidean μ-homeomorphism is a spherical μ-homeomorphism but not necessarily the other way round.

There are a few occasions when we use a normalized measure ν such that a certain set B, usually a disk, has measure one. If we use this normalized measure ν in the definition of a μ-homeomorphism we indicate this by saying that f is a μ-homeomorphism with respect to ν.

It is also useful to have a local definition. We say that f is a local μ-homeomorphism if every $x \in D$ has a neighbourhood $U \subset D$ such that $f|U$ is a spherical μ-homeomorphism; if $x \neq \infty$, then of course this is equivalent to the existence of a neighbourhood V such that $f|V$ is a euclidean μ-homeomorphism.

We will now extend David's theorem on the existence of solutions of the Beltrami equation (1a). David's theorem combined with the fact that topologically planar Riemann surfaces are conformally planar gives the following general existence theorem. Let μ be a complex function of a domain $D \subset \bar{\mathbf{C}}$ such that $|\mu(z)| < 1$ for a.e. z and set

$$K_\mu(x) = \frac{1 + |\mu(x)|}{1 - |\mu(x)|}; \tag{2b}$$

if there is f such that $\mu_f = \mu$, then K_μ is the dilatation of f. We say that μ satisfies a *local exponential condition* if f would be a local μ-homeomorphism, that is, if for every $z \in D$ there is a neighbourhood $U \subset D$ and numbers $\alpha > 0$, $C > 0$ and $K_0 \geq 1$ such that if σ is the spherical measure, then

$$\sigma\left(\{x \in U : K_\mu(x) > K\}\right) \leq Ce^{-\alpha K} \tag{2c}$$

for all $K \geq K_0$.

Theorem 2A. *Let μ: $D \to \mathbf{C}$ satisfy a local exponential condition. Then there is an ACL embedding f: $D \to \bar{\mathbf{C}}$ satisfying $f_{\bar{z}} = \mu f_z$ a.e. in D. Any other embedding g of D satisfying these conditions is of the form $g = hf$ where h is conformal.*

Proof. Let U_i, $i \in I$, be an open cover of D such that $\bar{U}_i \neq \bar{\mathbf{C}}$ and that $\mu|U_i$ satisfies (2c) for some α, C, and K_0. By David's theorem, there is a μ-homeomorphism f_i: $U_i \to \mathbf{C}$ satisfying (1a) a.e. for $f = f_i$ and $\mu = \mu_i$ in U_i; if $\infty \in \bar{U}_i$, we may have to use auxiliary conformal mappings. The uniqueness part of David's theorem implies that (U_i, f_i), $i \in I$, is a conformal atlas of D. Since D is a planar surface, D is in this conformal structure equivalent to a planar domain D'. This conformal equivalence gives the map f: $D \to \bar{\mathbf{C}}$ solving the Beltrami equation (1a).

The essential uniqueness of f is an immediate consequence of David's theorem.

There are situations where we need to assume that the exponent α of a μ-homeomorphism is sufficiently big. This situation can always be obtained if we allow the composition into a quasiconformal and μ-homeomorphic part. We can achieve this by the next lemma which is our version of [D, Section 4].

Lemma 2B. *Let f be a euclidean or spherical (α, C, K_0)-homeomorphism of a domain D and let $K' \geq 1$. Then there is a K'-quasiconformal map g and a (euclidean or spherical, respectively) $(\alpha K', C, \max\ (K_0/K', 1))$-homeomorphism h such that $f = gh$ and such that the dilatation of h is given a.e. in D by*

$$K_h(x) = \max\left(1, K_f(x)/K'\right). \tag{2d}$$

Proof. To obtain the decomposition $f = gh$ we simply define the maps so that g will take as much as possible of the dilatation of f while still being K'-quasiconformal (so that h is conformal if f is K'-quasiconformal). We let h be any map such that a.e. in D

$$\mu_h(z) = t(z)\mu_f(z)$$

where $t(z) \in [0,1)$ and $t(z) = 0$ if $K_f(z) \leq K'$ and if $K_f(z) > K'$, $t(z)$ is defined by the condition that $K_h(z)$ will be the number $K_f(z)/K'$. By David's theorem h exists. Thus the formula for K_h is (2d) and hence h is is indeed an $\big(\alpha K', C, \max(K_0/K', 1)\big)$-homeomorphism.

So we have the map h. To define g, we first note that both f and h are μ-homeomorphisms and hence they, and their inverses, preserve null-sets and are a.e. differentiable with non-singular derivative. It follows that fh^{-1} is a.e. differentiable with non-singular derivative and so $\mu = \mu_{fh^{-1}}$ is defined a.e. One can calculate as in the proof of [L3, Theorem 4.7] that $\|\mu\|_\infty \leq (K'-1)/(K'+1)$ and hence any ACL embedding with complex dilatation μ is K'-quasiconformal. Let g_0 be such a map. Then $g_0 h$ is a μ-homeomorphism by [D, Section 9] and has the same complex dilatation as f. The uniqueness part of David's theorem implies that $f = g_1 g_0 h$ where g_1 is conformal. Setting $g = g_1 g_0$ we have found K'-quasiconformal g such that $f = gh$.

3. Normal family properties of μ-homeomorphisms

We now come to the normal family properties of μ-homeomorphisms which are the same as the corresponding properties of quasiconformal mappings. This section is the equivalent of [LV, II.5] for μ-homeomorphisms. A family $\mathcal{F}$ of continuous mappings of a domain D is *normal* if any sequence of elements of $\mathcal{F}$ contains a subsequence converging uniformly on every compact subset of D towards some mapping of D. In the situations we consider the maps are maps of D into $\bar{\mathbf{C}}$ and the uniform convergence is with respect to the spherical metric.

We need the following consequence of David's Theorem 1. Recall that a map of a domain $D \subset \bar{\mathbf{C}}$ which contains $0, 1$ and ∞ is *normalized* if it fixes 0, 1 and ∞; if f is a homeomorphism of $\mathbf{C}$, the extension of f to $\bar{\mathbf{C}}$ fixes in any case ∞ and so in this case we say that f is normalized if it fixes 0 and 1.

Lemma 3A. *Let $f_j\colon \mathbf{C} \to \mathbf{C}$ be normalized (α, C, K_0)-homeomorphisms. Then there is a subsequence f_{n_i} such that f_{n_i} converge uniformly on compact subsets of $\mathbf{C}$ towards an embedding $\mathbf{C} \to \mathbf{C}$.*

Proof. The maps f_n are equicontinuous (cf. [LV, II.4.1]) by [D, Eq. (4) in Theorem 1] and hence we can pass to a subsequence which converges uniformly on compact subsets to a continuous function h as in [LV, II.5.1]. Since the numbers $|f_n(z) - f_n(z')|$ have a uniform positive minorant [D, Eq. (5)], depending only on α, C, K_0, $|z|$ and $|z - z'|$ (but not on n), h is in fact injective.

Note that at this stage we know only that the limit is injective but it will later follow that it is a μ-homeomorphism, and hence a homeomorphism, of $\mathbf{C}$ ([D, Proposition 1 and Section 4] also imply that the limit is a homeomorphism).

In the next theorem, $k(z,z')$ is the spherical distance of two points $z,z' \in \bar{\mathbf{C}}$.

Theorem 3B. *Let $\mathcal{F}$ be a family of spherical (α, C, K_0)-homeomorphisms of a domain D of $\bar{\mathbf{C}}$. The family $\mathcal{F}$ is normal if there is $d > 0$ such that one of the following conditions is true.*

1. Every $f \in \mathcal{F}$ omits two points whose spherical distance is at least d.

2. There are $z_1, z_2 \in D$ and $a \in \bar{\mathbf{C}}$ such that $k\big(f(z_j), a\big) > d$ for every $f \in \mathcal{F}$.

3. There are $z_1, z_2, z_3 \in D$ such that $k\big(f(z_j), f(z_k)\big) > d$ for all $f \in \mathcal{F}$ and $j,k \leq 3$, $j \neq k$.

Proof. This is Theorem II.5.1 of [LV] for μ-homeomorphisms. We start from the fact that the theorem is true if every element of $\mathcal{F}$ is conformal. We assume first that $\bar{D} \neq \bar{\mathbf{C}}$. Thus we can assume that $\infty \notin \bar{D}$ and that $f \in \mathcal{F}$ are euclidean μ-homeomorphisms with uniform parameters.

Let $f_j \in \mathcal{F}$. Let g_j be the normalized μ-homeomorphism which is conformal outside D and has the same complex dilatation as f_j in D. By David's theorem the maps g_j exist and by the preceding lemma we can pass to a subsequence, denoted in the same manner, in such a way that g_j converge uniformly on compact subset of D towards an embedding $g\colon \mathbf{C} \to \mathbf{C}$.

It follows by the uniqueness part of David's theorem that $f_j = h_j g_j | D$ where $h_j\colon g_j D \to \bar{\mathbf{C}}$ is conformal. Let $D' = gD$. If D'' is a domain such that $\bar{D}'' \subset D'$, then $g_j D \supset D''$ for big j. We note that the maps $h_j | D''$ (j big) satisfy also the same condition of the present theorem which f_j satisfy. Since the theorem is true for conformal maps, we can infer that there is a subsequence (again denoted in the same manner) such that $h_j | D''$ converge to some map of D'' uniformly on compact subsets. Considering bigger and bigger D'', we can find by the Cantor diagonal process the subsequence such that $h_j(g_j | D) = f_j$ converge uniformly on compact subsets towards a map of D.

In the general case we express D as a union of two domains D_1 and D_2 such that $\bar{D}_j \neq \bar{\mathbf{C}}$; if condition 2 or 3 is true, then we assume that $z_j \in D_1 \cap D_2$. Applying the theorem first to the maps $f_j | D_1$ and then to $f_j | D_2$, we find the desired subsequence.

The next theorem is the equivalent of [LV, Theorem II.5.3] and shows that the convergence theory for μ-homeomorphisms is much the same as the one for quasiconformal mappings. It will follow from Theorem 5D that in case (c) the limit map is in fact a μ-homeomorphism.

Theorem 3C. *Let f_j be an (α, C, K_0)-homeomorphism of a domain D into $\bar{\mathbf{C}}$ such that f_j converge pointwise towards a function f. Then f is either* (a) *a* *constant,* (b) *a map of* D *onto two points, or* (c) *an embedding of* D. *In case* (c) *the convergence is uniform on every compact subset of* D *and in case* (b) *the limit function takes one of its two values in just one point* $a \in D$ *and the convergence is uniform on compact subsets of* $D \setminus \{a\}$.

Proof. The argument in [LV, proof of Theorem II.5.2] proves the theorem otherwise except that in case (c) one only knows that the limit is a continuous function on D. We will show that if the limit is continuous and non-constant, then it is an embedding. It suffices to prove this in the case that $\infty \notin \bar{D}$.

As above, we compose each f_j as $h_j(g_j|D)$ where h_j is conformal and g_j is a normalized (α, C, K_0)-homeomorphism of $\bar{\mathbf{C}}$. We can assume by Lemma 3A, possibly by passing to a subsequence, that g_j converge uniformly towards an embedding g of $\bar{\mathbf{C}}$. By [LV, Theorem II.5.2], h_j converge towards a map h of gD which is either a constant, a map of gD onto two points or an embedding of gD (the fact that h_j are defined on g_jD which vary but "converge" to gD causes some minor difficulties which we by-pass). Furthermore, the convergence is uniform on compact subsets of gD or of $gD \setminus \{\text{point}\}$. Consequently $h_j(g_j|D)$ converge to f locally uniformly either on D or on $D \setminus \{\text{point}\}$. It follows that if f is continuous but not an embedding, it must be a constant.

A particular instance of these theorems is the next lemma which is the form in which we will make use of the compactness. Again, it will be later shown that the limit map f is actually a μ-homeomorphism.

Lemma 3D. *Let* $f_j\colon D \to \mathbf{C}$, D *a domain of* $\mathbf{C}$, *be* (α, C, K_0)*-homeomorphisms which fix two distinct points* $a, b \in D$. *Then there is a subsequence which converges towards an embedding* f *of* D *into* $\mathbf{C}$ *uniformly on compact subsets of* D.

Proof. By Case 1 of Theorem 3B, there is a subsequence which converges towards a map f of D uniformly on compact subsets of D. By Theorem 3C, f is either a constant, a map of D onto two points or an embedding. Since every f_j fixes a and b, f cannot be a constant.

If f is a map of D onto two points, then $fD = \{a, b\}$ and furthermore, we know that the convergence is uniform on compact subsets of $D \setminus \{a\}$ or of $D \setminus \{b\}$. Suppose that the first case occurs. Let $S \subset D$ be a topological circle separating a from b and from $\bar{\mathbf{C}} \setminus D$. Then for big j, $f_j(S)$ is in a given neighbourhood U of b and, as $f_j(S)$ separates a from b and from $\bar{\mathbf{C}} \setminus f_jD$, it would follow that $\bar{\mathbf{C}} \setminus f_jD$ is contained in an arbitrarily small neighbourhood of b. This is a contradiction since $\infty \notin f_jD$.

Finally, we will obtain a bound for the distortion of the measure by a μ-homeomorphism.

Lemma 3E. *Let* f *be an* (α, C, K_0)*-homeomorphism of a domain* D *of* $\mathbf{C}$ *into* $\mathbf{C}$. *Suppose that* $0, 1 \in D$ *and that* f *fixes* 0 *and* 1. *Let* $F \subset D$ *be* *compact. Then there is a universal constant $\theta > 0$ such that if $\varrho < \theta\alpha$, there is $C' = C'(\alpha, C, \varrho, K_0, F, D)$ such that*

$$m(fE) < C'\big(\log m(E)^{-1}\big)^{-\varrho}$$

for all measurable $E \subset F$ such that $m(E) \le \frac{1}{2}$.

Proof. This is Eq. (6) of Theorem 1 of [D]. We need only to compose f as $h(g|D)$ where $h\colon gD \to \mathbf{C}$ is conformal and $g\colon \mathbf{C} \to \mathbf{C}$ is a μ-homeomorphism which is conformal outside D and whose dilatation coincides with that of f a.e. in D. We assume that both h and g fix 0 and 1. By the preceding lemma the set of μ-homeomorphisms $g\colon \mathbf{C} \to \mathbf{C}$ fixing 0 and 1 is compact and so is the set of conformal maps $h\colon D' \to \mathbf{C}$ fixing 0 and 1 for any domain D' containing 0 and 1. A normal family argument easily shows that there is $M > 0$, $M = M(\alpha, C, \varrho, K_0, F, D)$ such that $|h'(z)| \le M$ for all $z \in gF$ whenever g is a normalized (α, C, K_0)-homeomorphism of $\mathbf{C}$ and $h\colon gD \to \mathbf{C}$ is conformal. Hence $m(hE') \le M^2 m(E')$ for all measurable $E' \subset gF$. The result now follows by (6) of Theorem 1 of [D].

Remark. Note that since the exponents may differ in our and David's definition of a μ-homeomorphism (see Section 2), our number θ is twice the number θ in David's theorem.

4. Some measure theoretic results

Our aim is to prove that limits of μ-homeomorphisms with uniform parameters are still μ-homeomorphisms (unless degenerate). We will present this theorem in the next section. Meanwhile we will prove some measure theoretic results needed later.

Let A be a metric measure space with measure m and metric d. If $h\colon A \to \mathbf{R}$ is measurable, we say that h is *exponential* if there are numbers $\alpha > 0$, $C > 0$ and $K_0 \ge 1$ such that h satisfies the growth condition

$$m\left(\{z \in A : h(z) > K\}\right) \le Ce^{-\alpha K} \tag{4a}$$

for all $K \ge K_0$; if we want to be more specific we say that h is (α, C, K_0)-exponential or α-exponential (if (4a) is true with this α for some C and K_0). Let $B(z,r)$ be the open ball with center z and radius r, and denote by m_{zr} for each $z \in A$ and $r > 0$ the measure

$$m_{zr}(E) = \frac{m(E)}{m\big(B(z,r)\big)} \tag{4b}$$

on $B(z,r)$ if $m\big(B(z,r)\big) > 0$. A point $z \in A$ is an *exponential point* of h if there are α, C and K_0 such that one can find arbitrarily small r for which $h|B(z,r)$

is (α, C, K_0)-exponential with respect to m_{zr}; when referring to this particular α we say that z is an *α-point* of h. Thus C and K_0 may depend on z.

We formulate the following theorems for the case at hand so that A will be a subset of $\mathbf{C} = \mathbf{R}^2$. The metric will be the euclidean metric and the measure either the areal measure on $\mathbf{C}$ or the linear measure on an interval but it is obvious that they can be generalized at least to subsets of $\mathbf{R}^n$.

If h is exponential, then it will turn out that a.e. point is an exponential point of h with an exponent not depending on the point. We will give two lemmas on this, the first one for single maps and the second for sequences.

Lemma 4A. *Let $A \subset \mathbf{C}$ and suppose that $h\colon A \to \mathbf{R}$ is (α, C, K_0)-exponential. Let β be a number such that $0 < \beta < \alpha$. Then a.e. point of A is a β-point of h. More precisely, a.e. $x \in A$ is a $\big(\beta, 1, K_1(x)\big)$-point of h where K_1 satisfies*

$$m\left(\left\{x \in A : K_1(x) > K\right\}\right) \le C' e^{(\beta-\alpha)K} \tag{4c}$$

for $K \ge K_0$ when $C' = 9^2 C e^{\beta}\big[\pi(1 - e^{\beta-\alpha})\big]^{-1}$. In other words, K_1 is $(\alpha - \beta, C', K_0)$-exponential.

Lemma 4B. *Let $h_k\colon A \to \mathbf{R}$ be a sequence of (α, C, K_0)-exponential maps. Let $0 < \beta < \alpha$. Then for a.e. point $x \in A$ it is true that there are $K_1 \ge 1$, a sequence $r_i > 0$ such that $r_i \to 0$ as $i \to \infty$, and a sequence $j_1 < j_2 < \cdots$ such that the maps $h_{j_i}|B(x, r_q)$ are $(\beta, 1, K_1)$-exponential with respect to m_{xr_q} for all $i \ge q$. Here $K_1 = K_1(x)$ is $(\alpha - \beta, e^{\beta} C', K_0)$-exponential with C' as above.*

Remark. Actually, the proof shows that, at least if A is bounded, there are numbers $j_1 < j_2 < \cdots$ independent of $x \in A$ such that for a.e. x there is a sequence $r_i > 0$ such that $\lim_{i\to\infty} r_i = 0$ and that the maps $h_{j_i}|B(x, r_q)$ are $\big(\beta, 1, K_1(x)\big)$-exponential with respect to m_{xr_q} if $2^{-i} \le r_q$. However, in the situations where we apply Lemma 4B, it is no restriction to let the subsequence h_{j_1}, h_{j_2}, ..., of h_i's to depend on x.

Proof. We will first consider the case for fixed h, that is, we will prove Lemma 4A. It is clear that we can assume firstly that A is bounded and secondly that A is a square of integral sidelength (extend h by 0 the points where it was not defined).

In order to facilitate the proof of Lemma 4B, we can clearly assume that h is one of the maps in the sequence h_1, h_2, ... of Lemma 4B and so we will relabel h as h_j though in the first part of the proof j will be fixed. We will, however, aim to estimates depending only on the parameters (α, C, K_0) and not on the particular map $h = h_j$.

Define for $K \ge K_0$

$$A_{Kj} = \left\{x \in A : h_j(x) > K\right\}$$

and let $\mathcal{K}_n$ be the set of squares which are obtained by subdividing the square A into equal squares of sidelength 2^{-n}. If $Q \in \mathcal{K}_n$, let

$$Q^* = \cup\{P \in \mathcal{K}_n : P \cap Q \neq \emptyset\},$$

set $c_0 = \pi 9^{-1} e^{-\beta}$, and define

$$\mathcal{K}_{Knj} = \{Q \in \mathcal{K}_n : m(Q^* \cap A_{Kj})/m(Q^*) > c_0 e^{-\beta K}\},$$

$$A_{Knj} = \cup \mathcal{K}_{Knj}.$$

Now every $x \in A$ is in the interior of at most nine cubes Q^*, $Q \in \mathcal{K}_n$. Hence

$$\begin{aligned} Ce^{-\alpha K} &\geq m(A_{Kj}) \geq 9^{-1} \sum_{Q \in \mathcal{K}_{Knj}} m(Q^* \cap A_{Kj}) \\ &\geq 9^{-1} \sum_{Q \in \mathcal{K}_{Knj}} m(Q^*) c_0 e^{-\beta K} \geq 9^{-1} \sum_{Q \in \mathcal{K}_{Knj}} m(Q) c_0 e^{-\beta K} \\ &\geq 9^{-1} c_0 e^{-\beta K} m(A_{Knj}), \end{aligned}$$

if $K \geq K_0$ and consequently

$$m(A_{Knj}) \leq 9 c_0^{-1} C e^{(\beta - \alpha)K} \tag{4d}$$

if $K \geq K_0$. We set for $K \geq K_0$

$$B_{Knj} = \bigcup_{q=0}^{\infty} A_{K+q,n,j}.$$

By (4d) we have the following upper bound $A(K)$ for $m(B_{Knj})$ when $K \geq K_0$, depending neither on j nor on n,

$$m(B_{Knj}) \leq 9 C c_0^{-1} \left[1 - e^{(\beta - \alpha)}\right]^{-1} e^{(\beta - \alpha)K} = A(K). \tag{4e}$$

Suppose that $x \in A \setminus B_{Knj}$. Then, if $x \in Q \in \mathcal{K}_n$ and $K \geq K_0$,

$$m(Q^* \cap A_{Lj})/m(Q^*) \leq c_0 e^{-\beta L}. \tag{4f}$$

for $L = K, K+1, K+2, \ldots$. If

$$C_{Kj} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_{Knj},$$

then $m(C_{Kj}) \le A(K)$ and if $x \notin C_{Kj}$, there are arbitrarily big n such that $x \notin B_{Knj}$. Hence there are arbitrarily big n such that $x \in Q$ for some $Q \in \mathcal{K}_n$ for which (4f) is true when $L = K, K+1, K+2, \dots$. Clearly, $h_j|B(x, 2^{-n})$ is (β, C'', K)-exponential with respect to $m_{x,2^{-n}}$ when $C'' = 9\pi^{-1}e^{\beta}c_0 = 1$ since $c_0 = \pi 9^{-1} e^{-\beta}$. The limit $A(K)$ for C_{Kj} in (4e) implies (4c).

This proves Lemma 4A. In order to have Lemma 4B, we note the set of squares $\mathcal{K}_n$ is finite and hence so is the number of its subsets. Hence, if K is fixed, the sets $\mathcal{K}_{Knj} \subset \mathcal{K}_n$ coincide for an infinite number of j's. Furthermore, if $m(A_{Knj}) > 0$, then $m(A_{Knj}) \ge 4^{-n}$ and hence $\mathcal{K}_{Knj} = \emptyset$ by (4d) if $9c_0^{-1}Ce^{(\beta-\alpha)K} < 4^{-n}$ and $K \ge K_0$. It now follows easily that for each n there is a sequence $j_{n1} < j_{n2} < \cdots$ such that

$$\mathcal{K}_{Knj_{ni}} = \mathcal{K}_{Knj_{nk}}$$

for all i, k and $K \in \mathbf{N}$. We can assume that each $(j_{n+1,i})$ is a subsequence of (j_{ni}) (n fixed, i varies). By the Cantor diagonal process we can now find a sequence $j_1 < j_2 < \cdots$ such that

$$\mathcal{K}_{Knj_i} = \mathcal{K}_{Knj_q} \tag{4g}$$

for all $n, K \in \mathbf{N}$ and i, q such that $i, q \ge n$.

We now assume for the moment that $K \in \mathbf{N}$ and $K \ge K_0$. So by (4g), $B_{Knj_i} = B_{Knj_q}$ if $i, q \ge n$. Hence we can define

$$B_{Kn} = B_{Knj_i}$$

indepedently of $i \ge n$. The number $A(K)$ in (4e) is an upper bound also for $m(B_{Kn})$ and hence $A(K)$ is an upper bound for the measure of

$$C_K = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_{Kn}.$$

If $x \notin C_K$, then there is a sequence $n_1 < n_2 < \cdots$ such that $x \notin B_{Kn_q}$ for all q. That is, (4f) is true whenever $n = n_q$, $x \in Q \in \mathcal{K}_{n_q}$, and $j = j_i$, $i \ge n_q$, for all $L = K, K+1, K+2, \dots$ Setting $r_q = 2^{-n_q}$, we have as above that $h_{j_i}|B(x, r_q)$ is $(\beta, 1, K)$-exponential with respect to m_{xr_q} if $i \ge n_q$. Passing to a subsequence of (j_i), we obtain that this is true if $i \ge q$.

We have now proved Lemma 4B with the exception that we only know that K_1 satisfies (4c) if $K \ge K_0$ *and* $K \in \mathbf{N}$. We can allow all values $K \ge K_0$ if we multiply C' in (4c) by e^{β}. This done, the proof is complete.

The next lemma is similar but simpler. Its formulation is related to the ACL property of quasiconformal mappings, as we consider a property of a map on line segments parallel to the coordinate axes and require that this property is true for almost all such line segments. There are actually two versions, one for horizontal line segments and a similar one for vertical line segments, exactly like in the definition of the ACL property. We will formulate and prove only for the "vertical" version, the formulation and the proof for the "horizontal" version being the same. The measure on line segments of the next lemma is the natural linear measure denoted by λ.

Lemma 4C. *Let $Q = [a,b] \times [c,d]$ be a quadrilateral. Let $h\colon Q \to \mathbf{R}$ be (α, C, K_0)-exponential. If $0 < \beta < \alpha$, then for a.e. $x \in [a,b]$ there is $C' = C'(x)$ such that $h|J_x$ is $(\beta, C', 1)$-exponential with respect to the linear measure when $J_x = x \times [c,d]$. Furthermore, there is a function $\theta(M)$, depending only on α, β, C, K_0 and Q such that $\theta(M) \to 0$ as $M \to \infty$ and such that*

$$\lambda\left(\left\{x \in [a,b] : C'(x) > M\right\}\right) \leq \theta(M).$$

Proof. We can assume that h is $(\alpha, C, 1)$-exponential for some $C > 0$. Let

$$A_K = \left\{x \in [a,b] : \lambda\left(\left\{y \in [c,d] : h(x,y) > K\right\}\right) \geq e^{-\beta K}\right\}.$$

Since $m\left(\left\{(x,y) : h(x,y) > K\right\}\right) \leq Ce^{-\alpha K}$, the Fubini theorem gives that

$$\lambda(A_K) \leq Ce^{(\beta-\alpha)K}.$$

Hence, if $B_K = A_K \cup A_{K+1} \cup A_{K+2} \cup \cdots$, then

$$\lambda(B_K) \leq C\left[1 - e^{\beta-\alpha}\right]^{-1} e^{(\beta-\alpha)K} = \theta(K). \tag{4h}$$

If $x \notin B_K$, then

$$\lambda\left(\left\{y \in [c,d] : h(x,y) > L\right\}\right) < e^{-\beta L}$$

for $L = K, K+1, K+2, \ldots$. Clearly, $h|J_x$ is (β, e^{β}, K)-exponential and hence $(\beta, (d-c)e^{\beta K}, 1)$-exponential. Since $\lambda(B_K) \leq \theta(K)$ where $\theta(K) \to 0$ as $K \to \infty$ by (4h), the lemma follows.

Finally, we will prove two simple lemmas of a more general character.

Lemma 4D. *Let X be a space with measure m. Let $h_{j1}, h_{j2}, \ldots$ for $j = 1, \ldots, n$ be measurable functions $X \to \mathbf{R}$ such that*

$$m\left(\left\{x \in X : h_{jk}(x) > M\right\}\right) \leq \theta_j(M)$$

for some functions $\theta_j(M)$ such that $\theta_j(M) \to 0$ as $M \to \infty$. Then for a.e. $x \in X$ there is a sequence $k_1 < k_2 < \cdots$ (which may depend on x) such that $h_{jk_i}(x)$ are bounded for all $i \geq 1$ and $j = 1, \ldots, n$.

Proof. Let

$$E_{Mk} = \{x \in X : h_{jk}(x) > M \text{ for some } j = 1, \ldots, n\},$$

$$E_M = \bigcup_{p=1}^{\infty} \bigcap_{k=p}^{\infty} E_{Mk}.$$

Clearly $m(E_{Mk}) \leq \theta(M) = \theta_1(M) + \cdots + \theta_n(M)$ for all M and k and hence also $m(E_M) \leq \theta(M)$. If $x \notin E_M$, then obviously there is a sequence $k_1 < k_2 < \cdots$ (depending on x) such that $x \notin E_{Mk_i}$ and hence $h_{jk_i}(x) \leq M$ for all $j \leq n$. Since $\theta(M) \to 0$ as $M \to \infty$, the lemma follows.

Lemma 4E. *Let f_k: $[a,b] \to \mathbf{C}$ be embeddings. Suppose that they are uniformly absolutely continuous, that is, given $\varepsilon > 0$, there is $\delta > 0$ such that $\lambda(f_k E) < \varepsilon$ for all k whenever $E \subset I$ is measurable and $\lambda(E) < \delta$ where λ is the linear measure. If f: $I \to \mathbf{C}$ is an embedding such that $f_k \to f$ pointwise, then f is absolutely continuous with respect to the linear measure.*

Proof. Choose $\varepsilon > 0$. Let $\delta > 0$ be a number such that if $\lambda(E) < \delta$, then $\lambda(f_k E) \leq \varepsilon$ for all k. We claim that then in fact

$$\lambda(fE) \leq \varepsilon. \tag{4i}$$

Suppose first that E is a finite union of closed intervals $I_j = [a_j, b_j]$, $j \leq n$. Suppose that $\lambda(fE) > \varepsilon$. Then there are points $x_{j0} = a_j < x_{j1} < \cdots < x_{jn_j} = b_j$ for $j = 1, \ldots, n$ such that

$$\sum_{j,i} \left| f(x_{ji}) - f(x_{j,i-1}) \right| > \varepsilon.$$

By the pointwise convergence, this is true for big k if we replace f by f_k. This contradicts the assumption that $\lambda(f_k E) < \varepsilon$. This easily implies the lemma.

5. The limit of μ-homeomorphisms

We can now prove that embeddings which are limits of μ-homeomorphisms with uniform parameters are still μ-homeomorphisms. We start with two lemmas in which the notion of the *linear dilatation* is crucial. We denote the linear dilatation of a map f at a point z by $H_f(z)$ and it is defined by the formula

$$H_f(z) = \limsup_{r \to 0} H_f(z, r)$$

when $H_f(z,r) \geq 1$ is the quotient of the maximum and minimum of $\left| f(z') - f(z) \right|$ on the circle $|z' - z| = r$.

If f is differentiable at z with a non-singular derivative, then $H_f(z)$ is just the dilatation $K_f(z)$ defined earlier. If f is a μ-homeomorphism, this happens at a.e. point z and hence H_f is (α, C, K_0)-exponential if K_f is.

Lemma 5A. *Let $w\colon G \to G'$ be a homeomorphism of two domains G and G' of $\mathbf{C}$. Let $R = (a,b) \times (c,d) \subset G$ be a quadrilateral such that $\bar{R} \subset G$ and that the area function*

$$A(y) = m\big(w\big[(a,b) \times (c,y)\big]\big)$$

has a finite derivative $A'(y_0)$ at the point y_0. If $I = I_{y_0} = (a,b) \times y_0$ and $F \subset I$ is a countable union of compact sets such that the linear dilatation $H_w(z) \le N$ for all $z \in F$, then the linear measure λ satisfies

$$\lambda(wF)^2 \le \frac{16}{\pi}(N+1)^2 A'(y_0)\lambda(F). \tag{5a}$$

Proof. If F is compact, the proof is contained in the proof Theorem IV.4.2 of [LV], starting from the fourth paragraph of the proof of this theorem and ending in the paragraph containing formula (4.6); inequality (5a) is obtained from this formula (4.6) by substituting $A'(y_0)$ for M. A limit process gives (5a) if F is a countable union of compact sets.

Lemma 5B. *Let $f_k\colon D \to \mathbf{C}$ be (α, C, K_0)-homeomorphisms and suppose that $f_k \to f$ uniformly on compact subsets of D where $f\colon D \to \mathbf{C}$ is an embedding. Then f is ACL.*

Proof. Let $R = (a,b) \times (c,d)$ and suppose that $\bar{R} \subset D$. We will prove that $f|I_y$ is absolutely continuous for a.e. $y \in (c,d)$. The proof for vertical line segments is the same.

Let $A_k(y)$ be the area function

$$A_k(y) = m\left(f_k\big[(a,b) \times (c,y)\big]\right).$$

Each A_k is differentiable with finite derivative a.e. in (a,b). Since $f_k \to f$ uniformly on $\bar{R}$, there is a constant $B > 0$ such that $A_k(d) \le B$ for all k. Let $E_{kM} = m\big(\{y : A_k'(y) \ge M\}\big)$. Then

$$B \ge \int_c^d A_k'(y)\,dy \ge M\lambda(E_{kM});$$

hence $\lambda(E_{kM}) \le B/M$ which tends to 0 as $M \to \infty$.

Now every H_{f_n} coincides with K_{f_n} a.e. and hence is (α, C, K_0)-exponential. Let $0 < \beta < \alpha$. By Lemma 4C there are functions $C_n(y)$ such that $H_{f_n}|I_y$ is $(\beta, C_n(y), 1)$-exponential for a.e. $y \in (c,d)$. In addition, there is a function θ such that $\theta(M) \to 0$ as $M \to \infty$ and that

$$\lambda\left(\{y \in (c,d) : C_n(y) > M\}\right) \le \theta(M)$$

for every n.

We can now conclude by Lemma 4D that there is a subset $E \subset (c,d)$ of full measure such that for every $y \in E$ there is a sequence $k_1 < k_2 < \cdots$ (depending on y) such that both $A'_{k_i}(y)$ and $C_{k_i}(y)$ are bounded. Removing a null-set, we can assume that each $f_k|I_y$, $y \in E$, is absolutely continuous.

Fix $y \in E$. We will show that $f|I_y$ is absolutely continuous. As we have seen, we can assume, possibly by passing to a subsequence, that there is $M > 0$ such that for all n, $A'_n(y) < M$ and that $H_{f_n}|I_y$ is $(\beta, M, 1)$-exponential.

By Lemma 4E, it suffices to show that $f_j|I_y$, $j > 0$, are uniformly absolutely continuous, $I_y = (a,b) \times y$ is as above. Denote

$$J_{ny} = \{z \in I_y : H_{f_n}(z) < \infty\},$$

$$J_{ny}(N) = \{z \in J_y : H_{f_n}(z) < N\},$$

and

$$L_{ny}(N) = J_{ny}(N+1) \setminus J_{ny}(N).$$

Since $H_{f_n}|I_y$ was exponential, J_{ny} has full measure in I_y. Obviously, $\cup_{N>0} L_{ny}(N) = J_{ny}$.

By Lemma 5A, if $F \subset J_{ny}(N)$ is a countable union of compact sets, then

$$\lambda(f_n F)^2 \le \frac{16}{\pi} M(N+1)^2 \lambda(F) \tag{5b}$$

for all n. Since every measurable subset of I_y contains a countable union of compact sets with the same linear measure, absolute continuity of $f_n|I_y$ implies that (5b) is in fact valid whenever $F \subset J_{ny}(N)$ is measurable.

In particular, since $H_{f_n}|I_y$ is $(\beta, M, 1)$-exponential, $\lambda(L_{ny}(N)) \le Me^{-\beta N}$ and hence

$$\lambda(f_n L_{ny}(N)) \le M_0(N+1)e^{-\beta N/2} \tag{5c}$$

where $M_0 = 4M/\sqrt{\pi}$. Hence there is a constant c_N, indepedent of n such that $c_N \to 0$ as $N \to \infty$ and that

$$\lambda(f_n(J_{ny} \setminus J_{ny}(N)) \le c_N. \tag{5d}$$

Since $I_y \setminus J_{ny}$ is a null-set, so is $\lambda(f_n(I_y \setminus J_{ny}))$ by the absolute continuity. Consequently, c_N of (5d) is an upper bound also for $\lambda(f_n(I_y \setminus J_y(N))$. Since $c_N \to 0$ as $N \to \infty$, this fact and (5b) imply the uniform absolute continuity of the maps $f_n|I_y$. As we have observed, this implies the lemma.

So we know that the limit map is ACL. Next we will show that the dilatation of the limit map satisfies the exponential condition and hence is a μ-homeomorphism. We start with

Lemma 5C. *Let $f_j\colon B(0,2) \to \mathbf{C}$ be (α, C, K_0)-exponential. Suppose that $f_j \to f$ where f is an affine and non-constant map of the disk $B(0,2)$. Then f is non-singular and*

$$K_f \le LK_0 \tag{5e}$$

where L is a constant depending only on C and α.

Proof. We can assume that f and each f_j fix 0 and 1. Composing f and f_j with suitable orthogonal maps, we can assume that $f(x+iy) = x + iK_f y$. Let Q be the square with center 0 and sides parallel to the coordinate axes such that the vertices of Q are on $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. We need the notion of the modulus of a quadrilateral in the sense of [LV, I.2.3] and for it we need to distinguish two pairs of opposite sides of Q, called a-sides and b-sides as in [LV, I.2.3]. We let the sides parallel to the y-axis be the a-sides of Q. Let $M(P)$ denote the modulus of a general quadrilateral in the sense of [LV, I.2.4]. Then $M(Q) = 1$ and $M(fQ) = K_f$. Thus it suffices to estimate $M(fQ)$.

By Theorem 3C, $f_j \to f$ uniformly on compact subsets. Hence the quadrilaterals f_jQ converge to fQ in the sense of [LV, I.4.9] and so

$$K_f = M(fQ) = \lim_{j\to\infty} M(f_jQ), \tag{5f}$$

by the continuity of the modulus [LV, I.4.9].

We first compose each f_j as $f_j = g_j h_j$ where g_j is K_0-quasiconformal and h_j is an $(\alpha K_0, C, 1)$-homeomorphism (see Lemma 2B) and hence an $(\alpha, C, 1)$-homeomorphism. Clearly, we can still assume that also g_j and h_j fix 0 and 1. Now the set of $(\alpha, C, 1)$-homeomorphisms $B(0,2) \to \mathbf{C}$ which fix 0 and 1 is compact by Lemma 3D. The compactness and the continuity of moduli of quadrilaterals, as expressed in equation (5f), imply

$$L = \sup M(hQ) < \infty$$

where the supremum is taken over $(\alpha, C, 1)$-homeomorphisms of $B(0,2)$ into $\mathbf{C}$ which fix 0 and 1. Since each g_j is K_0-quasiconformal,

$$M(f_jQ) = M(g_jh_jQ) \le K_0 M(h_jQ) \le LK_0.$$

Remark. In the above proof, h_j is an $(\alpha K_0, C, 1)$-homeomorphism and hence

$$m\left(\{x \in B(0,2) : K_{h_j}(x) > 1\}\right) \le Ce^{-\alpha K_0}$$

which tends to 0 as $K_0 \to \infty$ for fixed α and C. In view of the compactness properties of μ-homeomorphisms (Section 3) and the good approximation theorem to be proved later in Corollary 6C, a normal family argument would show that h_j becomes arbitrarily close to a conformal map if K_0 is big enough (and α and C are fixed). It would follow that if we regard in (5e) L also as a function of K_0, then $L \to 1$ as $K_0 \to \infty$ for fixed α and C.

Theorem 5D. *Let $f_j\colon D \to \bar{\mathbf{C}}$ be (α, C, K_0)-homeomorphisms for $j > 0$. Suppose that f_j converge towards an embedding $f\colon D \to \bar{\mathbf{C}}$ as $j \to \infty$. Then the convergence is uniform on compact subsets of D and there are $L = L(\alpha) > 1$ and $C' = C'(\alpha, C) > 0$ such that f is an $(\alpha/2L, C', LK_0)$-homeomorphism.*

Proof. By Theorem 3C the convergence is uniform on compact subsets. We know that f is ACL by Lemma 5B and so is differentiable a.e. [LV, III.3.2]. It suffices to show that it satisfies (1b).

Let E be the set of points of D such that f is differentiable in E and such that there are such sequences as in Lemma 4B, i.e. for each $x \in E$ there are a sequence $n_1 < n_2 < \cdots$ of integers and a sequence $r_i > 0$ of numbers with $r_i \to 0$ such that $f_{n_i}|B(x, r_q)$ is an $\big(\alpha/2, 1, K_1(x)\big)$-homeomorphism with respect to the measure m_{xr_q} of (4b) for all $i > q$. Here K_1 is $(\alpha/2, C', K_0)$-exponential with

$$C' = C'(\alpha, C) = 9^2 C e^{\alpha/2}/\pi\big(1 - e^{-\alpha/2}\big). \tag{5g}$$

Let $x \in E$. We will show that if the differential $Df(x) \neq 0$, then $Df(x)$ is non-singular and $K_f(x) \leq LK_1(x)$ where $L = L(\alpha)$.

Choose numbers $r_j > 0$ and integers $n_1 < n_2 < \cdots$ such that $r_j \to 0$ and that $K_{n_i}|B(x, 2r_j)$ are $\big(\alpha/2, 1, K_1(x)\big)$-exponential with respect to $m_{x,2r_i}$ in the balls $B(x, 2r_j)$ for $j \geq i$. We assume for simplicity that $x = 0 = f(x) = f_j(x)$ and define maps $g_j, g_{kj}\colon B(0,2) \to \mathbf{C}$, $j, k > 0$, by

$$g_j(y) = r_j^{-1} f(r_j y),$$
$$g_{kj}(y) = r_j^{-1} f_{n_k}(r_j y).$$

Then, as $j \to \infty$, $g_j \to Df(x)$ and, for fixed j, $g_{kj} \to g_j$ as $k \to \infty$. Furthermore, the convergences are uniform and every g_{kj} is an $\big(\alpha/2, m\big(B(0,2)\big), K_1(x)\big)$-homeomorphism if $k \geq i$.

Since for every fixed j, $g_{kj} \to g_j$ as $k \to \infty$, we can find a sequence $m_1 < m_2 < \cdots$ such that $h_j = g_{m_j j} \to Df(x)$ as $j \to \infty$. As $m_j \geq j$, each h_j is an $\big(\alpha/2, 4, K_1(x)\big)$-homeomorphism. Since $Df(x)$ is affine, we can apply Lemma 5C and conclude that $Df(x)$ is non-singular and that

$$K_{Df(x)} = K_f(x) \leq LK_1(x)$$

where $L = L(\alpha/2, 4) = L(\alpha)$. Now K_1 is $(\alpha/2, C', K_0)$-exponential, C' as in (5g), and hence LK_1 is $(\alpha/2L, C', LK_0)$-exponential.

If $Df(x) = 0$, then $K_f(x)$ is not defined. It follows that (1b) is true for a.e. x with substitution $\alpha \mapsto \alpha/2L$, $C \mapsto C'$, and $K \mapsto LK_0$.

Remark. We have formulated the theorem so as to get it as simple as possible. However, it is possible to obtain more precise information on the exponent of the limit map, at the cost of increasing other parameters. In fact any $\alpha' \in (0,\alpha)$ is an exponent of f.

In order to have this result, we have to choose first a very small $\beta \in (0,\alpha)$ (above we have chosen $\beta = \alpha/2$). Then we would have as above by Lemma 4B numbers r_i and n_i such that $f_{n_i}|B(x,2r_i)$ is a $(\beta,1,K_1)$-homeomorphism with respect to $m_{x,2r_i}$ and where K_1 is $(\alpha-\beta,C',K_0)$-exponential with $C' = C'(\alpha,\beta,C)$. Now the limit $Df(x)$ of the maps h_j defined above is $LK_1(x)$-quasiconformal and by the remark after the preceding lemma, $L = L(\alpha-\beta,C',K_0) = L(\alpha,\beta,C,K_0)$ where $L \to 1$ as $K_0 \to \infty$ if other parameters are fixed. Thus we can obtain by a proper choice of first β and then, after having chosen β, of K_0, that L is arbitrarily close to 1 and as f is an $\big((\alpha-\beta)/L, C', LK_0\big)$-homeomorphism, we have that any $\alpha' \in (0,\alpha)$ is a exponent of f.

6. Dilatation estimates and the good approximation

Knowing that non-degenerate limits of K-quasiconformal maps are still K-quasiconformal, Theorem 5D is somewhat unsatisfactory in the respect that we may have to change the parameters of the limit map. However, there are situations where we can obtain more precise information on the dilatation of the limit map. We present here some results in this direction as well as the related good approximation theorem.

We start with the following lemma which is a special case of the more general Theorem 6B. If all f_j are K_1-quasiconformal for some fixed number K_1, then this is a 2-dimensional version of Lemma B2 of [T] proved in much the same manner.

Lemma 6A. *Let f_j and f be embeddings of a domain D of $\mathbf{C}$ into $\mathbf{C}$ such that $f_j \to f$ and that f_j are (α,C,K_0)-homeomorphisms. Suppose that for some number $K \ge 1$,*

$$m\big(\{x \in D : K_{f_j}(x) > K\}\big) \to 0$$

as $j \to \infty$. Then f is K-quasiconformal.

Proof. We first assume that $\alpha > 3/\theta$ where $\theta > 0$ is the constant of Lemma 3E.

Let P be a rectangular quadrilateral of fD, that is P is a geometric quadrilateral whose sides intersect orthogonally and where we have in addition two pairs of opposite sides called a-sides and b-sides like in [LV, I.2.3]. The pre-image $f^{-1}P$ is a quadrilateral in the sense of [LV, I.2.3] with a-sides and b-sides that are pre-images of the a-sides and b-sides of P. If $M(P)$ and $M(f^{-1}P)$ denote the moduli of the quadrilaterals [LV, I.2.4], we will show that

$$M(f^{-1}P) \le KM(P) \tag{6a}$$

for all such P which will imply that f^{-1}, and hence f, is K-quasiconformal [LV, Theorem IV.3.3].

We will use the characterization of the modulus of a quadrilateral by means the modulus of path families. Let $\mathcal{C}$ be the family of arcs in P joining the a-sides of P. Then the modulus $M(\mathcal{C})$ of $\mathcal{C}$ and the modulus $M(P)$ of P coincide (cf. [LV, I.4.1 and III.4.1]).

Let s_a be the length of the a-sides of P and s_b the length of the b-sides of P. Then $M(P) = s_a/s_b$ and if we set

$$\varrho(x) = s_b^{-1}$$

for $x \in P$, we have, when λ_γ is the linear measure on γ,

$$\int_\gamma \varrho \, d\lambda_\gamma \geq 1 \tag{6b}$$

for all locally rectifiable $\gamma \in \mathcal{C}$ and

$$\int_P \varrho^2 \, dm = \frac{s_a}{s_b} = M(P) = M(\mathcal{C}). \tag{6c}$$

Let $Q_j = f_j^{-1}P$ and $Q = f^{-1}P$. Since $f_j \to f$ uniformly on compact sets by Theorem 3C, Q_j converge to $Q = f^{-1}P$ and the continuity of the modulus of a quadrilateral (cf. (5f)) implies that $M(Q_j) \to M(Q)$ as $j \to \infty$. We define a function ϱ_j in Q_j by setting $\varrho_j(x) = \infty$ if x is not a regular point of f_j and if f_j is differentiable at x, we set

$$\varrho_j(x) = \varrho\big(f_j(x)\big)\big|Df_j(x)\big| = s_b^{-1}\big|Df_j(x)\big|$$

where $\big|Df_j(x)\big|$ is the operator norm of the derivative of f_j at x. We will need the fact that

$$\big|Df_j(x)\big|^2 = J_{f_j}(x)K_{f_j}(x) \tag{6d}$$

a.e. in Q_j when J_{f_j} is the Jacobian.

Let $\mathcal{C}_j = f_j^{-1}\mathcal{C}$ which is the set of arcs in Q_j connecting the a-sides of Q_j. If $\gamma \in \mathcal{C}_j$, then $f_j\gamma \in \mathcal{C}$, and (6b) easily implies that

$$\int_\gamma \varrho_j \, d\lambda_\gamma \geq 1$$

for all locally rectifiable $\gamma \in \mathcal{C}_j$. Using (6d) we obtain

$$\begin{aligned} M(Q_j) = M(\mathcal{C}_j) &\leq \int_{Q_j} \varrho_j^2 \, dm = \int_{Q_j} s_b^{-2} J_{f_j}(x)K_{f_j}(x) \, dm \\ &= \int_P s_b^{-2} L_j(x) \, dm \end{aligned} \tag{6e}$$

where $L_j(x) = K_{f_j}\left(f_j^{-1}(x)\right)$. Let

$$E_{jn} = \{x \in P : K < L_j(x) \le n\},$$
$$F_{jn} = \{x \in P : n < L_j(x) \le n+1\},$$

and substitute these into (6e) to obtain

$$M(Q_j) \le \int_P s_b^{-2} K\, dm + \int_{E_{jn}} s_b^{-2} n\, dm + \sum_{p=n}^{\infty} \int_{F_{jp}} s_b^{-2}(p+1)\, dm$$

when $n > K_0$. The value of the first integral is $K(s_a/s_b) = KM(P)$ and hence

$$M(Q_j) \le KM(P) + s_b^{-2} nm(E_{jn}) + \sum_{p=n}^{\infty} s_b^{-2}(p+1)m(F_{jp}).$$

If $p > K_0$, then $m(f_j^{-1}F_{jp}) \le Ce^{-\alpha p}$. Since $\alpha > 3/\theta$, it follows by Lemma 3E that $m(F_{jp}) \le C'p^{-3}$ for some $C' > 0$ and hence

$$M(Q_j) \le KM(P) + s_b^{-2} nm(E_{jn}) + \sum_{p=n}^{\infty} C'(p+1)p^{-3}.$$

For fixed n, $m(f_j^{-1}E_{jn}) \to 0$ as $j \to \infty$ by assumption and hence so does $m(E_{jn})$ by Lemma 3E. Hence keeping n fixed and letting $j \to \infty$, the middle term in the above sum vanishes and the left hand side tends to $M(f^{-1}P)$ by the continuity of the modulus of quadrilaterals (cf. (5f)). Hence

$$M(f^{-1}P) \le KM(P) + \sum_{p=n}^{\infty} C'(p+1)p^{-3}$$

for every n and letting $n \to \infty$ we obtain (6a).

If $\alpha < 3/\theta$, we choose a number $K' > K$ such that $\alpha K' > 3/\theta$ and find the decomposition of f_j as $f_j = g_j(h_j|D)$ (see Lemma 2B) where g_j is K'-quasiconformal and h_j: $\mathbf{C} \to \mathbf{C}$ is conformal outside D and each $h_j|D$ is an $(\alpha K', C, 1)$-homeomorphism. We can assume that the maps h_j fix 0 and 1 and so we can pass by Lemma 3D to a subsequence so that there is a limit $h = \lim_{j\to\infty} h_j$ and hence also $g_j \to g = fh^{-1}$. Formula (2d) of Lemma 2B shows that

$$m\left(\{y \in D : K_{h_j}(y) > 1\}\right) = m\left(\{y \in D : K_{f_i}(y) > K'\}\right) \to 0$$

as $j \to \infty$ and hence $h = \lim_{j\to\infty} h_j$ is conformal as we have just proved. The maps g_j are K'-quasiconformal and furthermore, the uniform absolute continuity of the maps h_j (Lemma 3E) shows that

$$m\left(\{x \in h_j D : K_{g_j}(x) > K\}\right) = m\left(h_j\{x \in D : K_{f_j}(x) > K\}\right) \to 0$$

as $j \to \infty$ and it follows from the first part of the proof (or from [T, Lemma B2]) that $g = \lim_{j\to\infty} g_j$ is K-quasiconformal. Hence $f = gh$ is K-quasiconformal.

Now we can prove

Theorem 6B. *Let $f_j, f\colon D \to \mathbf{C}$ be embeddings such that all f_j are (α, C, K_0)-homeomorphisms and that $f = \lim_{j\to\infty} f_j$. Suppose that there is a measurable function $K\colon D \to [1,\infty)$ such that*

$$m\left(\{x \in D : K_{f_j}(x) > K(x) + \varepsilon\}\right) \to 0 \tag{6f}$$

for every $\varepsilon > 0$ as $j \to \infty$. Then f is a μ-homeomorphism such that $K_f(x) \leq K(x)$ a.e. in D.

Proof. The initial setup is almost the same as in the proof of Theorem 5D. Let E be the set of points $x \in D$ such that f is differentiable with a non-singular derivative at x, that K is approximately continuous (for this notion see [F, 2.9]) at x and that the following condition is true: There are numbers α', C' and K_0', sequences $n_1 < n_2 < \cdots$ and numbers $r_i > 0$ with $\lim_{i\to\infty} r_i = 0$ such that $f_{n_i}|B(x, 2r_q)$ is an (α', C', K_0')-homeomorphism with respect to the measure $m_{x,2r_i}$ of (4b) if $i > q$. Remembering that measurable maps are approximately continuous a.e., cf. [F, 2.9], the set E has full measure in D by Lemma 4B and Theorem 5D.

We will show that $K_f(x) \leq K(x)$ if $x \in E$ which implies the theorem. Assume that $x = 0 = f(x) = f_j(x)$ and define the maps g_j and g_{kj} of $B(x,2) = B(0,2)$ as in the proof of Theorem 5D. So we can again find a sequence $p_1 < p_2 < \cdots$ such that setting

$$h_j = g_{p_j, j},$$

then h_j is a map of $B(0,2)$ such that $h_j \to Df(x)|B(0,2)$ uniformly; in addition, since we have (6f) and K is approximately continuous at x, it is possible to choose p_j in such a way that, as $j \to \infty$,

$$m\left(\{y \in B(0,2) : K_{h_j}(y) > K(x) + \varepsilon\}\right) \to 0 \tag{6g}$$

for every $\varepsilon > 0$. Lemma 6A implies that the limit $Df(x)$ of the maps h_j is $\bigl(K(x) + \varepsilon\bigr)$-quasiconformal for every $\varepsilon > 0$ and hence $K(x)$-quasiconformal. So $K_f(x) \leq K(x)$.

Finally we present the good approximation theorem for μ-homeomorphisms.

Corollary 6C. *Let f_j be (α, C, K_0)-homeomorphisms of a domain D. If $f_j \to f$ where f is an embedding of D and if there is a complex map μ of D such that*

$$\mu_{f_j} \to \mu \tag{6h}$$

in measure as $j \to \infty$, then f is a μ-homeomorphism such that $\mu_f = \mu$ a.e. in D.

Proof. Let E be defined as in the proof of Theorem 6B with the exception that the condition that K is approximately continuous at x is changed so that μ is approximately continuous at x. It suffices to show that $\mu_f(x) = \mu(x)$ for $x \in E$.

Let $K = K_\mu$ be defined by (2b), i.e. K is the dilatation corresponding to μ. Then K_{f_i} satisfy (6f) for $K = K_\mu$ and K is approximately continuous at points where μ is approximately continuous. If $x \in E$ is such a point and $\mu(x) = 0$, then $K(x) = 1$ and hence $K_f(x) \le K(x) = 1$ by Theorem 6B (see the first paragraph of the proof where we gave the exact conditions for the validity of the inequality). That is, $\mu_f(x) = 0 = \mu(x)$.

So if $x \in E$ and $\mu(x) = 0$, then we are done. If $\mu(x) \neq 0$, then we can obtain this situation as follows. Replace each f_j by $f_j h$ where h is a suitable affine map, chosen so that $h(x) = x$ and that $\mu^*(x) = 0$ if μ^* is given by the following rule, obtained by imitating the composition rule for the complex dilatation [LV, IV.5.2]:

$$\text{(6i)} \qquad \mu^*(y) = \frac{\mu_h + \mu\big(h(y)\big)\exp(-2i\arg h_z)}{1 + \mu_h\mu\big(h(y)\big)\exp(-2i\arg h_{\bar z})};$$

the condition that $\mu^*(x) = 0$ is equivalent to the condition that $\mu_{h^{-1}} = \mu(x)$. Note that since h is affine μ_h, h_z and $h_{\bar z}$ are constants. Then $f_j h \to fh$ and $\mu_{f_j h} \to \mu^*$ in measure. Define the set E as in the beginning of this proof but replacing μ by μ^*, f by fh and f_j by $f_j h$. Clearly, still $x \in E$ and furthermore $\mu^*(x) = 0$. Hence $\mu_{fh}(x) = 0$ as we have just proved. Consequently $\mu_f(x) = \mu_{fhh^{-1}}(x) = \mu_{h^{-1}} = \mu(x)$.

7. Linear dilatation

It is well known that quasiconformality can be characterized by means of the linear dilatation which we have defined in the beginning of Section 5 [LV, Theorem IV.4.2]. We can obtain something similar in this direction for μ-homeomorphisms although complete characterization of μ-homeomorphisms by means of the linear dilatation seems unlikely as probably the linear dilatation $H_f(x)$ can be infinite in an uncountable set for a given μ-homeomorphism f.

Theorem 7. *Let $f\colon D \to \mathbf{C}$ be a map such that $H_f(z)$ is finite outside a countable set and that H_f is (α, C, K_0)-exponential. Then f is an (α, C, K_0)-homeomorphism.*

Proof. We must only show that f is ACL. We only recall the proof of Lemma 5B. Let $R = (a,b) \times (c,d) \subset D$ be a quadrilateral and $I_y = (a,b) \times y$. By Lemma 4C, $H_f|I_y$ is β-exponential if $0 < \beta < \alpha$ for a.e. y. Fix such y and assume in addition that $H_f(z)$ is finite at every $z \in I_y$. Formulas (5b) and (5d) of Lemma 5B show that there is a function $\nu(t)$, depending only on the parameters of $H_f|I_y$, such that $\nu(t) \to 0$ as $t \to 0$ and that

$$\text{(7a)} \qquad \lambda(fE) \le \nu\big(\lambda(E)\big)$$

if $E \subset I_y$ is a countable union of compact sets. It follows now like in the latter part of the proof of Theorem IV.4.2 in [LV] that (7a) is in fact true for all measurable $E \subset I_y$.

Remark. In order to conclude that f is ACL, the exponential condition for H_f might be too strong. If, for instance, H_f is finite outside a countable set and

$$m\left(\{x \in D : H_f(x) > K\}\right) < CK^{-\alpha} \tag{7b}$$

if $K \geq K_0$ for some $K_0 \geq 1$, $C > 0$ and $\alpha > 3$, then this would seem sufficient for the ACL property. By a variant of Lemma 4D this would imply that H_f would satisfy a condition analogous to (7b) on almost all lines parallel to the coordinate axes for some power $\alpha > 2$ which would suffice in the variant of (5c) needed.

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Received 10 October 1989