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MONOTONE FUNCTIONS AND EXTREMAL FUNCTIONS FOR CONDENSERS IN $\overline{\mathbf{R}}^n$

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0. Introduction

In this paper, we study some properties of monotone functions and extremal functions for capacities of condensers. After introducing some notations and preliminary results in Section 1, we shall give the construction of a monotone function and prove an oscillation lemma. In Section 2 we also prove that a relative quasiextremal distance exceptional set with n-dimensional measure zero is removable for monotone ACL-functions. These are generalizations and modifications of some results due to A. Aseev and A. Syčev [AS]. In Section 3, by using the results obtained in Section 2 and some results on conformally invariant variational integrals, we prove a general existence and uniqueness theorem for the extremal function of the conformal capacity of a condenser R and study the boundary behavior of the extremal. The corresponding results for the special case where R is a ring are due to F.W. Gehring [G2] and G.D. Mostow [M6]. In Section 4 we establish the corresponding results for the extremal functions of p-capacities of condensers. Some results obtained here are needed to characterize quasiextremal distance domains and null sets for extremal distances in $\overline{\mathbf{R}}^n$ (see [Y] for applications).

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1. Notation and preliminary results

We use the following notation for Euclidean n-space $\mathbf{R}^n$ and its one point compactification $\overline{\mathbf{R}}^n$. Given $x \in \mathbf{R}^n$ and $0 < r < \infty$, we let $B^n(x,r)$ denote the open n-ball with center x and radius r and $S^{n-1}(x,r)$ its boundary. We also let $e_1, \ldots, e_n$ denote the unit vectors in the directions of the rectangular coordinate axes in $\overline{\mathbf{R}}^n$.

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1.1. ACL-functions and Sobolev spaces. Suppose D is an open set in $\mathbf{R}^n$. A function $u\colon D \to \mathbf{R}^1$ is said to be ACL or to be in the class ACL(D) if u is continuous in D and if for each closed n-interval Q in D, u is absolutely continuous on almost every line segment in Q parallel to the coordinate axes. When $\infty \in D$ we say that u is in ACL(D) if u is in $C(D) \cap \mathrm{ACL}(D \setminus \{\infty\})$. It is well-known that an ACL-function has partial derivatives almost everywhere and that the ACL-property is a local property. Let $1 < p < \infty$. If u is in ACL(D) and ∇u is locally L^p-integrable or L^p-integrable in D, then u is said to be in $\mathrm{ACL}^p_{\mathrm{loc}}(D)$ or $\mathrm{ACL}^p(D)$, respectively. We let $\mathrm{ACL}^p_0(D)$ denote the closure of $C_0^\infty(D)$ in $\mathrm{ACL}^p(D)$ under the norm

$$\|u\|_{\mathrm{ACL}^p(D)} = \|\nabla u\|_{L^p(D)} = \Big(\int_D |\nabla u|^p\, dm\Big)^{1/p}. \tag{1.2}$$

We further let $W^1_p(D)$ and $W^1_{p,\mathrm{loc}}(D)$ denote the Sobolev space and the local Sobolev space in D, respectively, and $W^1_{p,0}(D)$ the closure of $C_0^\infty(D)$ in $W^1_p(D)$. Another fact we need about ACL functions and Sobolev spaces is that

$$\mathrm{ACL}^p_{\mathrm{loc}}(D) = C(D) \cap W^1_{p,\mathrm{loc}}(D) \tag{1.3}$$

for any D in $\mathbf{R}^n$. For the proof of (1.3) and more details about Sobolev spaces and ACL-functions we refer the reader to [M3] and [M5].

1.4. Conformal capacity and moduli of curve families. Suppose that D is an open set in $\overline{\mathbf{R}}^n$, that F_0 and F_1 are two disjoint compact sets in $\overline{D}$ and that $W(F_0, F_1; D)$ is the set of all real valued functions u such that

(1) u is continuous in $D \cup F_0 \cup F_1$,
(2) $u(x) \le 0$ if $x \in F_0$ and $u(x) \ge 1$ if $x \in F_1$,
(3) u is ACL in D.

Then the *p-capacity* of F_0 and F_1 relative to D, denoted by $\mathrm{cap}_p(F_0, F_1; D)$, is defined as

$$\mathrm{cap}_p(F_0, F_1; D) = \inf \int_D |\nabla u|^p\, dm, \tag{1.5}$$

where the infimum is taken over all u in $W(F_0, F_1; D)$.

1.6. Remark. If u is in $W(F_0, F_1; D)$, then

$$v(x) = \begin{cases} u(x) & \text{if } 0 \le u(x) \le 1, \\ 0 & \text{if } u(x) \le 0, \\ 1 & \text{if } u(x) \ge 1 \end{cases}$$

is also a function in $W(F_0, F_1; D)$ such that $0 \le v(x) \le 1$, $v(x) = i$ on F_i for $i = 0, 1$, and

$$\int_D |\nabla v|^p\, dm \le \int_D |\nabla u|^p\, dm$$

for $0 < p < \infty$. Therefore, in (1.5) one can replace $W(F_0, F_1; D)$ by a subclass $W' = W'(F_0, F_1; D)$ without changing the capacity, where $W'(F_0, F_1; D)$ is the set of all functions u in $W(F_0, F_1; D)$ such that $0 \le u(x) \le 1$, $u(x) = 0$ if $x \in F_0$ and $u(x) = 1$ if $x \in F_1$. A function u is said to be *admissible* for $\mathrm{cap}_p(F_0, F_1; D)$ if u is in $W'(F_0, F_1; D)$. The n-capacity $\mathrm{cap}_n(F_0, F_1; D)$ is called the *conformal capacity*, since it is conformally invariant, and it is usually denoted by $\mathrm{cap}(F_0, F_1; D)$.

Furthermore, we let $\Delta(F_0, F_1; D)$ denote the family of curves joining F_0 and F_1 in D. Given a curve family Γ we let $\mathrm{mod}_p(\Gamma)$ denote its p-modulus. We denote the n-modulus by $\mathrm{mod}(\Gamma)$ instead of $\mathrm{mod}_n(\Gamma)$. For the definition and basic facts about modulus, we refer the reader to [G1] and [V]. The next lemma is due to J. Hesse [H, Lemma 5.2 and Theorem 5.5].

1.7. Lemma. *If F_0 and F_1 are two disjoint compact sets in $\overline{D}$, then*

$$\mathrm{cap}_p(F_0, F_1; D) \ge \mathrm{mod}_p\left(\Delta(F_0, F_1; D)\right). \tag{1.8}$$

Furthermore, if F_0 and F_1 lie in D, then (1.8) holds with equality.

1.9. Condensers and their capacities. A *condenser* is a domain in $\overline{\mathbf{R}}^n$ whose complement consists of two disjoint compact sets F_0 and F_1. It is usually denoted by $R(F_0, F_1)$ or R. We say that R is a *ring* if, in addition, F_0 and F_1 are connected. A compact set F is said to be *nondegenerate* if it contains a nondegenerate component. A condenser R is said to be nondegenerate if F_0 and F_1 are nondegenerate. The p-capacity of R, denoted by $\mathrm{cap}_p(R)$, is defined as the p-capacity of F_0 and F_1 relative to $\overline{\mathbf{R}}^n$, that is

$$\mathrm{cap}_p(R) = \mathrm{cap}_p(F_0, F_1; \overline{\mathbf{R}}^n).$$

The conformal capacity of R is denoted by $\mathrm{cap}(R)$.

1.10. Relative QED exceptional set. A compact subset E in $\overline{\mathbf{R}}^n$ is said to be an *M-QED exceptional set relative to a domain D*, $1 \le M < \infty$, if $E \subset D$ and if for each pair of disjoint continua F_0 and F_1 in $D \setminus E$

$$\mathrm{mod}\left(\Delta(F_0, F_1; D)\right) \le M \,\mathrm{mod}\left(\Delta(F_0, F_1; D \setminus E)\right). \tag{1.11}$$

This definition is a generalization of the concept of QED exceptional sets introduced by Gehring and Martio [GM]. If $D = \overline{\mathbf{R}}^n$, then a relative M-QED exceptional set E is an M-QED exceptional set and $\overline{\mathbf{R}}^n \setminus E$ is an M-QED domain.

1.12. Let D be a domain in $\overline{\mathbf{R}}^n$ and $F \subset D$ be a compact set. We say that $F \cap \partial D$ is *locally non-isolated* if for each $x_0 \in F \cap \partial D$ there are arbitrarily small neighborhoods U of x_0 such that each component V of $U \cap D$ satisfies $\overline{V} \cap F \ne \emptyset$. It is easy to see that if D is locally connected at each point of $F \cap \partial D$ (see [V, 17.5] for definition), then $F \cap \partial D$ is locally non-isolated. But the converse is false.

2. Monotone functions

In this section we study the construction, distortion and extension of monotone ACL-functions.

2.1. Definition. A real function u defined in an open set $D \subset \overline{\mathbf{R}}^n$ is said to be *monotone above relative to a compact set* $F \subset \overline{D}$, if for any $\varepsilon > 0$ each point $x_0 \in D$ can be joined to the set F by a curve γ in D such that

$$u(x) \leq u(x_0) + \varepsilon$$

for all $x \in \gamma$. A function u is said to be *monotone relative to* F, if both u and $-u$ are monotone above relative to F.

2.2. Remarks. In the above definition, by a curve γ in D joining x_0 to F we mean that γ: $[0,1) \to \overline{\mathbf{R}}^n$ is a continuous map with $\gamma(0) = x_0$, $\gamma(t) \in D$ for all $t \in [0,1)$ and $\overline{\gamma} \cap F \neq \emptyset$. This definition is essentially equivalent to Lebesgue's definition of monotonicity. More precisely, we have the following lemma.

2.3. Lemma. *Suppose that D is a domain and that u is a continuous function in D. Then u is monotone in D in the sense of Lebesgue, i.e.*

$$\sup_{\partial\Delta} u = \sup_{\Delta} u, \qquad \inf_{\partial\Delta} u = \inf_{\Delta} u \tag{2.4}$$

for all domains Δ with $\overline{\Delta} \subset D$, if and only if u is monotone in D relative to a non-empty compact set $F \subset \partial D$.

Proof. We first assume that u is monotone in D in the sense of Lebesgue. For each $x_0 \in D$ and $\varepsilon > 0$, we let Δ denote the component of the open set $\{x \in D : u(x) > u(x_0) - \varepsilon\}$ containing x_0. One can show that Δ is a subdomain of D with $x_0 \in \Delta$ and $\partial\Delta \cap \partial D \neq \emptyset$. Thus $-u$ is monotone above in D relative to $F = \partial D$. In a similar manner, one can show that u is monotone above relative to F.

Next we assume that u is monotone in D relative to a compact set F in ∂D. Let Δ be any domain in D with $\overline{\Delta} \subset D$. Then for any $x_0 \in \Delta$ and $\varepsilon > 0$, there exists a curve γ in D joining x_0 to F such that

$$u(x) \geq u(x_0) - \varepsilon$$

for all $x \in \gamma$. Thus

$$\sup_{\partial\Delta} u \geq u(x_0) - \varepsilon,$$

since $\gamma \cap \partial\Delta \neq \emptyset$, and letting $\varepsilon \to 0$ we obtain $\sup_{\partial\Delta} u \geq u(x_0)$. Hence

$$\sup_{\partial\Delta} u \geq \sup_{\Delta} u$$

which implies the first equality in (2.4). Similarly, one can show that

$$\inf_{\partial\Delta} u \leq u(x) + \varepsilon$$

for any $x \in \Delta$ and $\varepsilon > 0$. This implies the second equality in (2.4). □

2.5. Theorem. *Suppose that D is a domain in $\overline{\mathbf{R}}^n$, F a compact set in $\overline{D}$ and u a function which is bounded in $D \cup F$ and ACL^n in D. Then there exists a function u' defined in $D \cup F$ such that*
(a) *$u' = u$ on F and $u' \in \mathrm{ACL}^n(D)$ with*

$$\int_D |\nabla u'|^n dm \leq \int_D |\nabla u|^n dm, \tag{2.6}$$

(b) *u' is monotone relative to F,*
(c) *if $F \cap \partial D$ is locally non-isolated and if u is in $C(D \cup F)$, then u' is also in $C(D \cup F)$.*

Proof. Let $|u| \leq M < \infty$ and $\{r_k\}$ be an ordering of the rationals on the interval $[-M, M]$. Using Lebesgue's method (see [AS], [G2, p. 359] and [M5, 4.3.3]), we first construct a sequence of functions $\{u_k\}$ as follows.

Set $u_0 = u$. If u_{k-1} has been constructed, then let

$$G_k = \{x \in D : u_{k-1}(x) > r_k\} \tag{2.7}$$

and let D_k be the union of all components of G_k whose closures do not intersect F. Set

$$u_k(x) = \begin{cases} u_{k-1}(x) & \text{if } x \in F \cup D \setminus D_k, \\ r_k & \text{if } x \in D_k. \end{cases} \tag{2.8}$$

The sequence $\{u_k\}$ is monotone decreasing and converges to the limit function v in $D \cup F$. Then applying the above process to $-v$, we obtain v' and set $u' = -v'$. We next show that u' has the desired properties.

For the proof of (a) it suffices to show that $v \in \mathrm{ACL}^n(D)$ with

$$\int_D |\nabla v|^n dm \leq \int_D |\nabla u|^n dm. \tag{2.9}$$

From the definition of $\{u_k\}$ and the assumption on u, one can show, by induction on k, that

$$\sup_{x,y \in B} |u_k(x) - u_k(y)| \leq \sup_{x,y \in B} |u(x) - u(y)| \tag{2.10}$$

for any connected set B in D and all k, that u_k is in $\mathrm{ACL}(D)$ and that

$$|\nabla u_k| \leq |\nabla u| \tag{2.11}$$

a.e. in D for all k. It follows from (2.10) that v is continuous in D.

Next by (2.11) and the hypothesis that $u \in \mathrm{ACL}^n(D)$,

$$\int_D |\nabla u_k|^n dm \le \int_D |\nabla u|^n dm < \infty.$$

Hence there is a subsequence of $\{u_k\}$, denoted again by $\{u_k\}$, such that $\{\nabla u_k\}$ converges weakly in $L^n(D)$ to a vector function $f = (f_1, \dots, f_n)$. Since u_k is $\mathrm{ACL}(D)$ for all k, the same argument as in [G2, p. 362] shows that v is ACL in D and that $\nabla v = f$ a.e. in D. Hence (2.9) follows from the fact

$$\int_D |\nabla v|^n dm = \int_D |f|^n \, dm \le \limsup_{k\to\infty} \int_D |\nabla u_k|^n dm \le \int_D |\nabla u|^n dm.$$

The same reasoning as in [AS] shows that (b) is true.

Finally for the proof of (c), it suffices to show that v is continuous at each point of $F \cap \partial D$. For this we may assume that $F \cap \partial D$ lies in $\mathbf{R}^n$.

Let $x_0 \in F \cap \partial D$. By the assumption, for any $r > 0$, there is a neighborhood U of x_0 such that $U \subset B^n(x_0, r)$ and that each component V of $U \cap D$ satisfies $\overline{V} \cap F \neq \emptyset$. Thus, as in the proof of (2.10), one can prove that

$$\sup_{U\cap D} \left|u_k(x) - u(x_0)\right| \le \sup_{U\cap D} \left|u(x) - u(x_0)\right| \tag{2.12}$$

for all k and the continuity of v at x_0 follows from (2.12) and the continuity of u at x_0. This completes the proof of Theorem 2.5. □

Next we establish an oscillation lemma for the monotone ACL-functions which will be needed in what follows.

2.13. Lemma. *Suppose that E is an M-QED exceptional set relative to a domain D in $\overline{\mathbf{R}}^n$. Let G be any open set in $\overline{\mathbf{R}}^n$ and F a compact set in $\overline{G}$. If a function u in $\mathrm{ACL}(G \setminus E)$ is monotone relative to F, then there exists a constant $t > 4$ depending only on n and M such that*

$$\left|u(x_1) - u(x_0)\right| \le \Big(\int_{B^n(x_0,tr)\setminus E} |\nabla u|^n dm\Big)^{1/n} \tag{2.14}$$

for all x_0, x_1 in $D\cap G\setminus(E\cup F)$ with $B^n(x_0,tr) \subset D\cap G\setminus F$, where $r = |x_0 - x_1|$.

Proof. Let x_0, x_1 be in $D \cap G \setminus (E \cup F)$ with $B^n(x_0, tr) \subset D \cap G \setminus F$ and set

$$r_1 = 2r, \qquad r_2 = \tfrac{1}{2}R, \qquad R = tr,$$

where $r = |x_0 - x_1|$ and $t > 4$ is a constant to be determined. Without loss of generality, we may assume that

$$\tfrac{1}{3}\big(u(x_1) - u(x_0)\big) = \delta > 0. \tag{2.15}$$

For any $0<\varepsilon<\delta$, by the monotonicity of $u(x)$, we can choose curves γ_0 and γ_1 in $G\setminus E$ joining x_0 and x_1 to F, respectively, such that

$$u(x)\le u(x_0)+\varepsilon \tag{2.16}$$

for all $x\in\gamma_0$ and that

$$u(x)\ge u(x_1)-\varepsilon \tag{2.17}$$

for all $x\in\gamma_1$. For $i=0,\ 1$, let $x_i^{(1)}$ be the last point where γ_i meets $S^{n-1}(x_0,r_1)$, let $x_i^{(2)}$ be the first point where γ_i meets $S^{n-1}(x_0,r_2)$ after $x_i^{(1)}$, and denote the subcurve of γ_i from $x_i^{(1)}$ to $x_i^{(2)}$ by γ_i'. Then $\gamma_0'\cap\gamma_1'=\emptyset$ by (2.15), (2.16) and (2.17). Next let

$$D_1=\{x: r<|x-x_0|<R\},\qquad D_2=\{x: r_1<|x-x_0|<r_2\},$$

and let Γ be the curve family each member of which contains a subcurve connecting the spheres $S^{n-1}(x_0,r_1)$ and $S^{n-1}(x_0,r)$ or $S^{n-1}(x_0,r_2)$ and $S^{n-1}(x_0,R)$. It is easy to see that

$$\Delta(\gamma_0',\gamma_1';D\setminus E)\subset\Delta(\gamma_0',\gamma_1';D_1\setminus E)\cup\Gamma. \tag{2.18}$$

Since the modulus of a curve family joining two spheres $S^{n-1}(x_0,a)$ and $S^{n-1}(x_0,b)$ is bounded above by $c_n\big(\log(b/a)\big)^{1-n}$ ([V,7.5]), by (2.18)

$$\operatorname{mod}\big(\Delta(\gamma_0',\gamma_1';D\setminus E)\big)\le\operatorname{mod}\big(\Delta(\gamma_0',\gamma_1';D_1\setminus E)\big)+2c_n(\log 2)^{1-n} \tag{2.19}$$

where c_n is a number depending only on n. On the other hand, by Theorem 10.12 of [V] and the hypothesis that E is an M-QED exceptional set relative to D,

$$\operatorname{mod}\big(\Delta(\gamma_0',\gamma_1';D\setminus E)\big)\ge\frac{1}{M}\operatorname{mod}\big(\Delta(\gamma_0',\gamma_1';D)\big)\ge\frac{c_n'}{M}\log\frac{r_2}{r_1}, \tag{2.20}$$

where c_n' is also a number depending only on n. Hence

$$\operatorname{mod}\Big(\Delta(\gamma_0',\gamma_1';D_1\setminus E)\Big)\ge\frac{c_n'}{M}\log\left(\tfrac14 t\right)-2c_n(\log 2)^{1-n}. \tag{2.21}$$

The right side of (2.21) tends to ∞ when t tends to ∞. Therefore, we can choose the positive constant t depending only on n and M such that $t>4$ and

$$\operatorname{mod}\big(\Delta(\gamma_0,\gamma_1;D_1\setminus E)\big)\ge\operatorname{mod}\big(\Delta(\gamma_0',\gamma_1';D_1\setminus E)\big)\ge 1. \tag{2.22}$$

Next set

$$v(x)=\frac{u(x)-u(x_0)-\varepsilon}{u(x_1)-u(x_0)-2\varepsilon}.$$

Then v is in $W(\gamma_0,\gamma_1;D_1\setminus E)$. Thus, by Lemma 1.7 and (2.22), we obtain

$$\int_{D_1\setminus E}|\nabla v|^n dm\ge\operatorname{cap}(\gamma_0,\gamma_1;D_1\setminus E)\ge\operatorname{mod}\big(\Delta(\gamma_0,\gamma_1;D_1\setminus E)\big)\ge 1. \tag{2.23}$$

Letting $\varepsilon\to 0$ in (2.23) yields (2.14) as desired. □

The main result in this section is the following extension theorem for monotone ACL-functions.

2.24. Theorem. *Suppose that E is an M-QED exceptional set relative to a domain D in $\overline{\mathbf{R}}^n$ with $m(E)=0$, that G is any open set in $\overline{\mathbf{R}}^n$ and that F is a compact set in $\overline{G}$. Suppose also that a function u is bounded and ACL^n in $G\setminus E$ and that u is monotone relative to F. Then u can be extended to be a function u^* in $\mathrm{ACL}^n\,(G\setminus(E\cap F))$ which is also monotone relative to F.*

Proof. For each $P\in G\cap E\setminus F$, Lemma 2.13 and the absolute continuity of Lebesgue integrals imply that there is a neighborhood V of P such that u is uniformly continuous in $V\setminus E$. Thus u can be extended to P continuously and hence it can be extended to be a continuous function u^* in $G\setminus(E\cap F)$. Furthermore, since $m(E)=0$, $\nabla u^*=\nabla u$ a.e. in G and it follows from (2.14) that

$$|u^*(y)-u^*(x)|\le|y-x|t\Big(\int_B f(x+trz)\,dm(z)\Big)^{1/n} \tag{2.25}$$

for all x, y in $D\cap G\setminus F$ with $B^n(x,tr)\subset D\cap G\setminus F$, where $B=B^n(0,1)$, $r=|x-y|$,

$$f(x)=|\nabla u^*(x)|^n=|\nabla u(x)|^n$$

a.e. in G and t is as in (2.14).

Next we show that u^* is ACL in $G\setminus(E\cap F)$. Since the ACL-property is a local property, it suffices to show that u^* is ACL in a neighborhood of each point P in $G\cap E\setminus F$. Given $P\in G\cap E\setminus F$, we let $d=d\big(P,\partial(D\cap G\setminus F)\big)$, $r_1=d/(2t+1)$ and $B_1=B^n(P,r_1)$. Then (2.25) holds for all x, y in B_1. We will prove that u^* is in $C(B_1)\cap W_n^1(B_1)$. For this we only need to show that the partial derivatives $\partial u^*/\partial x$, $i=1,\dots,n$, are the weak derivatives of u^*. To this end we let φ be any function in $C_0^\infty(B_1)$ and need to show that

$$\int_{B_1} u^*\frac{\partial\varphi}{\partial x_i}\,dm=-\int_{B_1}\varphi\frac{\partial u^*}{\partial x_i}\,dm,\qquad i=1,2,\dots,n,$$

that is

$$\int_{B_1}\frac{\partial u^*\varphi}{\partial x_i}\,dm=0. \tag{2.26}$$

The following is a sketch of the proof for (2.26). For more details we refer the reader to [AS, Lemma 4].

We extend φ to the whole space by letting $\varphi(x)=0$ for $x\notin B_1$ and assume that $\operatorname{supp}(\varphi)\subset B_2$, where $B_2=B^n(P,r_2)$ with $0<r_2<r_1$. Let s be any real number with $0<|s|<\min\{r_1-r_2,r_2\}$ and set

$$g(x)=u^*(x)\varphi(x),\qquad F(x,s)=\frac{g(x+se_i)-g(x)}{s},\qquad F(x)=\frac{\partial g(x)}{\partial x_i}.$$

Then

$$\int_{B_1} F(x,s)\,dm = 0. \tag{2.27}$$

On the other hand, for any $\varepsilon > 0$, we let

$$K = \{x \in B_1 : |F(x,s) - F(x)| \geq \varepsilon\}.$$

Then

$$\int_{B_1} |F(x,s) - F(x)|\,dm \leq \varepsilon m(B_1) + \int_K |F(x)|\,dm + \int_K |F(x,s)|\,dm. \tag{2.28}$$

Since $F(x,s) \to F(x)$ a.e. in B_1 as $s \to 0$, $m(K) \to 0$ as $s \to 0$ and

$$\lim_{s\to 0} \int_K |F(x)|\,dm = 0. \tag{2.29}$$

Next, it follows from (2.25) that

$$|g(y) - g(x)| \leq M_1|y-x| + M_2 t|y-x| \Big(\int_B f(x+trz)\,dm(z) \Big)^{1/n}$$

for all x, y in B_1 , where M_1 and M_2 are constants depending only on u^* and φ. Thus using Hölder's inequality and Fubini's theorem, we obtain

$$\int_K |F(x,s)|\,dm \leq M_1 m(K) + M_2 t m(K)^{(n-1)/n} \|\nabla u\|_{L^n(G)} m(B)^{1/n}, \tag{2.30}$$

which tends to 0 as s tends to 0. Hence by (2.28), (2.29) and (2.30),

$$\lim_{s\to 0} \int_{B_1} F(x,s)\,dm = \int_{B_1} F(x)\,dm.$$

This and (2.27) imply (2.26).

Finally, the monotonicity of u^* follows directly from the monotonicity of u, and the proof of Theorem 2.24 is completed. □

3. Extremal functions for the conformal capacity of condensers

In this section we generalize a result due to F.W. Gehring [G2] and G.D. Mostow [M6] which says that if $R(F_0, F_1)$ is a nondegenerate ring in $\overline{\mathbf{R}}^n$, then there exists a unique extremal function u in $W'(F_0, F_1; \overline{\mathbf{R}}^n)$ such that

$$\operatorname{cap}(R) = \int_R |\nabla u|^n dm.$$

Our proof makes use of results obtained in Section 2 and some results on the conformally invariant variational integral

$$I(u, D) = \int_D |\nabla u|^n dm. \tag{3.1}$$

In order to formulate our result we need the following definition.

3.2. Definition. Suppose that D is an open set in $\overline{\mathbf{R}}^n$ and that $x_0 \in \mathbf{R}^n$. For each $t > 0$ let

$$\varphi(t) = \operatorname{cap}(R_t)$$

where $R_t = R(F_0, F_1)$ is the condenser with $F_0 = \overline{\mathbf{R}}^n \setminus B^n(x_0, 2t)$ and $F_1 = \overline{B}^n(x_0, t) \setminus D$. We say that a point $x_0 \in \partial D$ is a *regular boundary point of* D if

$$W(x_0, D) = \int_0^1 \frac{\varphi(t)^{1/(n-1)}}{t} dt = \infty, \tag{3.3}$$

i.e. if x_0 satisfies the Wiener criterion with respect to D.

The main result of this section is the following theorem.

3.4. Theorem. *If $R = R(F_0, F_1)$ is a condenser in $\overline{\mathbf{R}}^n$, then there exists a function u in $\mathrm{ACL}(R)$ such that*

(a) $\operatorname{cap}(R) = \int_R |\nabla u|^n dm$;

(b) $\lim_{x \to x_0} u(x) = i$ *for each regular boundary point x_0 of R on ∂F_i, $i = 0, 1$;*

(c) *u is a weak solution to the partial differential equation*

$$\operatorname{div}\left(|\nabla u|^{n-2} \nabla u\right) = 0$$

in R, i.e.

$$\int_R |\nabla u|^{n-2} \nabla u \cdot \nabla w \, dm = 0$$

for all $w \in C_0^\infty(R)$;

(d) *u is unique in the sense that if v is in $\mathrm{ACL}(R)$ with $v - u$ in $\mathrm{ACL}_0^n(R)$ and*

$$\operatorname{cap}(R) = \int_R |\nabla v|^n dm,$$

then $u - v$ is identically constant in R.

The proof of Theorem 3.4 depends on several preliminary results. The first is a modification of several well-known facts about the integral (3.1). For the details, we refer the reader to [M4], [LM] and [GLM].

3.5. Lemma. *Suppose that D is a bounded domain in $\mathbf{R}^n$. If φ is in $C(\overline{D})\cap W_n^1(D)$, then there exists a unique extremal function u_D in $C(D)\cap W_n^1(D)$ with $u_D-\varphi$ in $W_{n,0}^1(D)$ such that*

$$I(u_D,D)\le I(v,D) \tag{3.6}$$

for all v in $\mathrm{ACL}^n(D)$ with $v-\varphi$ in $\mathrm{ACL}_0^n(D)$. Furthermore,

$$\lim_{x\to x_0} u_D(x)=\varphi(x_0) \tag{3.7}$$

for all regular boundary points x_0 of D.

Proof. By [LM, Section 2.7] there exists a unique extremal function u_D in $C(D)\cap W_n^1(D)$ with $u_D-\varphi$ in $W_{n,0}^1(D)$ such that (3.6) holds for all v in $C(D)\cap W_n^1(D)$ with $v-\varphi$ in $C(D)\cap W_{n,0}^1(D)$. Now let v be in $\mathrm{ACL}^n(D)$ with $v-\varphi$ in $\mathrm{ACL}_0^n(D)$. For any fixed $\varepsilon>0$, $v-\varphi\in\mathrm{ACL}_0^n(D)$ implies that there exists $w\in C_0^\infty(D)$ such that

$$\|\nabla(v-\varphi)-\nabla w\|_{L^n(D)}<\varepsilon.$$

Thus,

$$\begin{aligned}\|\nabla v\|_{L^n(D)}&=\big\|\nabla(w+\varphi)-\nabla\big(w-(v-\varphi)\big)\big\|_{L^n(D)}\\&\ge\|\nabla(w+\varphi)\|_{L^n(D)}-\big\|\nabla\big(w-(v-\varphi)\big)\big\|_{L^n(D)}>\|\nabla u\|_{L^n(D)}-\varepsilon.\end{aligned}$$

Letting $\varepsilon\to0$, we obtain the desired inequality (3.6).

Finally (3.7) holds if x_0 is a regular boundary point of D by [LM, p. 154]. □

3.8. Lemma. *Suppose D is a domain in $\mathbf{R}^n$ which contains the sphere $S^{n-1}(x_0,r_0)$. If u_1 and u_2 are ACL in D with $u_1=u_2$ on $S^{n-1}(x_0,r_0)$ then u is also ACL in D where*

$$u(x)=\begin{cases}u_1(x) & \text{if } x\in D\cap\overline{B}^n(x_0,r_0),\\ u_2(x) & \text{if } x\in D\setminus B^n(x_0,r_0).\end{cases}$$

Proof. Obviously u is continuous in D. Since the ACL-property is a local property, it suffices to show that u is ACL at every point on $S^{n-1}(x_0,r_0)$. Suppose $x\in S^{n-1}(x_0,r_0)$ and choose $r>0$ so that $B=B^n(x,r)\subset D$. Let Q be any closed n-interval $\{x\in\mathbf{R}^n: a_i\le x_i\le b_i, i=1,2,\dots,n\}$ with $Q\subset B$ and L be any line segment in Q parallel to x_i-axis on which u_1 and u_2 are absolutely continuous. Since L is divided into a finite number of subintervals by the sphere $S^{n-1}(x_0,r_0)$ on each of which u is equal to either u_1 or u_2, it follows that u is absolutely continuous on L. This completes the proof of Lemma 3.8. □

3.9. Proof of Theorem 3.4. We first prove the existence of u. Since F_0 and F_1 are disjoint, it is easy to see that

$$\operatorname{cap}(R) = \operatorname{mod}\left(\Delta(F_0, F_1; \overline{\mathbf{R}}^n)\right) < \infty.$$

Thus appealing to Theorem 2.5, we can choose a sequence of functions $\{u_j\} \subset W' \cap \mathrm{ACL}^n$ such that

$$\operatorname{cap}(R) = \lim_{j\to\infty} \int_R |\nabla u_j|^n dm$$

and that each u_j is monotone in R relative to ∂R. Since $\{\nabla u_j\}$ is a Cauchy sequence in $L^n(R)$, it converges in $L^n(R)$ to a vector function $f = (f_1, \dots, f_n)$. Hence,

$$\operatorname{cap}(R) = \lim_{j\to\infty} \int_R |\nabla u_j|^n dm = \int_R |f|^n dm. \tag{3.10}$$

Next, by Lemma 2.13 and (3.10), we see that $\{u_j\}$ is equicontinuous at each point $x \in R$. Then Ascoli's theorem implies that there is a subsequence, denoted again by $\{u_j\}$, which converges uniformly on each compact subset of R to a continuous function $u(x)$. By the same method as in [G2] and [M6], one can show that $u(x)$ is in $\mathrm{ACL}(R)$ and that $\nabla u(x) = f(x)$ a.e. in R. Therefore,

$$\operatorname{cap}(R) = \int_R |\nabla u|^n dm.$$

For the boundary behavior of u, without loss of generality, we may assume that $\overline{\mathbf{R}}^n \setminus R$ lies in a ball $B^n(0,r)$, $0 < r < 1$. Choose $s \in (r,1)$ and let

$$\varphi(x) = \begin{cases} u_1(x), & \text{if } |x| \le r, \\ (s-r)^{-1}\left((s-|x|)u_1(x) + (|x|-r)u(x)\right), & \text{if } r < |x| < s, \\ u(x), & \text{if } |x| \ge s, \end{cases}$$

where u_1 is the first function in the sequence $\{u_j\}$, $B = B^n(0,1)$, and $D = B \cap R$. By Lemma 3.8, φ is in $\mathrm{ACL}^n(\mathbf{R}^n)$ and hence in $C(\overline{D}) \cap W_n^1(D)$ with $\varphi = u$ on ∂B and $\varphi = i$ on ∂F_i for $i = 0, 1$. By Lemma 3.5, there is a unique extremal function u_D in $C(D) \cap W_n^1(D)$ with $u_D - \varphi$ in $W_{n,0}^1(D)$ such that

$$I(u_D, D) \le I(v, D) \tag{3.11}$$

for all v in $\mathrm{ACL}^n(D)$ with $v - \varphi$ in $\mathrm{ACL}_0^n(D)$.

We show next that u has the same extremal property as u_D in D and that $u - \varphi$ is in $\mathrm{ACL}_0^n(D)$.

For the extremal property of u, we must prove that

$$I(u,D) \le I(v,D) \tag{3.12}$$

for all v in $\mathrm{ACL}^n(D)$ with $v-\varphi$ in $\mathrm{ACL}_0^n(D)$. In fact, $v-\varphi \in \mathrm{ACL}_0^n(D)$ implies that for any fixed $\varepsilon > 0$, there exists w in $C_0^\infty(D)$ such that

$$\|\nabla(v-\varphi) - \nabla w\|_{L^n(D)} < \varepsilon. \tag{3.13}$$

On the other hand, since the function

$$u'(x) = \begin{cases} w(x)+\varphi(x), & \text{if } x \in D, \\ u(x), & \text{if } x \in R\setminus D, \end{cases}$$

is admissible for R, that is, $u' \in \mathrm{ACL}(R)\cap C(\overline{R})$ and $u' = i$ on ∂F_i for $i = 0,1$, (3.10) implies that

$$\|\nabla(\varphi + w)\|_{L^n(D)} \ge \|\nabla u\|_{L^n(D)}\,. \tag{3.14}$$

Thus, by (3.13) and (3.14),

$$\begin{aligned}\|\nabla v\|_{L^n(D)} &= \big\|\nabla(w+\varphi) - \nabla\big(w-(v-\varphi)\big)\big\|_{L^n(D)} \\ &\ge \|\nabla(w+\varphi)\|_{L^n(D)} - \big\|\nabla\big(w-(v-\varphi)\big)\big\|_{L^n(D)} > \|\nabla u\|_{L^n(D)} - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ yields (3.12). This shows that u has the same extremal property as u_D in D.

To show that $u-\varphi$ is in $\mathrm{ACL}_0^n(D)$, we let $A = \{x : s \le |x| \le 1\}$ and

$$v_j(x) = \begin{cases} u_j(x)-\varphi(x), & \text{if } |x| \le s, \\ (1-s)^{-1}(1-|x|)\big(u_j(x)-\varphi(x)\big), & \text{if } |x| \ge s, \end{cases}$$

for $j = 1,2,\ldots$, where $\{u_j\}$ is the sequence of monotone admissible functions for R chosen above. By Lemma 3.8 v_j is in $C(\overline{D})\cap W_n^1(D)$ with $v_j = 0$ on ∂D. Since $\nabla u_j \to \nabla u$ in $L^n(R)$ and since $u_j \to u$ uniformly on each compact subset of R, for any $\varepsilon > 0$ we can choose j such that

$$\|\nabla u_j - \nabla u\|_{L^n(R)} < \tfrac{1}{4}\varepsilon \tag{3.15}$$

and

$$\|u_j - \varphi\|_{L^n(A)} = \|u_j - u\|_{L^n(A)} \le \tfrac{1}{4}\varepsilon(1-s). \tag{3.16}$$

Then for such j,

$$\|\nabla(u_j-\varphi)-\nabla v_j\|_{L^n(D)} = \left\|\nabla\left((u_j-\varphi)\frac{|x|-s}{1-s}\right)\right\|_{L^n(A)} \tag{3.17}$$
$$\le \|\nabla u_j-\nabla u\|_{L^n(A)} + \frac{1}{1-s}\|u_j-\varphi\|_{L^n(A)} < \tfrac{1}{2}\varepsilon.$$

On the other hand, by [M1, Lemma 2.2], v_j is in $W^1_{n,0}(D)$, so there exists w in $C_0^\infty(D)$ such that

$$\|\nabla(w-v_j)\|_{L^n(D)} < \tfrac{1}{4}\varepsilon. \tag{3.18}$$

Then by inequalities (3.15), (3.17) and (3.18),

$$\|\nabla(u-\varphi)-\nabla w\|_{L^n(D)} \le \|\nabla(u-u_j)\|_{L^n(D)}$$
$$+\|\nabla(u_j-\varphi)-\nabla v_j\|_{L^n(D)} + \|\nabla v_j-\nabla w\|_{L^n(D)} < \varepsilon.$$

This shows that $u-\varphi$ is in $\mathrm{ACL}_0^n(D)$ as desired.

From the extremal property of u and u_D, we see that

$$I(u,D) = I(u_D,D). \tag{3.19}$$

Set $w = u-u_D$, and for each real number t, let

$$W(t) = \int_D |\nabla(u+tw)|^n dm.$$

Then

$$u+tw-\varphi = (1+t)(u-\varphi) - t(u_D-\varphi)$$

is in $\mathrm{ACL}_0^n(D)$ for each fixed t and (3.12) implies that $W(t) \ge W(0)$. Using Hölder's inequality and Lebesgue's dominated convergence theorem as in [GLM, p. 51] and [G2, p. 363], we can differentiate $W(t)$ with respect to t under the integral sign. Then setting $t=0$ yields

$$\int_D |\nabla u|^{n-2}\nabla u\cdot\nabla w\, dm = 0. \tag{3.20}$$

Hence, Hölder's inequality, (3.19) and (3.20) yield

$$I(u,D) = \int_D |\nabla u|^n dm = \int_D |\nabla u|^{n-2}\nabla u\cdot\nabla u_D\, dm \tag{3.21}$$
$$\le \int_D |\nabla u|^{n-1}|\nabla u_D|\, dm \le I(u,D)^{(n-1)/n} I(u_D,D)^{1/n} = I(u,D),$$

and we have equality throughout (3.21). This implies that $\nabla u = \nabla u_D$ a.e. in D and hence that $u-u_D$ is identically constant in D. Finally, the boundary value property of u_D in Lemma 3.5 yields the desired boundary value property of u.

For the proof of (c) and (d), we first show that

$$I(u,R) \le I(v,R) \tag{3.22}$$

for all v in $\mathrm{ACL}(R)$ with $v - u \in \mathrm{ACL}_0^n(R)$. This can be done by the same method as in the proof of (3.12). Then conclusions (c) and (d) will follow from an argument similar to that used to establish (3.20) and (3.21). This completes the proof of Theorem 3.4. □

3.23. Remark. Since all nonregular boundary points of a domain form a set of zero capacity [K, Corollary 5.6], Theorem 3.4 implies that $\mathrm{cap}(R) = 0$ if and only if either ∂F_0 or ∂F_1 contains no regular boundary points of R.

3.24. Remark. Theorem 3.4 does not guarantee that the extremal function u for a condenser R is admissible since it may not have right boundary values at nonregular boundary points of R. However, the following lemma gives a geometric condition which is sufficient to ensure that the extremal function is admissible. Essentially the same result was given by Martio in a different form [M2, Corollary 3.8].

3.25. Lemma. *If D is a domain and if x_0 lies in a nondegenerate component of ∂D, then x_0 is a regular boundary point of D.*

Proof. Suppose F is the nondegenerate component of ∂D containing x_0. Then there is a positive number $r > 0$ such that $\partial B^n(x_0,t) \cap F \ne \emptyset$ for any t with $0 < t \le r$. It is easy to see that

$$R_t \subset R(\overline{\mathbf{R}}^n \setminus B^n(x_0,2t), \overline{B}^n(x_0,t) \cap F)$$

where R_t is defined in 3.2. Therefore, by the monotonicity of capacity and the spherical symmetrization inequality for the moduli of condensers [G1, p. 225],

$$\begin{aligned}\varphi(t) = \mathrm{cap}(R_t) &\ge \mathrm{cap}\left(R(\overline{\mathbf{R}}^n \setminus B^n(x_0,2t), \overline{B}^n(x_0,t) \cap F)\right)\\ &\ge \mathrm{cap}\left(R_G(\tfrac{1}{2})\right) = c_n > 0,\end{aligned}$$

where for $0 < t < 1$, $R_G(t)$ is the Grötzsch ring. Thus,

$$W(x_0,D) = \int_0^1 \frac{\varphi(t)^{1/(n-1)}}{t}\,dt \ge c_n^{1/(n-1)} \int_0^1 \frac{1}{t}\,dt = \infty$$

and hence x_0 is a regular boundary point of D. □

3.26. Corollary. *If R is a condenser and if each component of ∂R is nondegenerate, then the extremal function for $\mathrm{cap}(R)$ can be extended to be an admissible function for R.*

4. Extremal functions for p-capacities

In this section we extend the results in Section 3 to the case where the conformal capacity is replaced by p-capacity, $1 < p < \infty$. For this case we say that a point $x_0 \in \partial D \setminus \{\infty\}$ is a *p-regular boundary point of D* if

$$W_p(x_0, D) = \int_0^1 \varphi_p(t)^{1/(p-1)} \frac{dt}{t} = \infty, \tag{4.1}$$

where

$$\varphi_p(t) = \frac{\operatorname{cap}_p(R_t)}{\operatorname{cap}_p(R'_t)} \tag{4.2}$$

and where $R_t = R(F_0, F_1)$ and $R'_t = R(F'_0, F'_1)$ are condensers with

$$F_0 = F'_0 = \overline{\mathbf{R}}^n \setminus B^n(x_0, 2t), \quad F_1 = \overline{B}^n(x_0, t) \setminus D \quad \text{and} \quad F'_1 = \overline{B}^n(x_0, t).$$

4.3. Theorem. *If F_0, F_1 are disjoint compact sets in $\mathbf{R}^n$ and if every boundary point of $R = \mathbf{R}^n \setminus (F_0 \cup F_1)$ in $\mathbf{R}^n$ is p-regular, then there exists a unique function u in $W'(F_0, F_1; \mathbf{R}^n)$ such that*

$$\operatorname{cap}(F_0, F_1; \mathbf{R}^n) = \int_R |\nabla u|^p dm. \tag{4.4}$$

Furthermore, u is a weak solution to the differential equation

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0. \tag{4.5}$$

4.6. Theorem. *If F_0, F_1 are disjoint compact sets in $\mathbf{R}^n$ and if $R = \mathbf{R}^n \setminus (F_0 \cup F_1)$ is a domain, then there exists a function u in* ACL(R) *such that*

(a) $\operatorname{cap}_p(F_0, F_1; \mathbf{R}^n) = \int_R |\nabla u|^p dm$;

(b) $\lim_{x \to x_0} u(x) = i$ *for each p-regular boundary point x_0 of R on ∂F_i, $i = 0, 1$;*

(c) *u is a weak solution to the partial differential equation*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

in R, i.e.

$$\int_R |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dm = 0 \tag{4.7}$$

for all $w \in C_0^\infty(R)$;

(d) *u is unique in the sense that if v is in* ACL(R) *with $v - u$ in* ACL$_0^p(R)$ *and*

$$\operatorname{cap}_p(F_0, F_1; \mathbf{R}^n) = \int_R |\nabla v|^p dm,$$

then $u - v$ is identically constant in R.

4.8. Outline of the proof for Theorem 4.3. First we choose a sequence $\{u_j\}$ in $W' \cap \mathrm{ACL}^p$ such that

$$\mathrm{cap}_p(F_0, F_1; \mathbf{R}^n) = \lim_{j\to\infty} \int_R |\nabla u_j|^p \, dm.$$

Let F_0, $F_1 \subset B^n(0,r)$, choose an increasing sequence $\{r_j\}$ so that $r_1 > r$ and $r_j \to \infty$ as $j \to \infty$, and let $D_j = R \cap B^n(0, r_j)$. Then by using a generalized form of Lemma 3.5, we obtain

$$w_j \in \mathrm{ACL}\left(B^n(0,r_j)\right) \cap C\left(\overline{B}^n(0,r_j)\right)$$

with $w_j = i$ on F_i for $i = 0, 1$, $w_j = u_j$ on $S^{n-1}(0, r_j)$ and

$$\int_{D_j} |\nabla w_j|^p \, dm \le \int_{D_j} |\nabla v|^p \, dm$$

for all $v \in \mathrm{ACL}(D_j)$ with $v - u_j \in \mathrm{ACL}_0^p(D_j)$.

Next let

$$v_j(x) = \begin{cases} u_j(x), & \text{if } |x| \ge r_j, \\ w_j(x), & \text{if } |x| \le r_j. \end{cases}$$

Then Theorem 4.7 in [GLM] implies that $\{v_j\}$ is an equicontinuous sequence in R. Furthermore, $\{v_j\} \subset W'(F_0, F_1; \mathbf{R}^n)$ and

$$\mathrm{cap}_p(F_0, F_1; \mathbf{R}^n) = \lim_{j\to\infty} \int_R |\nabla v_j|^p \, dm.$$

Finally, an argument similar to that of Theorem 3.4 shows that

$$u = \lim_{j\to\infty} v_j$$

is the desired extremal function. □

4.9. Outline of the proof for Theorem 4.6. Let $d = d(F_0, F_1) > 0$, choose a decreasing sequence $\{\delta_j\}$ so that $\delta_1 < \frac{1}{2}d$ and $\delta_j \to 0$ as $j \to \infty$, and let

$$F_i^{(j)} = F_i(\delta_j) = \{x \in \mathbf{R}^n : d(x, F_i) \le \delta_j\}, \qquad i = 0, 1, j = 1, 2, \ldots.$$

Applying Theorem 4.3 to $(F_0^{(j)}, F_1^{(j)}; \mathbf{R}^n)$, we obtain extremal functions $u_j \in W'(F_0^{(j)}, F_1^{(j)}; \mathbf{R}^n) \subset W'(F_0, F_1; \mathbf{R}^n)$. By [H, Theorem 3.3],

$$\lim_{j\to\infty} \mathrm{cap}_p(F_0^{(j)}, F_1^{(j)}; \mathbf{R}^n) = \mathrm{cap}_p(F_0, F_1; \mathbf{R}^n). \tag{4.10}$$

Thus $\{u_j\}$ is an equicontinuous minimizing sequence for $\mathrm{cap}_p(F_0, F_1; \mathbf{R}^n)$. Then as in the proof of Theorem 3.4, it follows that

$$u = \lim_{j\to\infty} v_j$$

is the desired extremal function. □

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