MEAN VALUES AND THERMIC MAJORIZATION OF SUBTEMPERATURES

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1. Introduction

A classical result of F. Riesz states that the mean values of subharmonic functions over concentric spheres of radius $r$, form convex functions of $\log r$ or $r^{2-n}$, depending on the dimension of the space [2, p. 24]. The corresponding result for subtemperatures, in which the mean values are taken over level surfaces of the Green function, was presented in [6], along with some of its consequences. In the present paper, we give a different, more elementary proof of the theorem for subtemperatures, as well as two new results on thermic majorization, one of which gives a criterion in terms of the mean values and depends upon consequences of the convexity theorem.

The principal result on thermic majorization, Theorem 3, is analogous to a well-known, elementary result on the harmonic majorization of subharmonic functions: a subharmonic function on $\mathbb{R}^n$ has a harmonic majorant there if and only if its mean values over all spheres centred at the origin form a bounded function. Again the mean values of a subtemperature over level surfaces of the Green function are used, and because of their geometry the whole space is replaced by a half-space $\mathbb{R}^n \times] - \infty, a[. The result requires the boundedness of means associated with a sequence of points rather than just one, and there are many technical difficulties which do not arise in the subharmonic case. For example, we have to prove that a subtemperature which has a thermic majorant on the sets $\mathbb{R}^n \times] - \infty, a_j[ for all $j \in \mathbb{N}$, must also possess one on the union of those half-spaces. This result has an illuminating generalization to arbitrary open sets, which is given in Theorem 2.

We work in $\mathbb{R}^{n+1}$, and denote a typical point by $p$ or $(x, t)$, as convenient. A particular point $p_0$ is assumed without comment to be $(x_0, t_0)$. A temperature is a solution of the heat equation

$$\sum_{i=1}^{n} D_i^2 u - D_t u = 0.$$
We use $\theta$ to denote the heat operator, and $\theta^*$ is adjoint (obtained from $\theta$ by changing the sign of $D_t$).

For all $x \in \mathbb{R}^n$, we put $W(x, t) = (4\pi t)^{-n/2} \exp(-\|x\|^2/4t)$ if $t > 0$, and $W(x, t) = 0$ if $t \leq 0$. Then the Green function $G$ for $\theta$ on $\mathbb{R}^{n+1}$ is given by $G(p, q) = W(x - y, t - s)$, where $p = (x, t)$ and $q = (y, s)$.

If $p_0 \in \mathbb{R}^{n+1}$ and $c > 0$, the fundamental domain $\Omega(p_0, c)$ is defined as $\{p \in \mathbb{R}^{n+1} : G(p_0, p) > (4\pi c)^{-n/2}\}$; it is convex and bounded. Its boundary is a smooth surface with equation $\|x_0 - x\| = \left[2n(t_0 - t) \log (c/(t_0 - t))\right]^{1/2}$, together with $\{p_0\}$. If $(x, t) \in \mathbb{R}^n \times ]0, \infty[$, we put

$$Q(x, t) = \|x\|^2 \left[4 \frac{\|x\|^2 t^2 + (\|x\|^2 - 2nt)^2}{t} \right]^{-1/2};$$

we also put $Q(0, 0) = 1$. For each fixed $c > 0$, the restriction to $\partial \Omega(p_0, c)$ of the function $(x, t) \mapsto Q(x_0 - x, t_0 - t)$ is continuous, and is positive except for a zero at $(x_0, t_0 - c)$. If $w$ is a function on $\partial \Omega(p_0, c)$, we put

$$\mathcal{M}(w, p_0, c) = (4\pi c)^{-n/2} \int_{\partial \Omega(p_0, c)} Q(x_0 - x, t_0 - t)w(x, t)\, d\sigma(x, t),$$

where $\sigma$ denotes surface measure, provided that the integral exists. If $u$ is a temperature on an open set $D$, and $\tilde{\Omega}(p_0, c) \subseteq D$, then $u(p_0) = \mathcal{M}(u, p_0, c)$. In particular, $\mathcal{M}(1, p_0, c) = 1$ for any $p_0$ and $c$.

If $D$ is an open set and $p_0 \in D$, we denote by $\Lambda(p_0)$ the set of all points $q \in D \setminus \{p_0\}$ which can be joined to $p_0$ by a polygonal line in $D$ along which $t$ is strictly increasing as the line is described from $q$ to $p_0$. A function $w$ on $D$ is called a subtemperature if it is upper semicontinuous, extended real valued but never $+\infty$, real valued on a sequence $\{p_i\}$ such that $D = \bigcup_{i=1}^{\infty} \Lambda(p_i)$, and satisfies $w(p_0) \leq \mathcal{M}(w, p_0, c)$ whenever $\tilde{\Omega}(p_0, c) \subseteq D$. If $w$ is a subtemperature on $D$, a thermic majorant of $w$ on $D$ is a temperature $u$ such that $w \leq u$ on $D$. If $w$ has a thermic majorant on $D$, then it has a least one. The basic properties of subtemperatures are given in [4] and [5], and the equivalent class of subparabolic functions is discussed in [2].

2. Convexity of mean values of subtemperatures

In this section we present a more elementary proof of [6, Theorem 2] than was given in [6]. We consider subtemperatures on a domain of the form $A(p_0, c_1, c_2) = \Omega(p_0, c_2) \setminus \tilde{\Omega}(p_0, c_1)$, where $p_0 \in \mathbb{R}^{n+1}$ and $0 < c_1 < c_2$. Such a domain corresponds to an annulus in the subharmonic case. We show that, if $w$ is a subtemperature on an open superset of $A(p_0, c_1, c_2)$ then $\mathcal{M}(w, p_0, c)$ is a convex function of $c^{-n/2}$ for $c \in [c_1, c_2]$. Our method is based on an idea due to Dinghas [1] in the subharmonic case.
By a smooth function, we mean one for which the partial derivatives that occur in \( \theta \) exist as continuous functions. For a smooth function \( u \) on a domain in \( \mathbb{R}^{n+1} \), we put \( \nabla_x u = (D_1 u, \ldots, D_n u) \) and \( \|\nabla_x u\| = (\sum_{i=1}^{n} (D_i u)^2)^{1/2} \). We use \( \langle \cdot, \cdot \rangle \) to denote the inner product in \( \mathbb{R}^n \).

It is convenient to first establish some notation and list some elementary formulas. In addition to the functions \( W \) and \( Q \) given above, for all \( (x,t) \in \mathbb{R}^n \times [0, \infty[ \) we put

\[
L(x,t) = \left[ 4 \|x\|^2 t^2 + (\|x\|^2 - 2nt)^2 \right]^{-1/2}
\]

and

\[
J(x,t) = 2nt \exp(-\|x\|^2/(2nt)) L(x,t);
\]

note that \( Q(x,t) = \|x\|^2 L(x,t) \). If \( F \in \{W, Q, L, J\} \) and \( (x_0, t_0) \in \mathbb{R}^{n+1} \), we put \( F_0(x,t) = F(x_0 - x, t_0 - t) \) for all \( (x,t) \in \mathbb{R}^n \times ]-\infty, t_0[ \). On \( \partial \Omega(p_0, c) \), where

\[
(t_0 - t) \exp(\|x_0 - x\|^2/(2nt_0 - t)) = c,
\]

the outward unit normal \( (\nu_x, \nu_t) \) is given by

\[
\nu_x = -2(t_0 - t)(x_0 - x)L_0(x,t), \quad \nu_t = (\|x_0 - x\|^2 - 2nt_0 - t)L_0(x,t).
\]

It is useful to have \( (\nu_x, \nu_t) \) in terms of \( J_0 \). Since

\[
(1) \quad cJ_0(x,t) = 2nt_0 - t^2 L_0(x,t)
\]

whenever \( (x,t) \in \partial \Omega(p_0, c) \), we have

\[
(2) \quad \nu_x = -\frac{c(x_0 - x)}{nt_0 - t} J_0(x,t),
\]

\[
(3) \quad \nu_t = \frac{c(\|x_0 - x\|^2 - 2nt_0 - t)}{2nt_0 - t^2} J_0(x,t)
\]

for such points \( (x,t) \). Next,

\[
(4) \quad \nabla_x W_0(x,t) = \frac{x_0 - x}{2(t_0 - t)} W_0(x,t),
\]

\[
(5) \quad D_t W_0(x,t) = \frac{2nt_0 - t - \|x_0 - x\|^2}{4(t_0 - t)^2} W_0(x,t),
\]
and if $g = W_0^{-2/n}$ we have

$$\theta^* g = \frac{2}{n} \left( \frac{2}{n} + 1 \right) \frac{\| \nabla_x W_0 \|^2}{W_0^2} g$$

since $\theta^* W_0 = 0$. Finally, on $\partial \Omega(p_0, c)$ we have

$$-\langle \nabla_x W_0, \nu_x \rangle = \| x_0 - x \|^2 L_0(x, t) W_0(x, t) = (4\pi c)^{-n/2} Q_0(x, t).$$

We need certain Green identities. If $v$ and $w$ are smooth functions, it is elementary that

$$v \theta w = \sum_{i=1}^n D_i (v D_i w) - \langle \nabla_x v, \nabla_x w \rangle - D_t (vw) + w D_t v$$

and

$$w \theta^* v = \sum_{i=1}^n D_i (w D_i v) - \langle \nabla_x v, \nabla_x w \rangle + w D_t v.$$

Therefore, if $A$ is any domain for which the divergence theorem is applicable,

$$\iint_A (v \theta w + \langle \nabla_x v, \nabla_x w \rangle - w D_t v) \, dx \, dt = \int_{\partial A} ((v \nabla_x w, \nu_x) - vw \nu_t) \, d\sigma$$

and

$$\iint_A (w \theta^* v + \langle \nabla_x v, \nabla_x w \rangle - w D_t v) \, dx \, dt = \int_{\partial A} (w \nabla_x v, \nu_x) \, d\sigma.$$

**Lemma 1.** Let $p_0 \in \mathbb{R}^{n+1}$, let $0 < c_1 < c_2$, let $u$ be a smooth function on an open superset of $\tilde{A}(p_0, c_1, c_2)$, and put $\Omega(c) = \Omega(p_0, c)$ and $M(c) = M(u, p_0, c)$ for all $c \in [c_1, c_2]$. Then, if $\kappa_n = 2^{n+1} \pi^{n/2} n^{-1}$ and $c \in [c_1, c_2]$, we have

$$\kappa_n c^{(n/2)+1} M_c(c) = \int_{\partial \Omega(c)} (\langle \nabla_x u, \nu_x \rangle - u \nu_t) \, d\sigma$$

and

$$\kappa_n \left( c^{(n/2)+1} M_c(c) \right)_c = \int_{\partial \Omega(c)} J_0 \theta u \, d\sigma.$$

**Proof.** Let $c \in [c_1, c_2]$, and put $A = A(p_0, c_1, c)$. We want to use (9) with this choice of $A$ and $w$ smooth, but with $v = W_0^{-2/n} / 4\pi$, so that the smoothness of
v breaks down at \( p_0 \). To prove that this is permissible, we use an approximation argument. Let \( t \in [t_0 - c_1, t_0] \), and put

\[
F_1(t) = \partial \Omega(p_0, c_1) \cap (\mathbb{R}^n \times [t_0 - c_1, t]),
\]

\[
F_2(t) = \partial \Omega(p_0, c) \cap (\mathbb{R}^n \times [t_0 - c, t]),
\]

\[
V(t) = A(p_0, c_1, c) \cap (\mathbb{R}^n \times [t_0 - c, t]).
\]

Applying (9) on \( V(t) \) with \( v = W_0^{-2/n}/4 \pi \), and using (6) and (7), we obtain

\[
\begin{aligned}
(12) \int \int _{V(t)} & \left( \frac{2}{n} \left( \frac{2}{n} + 1 \right) \left\| \nabla_x W_0 \right\|^2 v w - \frac{2v}{nW_0} \langle \nabla_x w, \nabla_x W_0 \rangle + \frac{2vw}{nW_0} D_t W_0 \right) dx dt \\
&= \int_{\partial V(t)} \left( -2vw \frac{nW_0}{nW_0} \langle \nabla_x W_0, \nu_x \rangle \right) d\sigma = \frac{2}{n} \left( c \int_{F_2(t)} -c_1 \int_{F_1(t)} \right) wQ_0 d\sigma.
\end{aligned}
\]

Since \( wQ_0 \) is bounded on \( \partial A(p_0, c_1, c) \), as \( t \to t_0 \)- the last expression tends to

\[
(13) \frac{2}{n} \left( c \int_{\partial A(p_0, c_1, c)} -c_1 \int_{\partial A(p_0, c_1, c)} \right) wQ_0 d\sigma.
\]

For the integral over \( V(t) \) in (12), the integrand is

\[
\begin{aligned}
\frac{2}{n} \left( \frac{2}{n} + 1 \right) & \left\| x_0 - x \right\|^2 v w - \frac{v}{n(t_0 - t)^2} \langle \nabla_x w, x_0 - x \rangle + \frac{vw(2n(t_0 - t) - \left\| x_0 - x \right\|^2)}{2n(t_0 - t)^2}
\end{aligned}
\]

by (4) and (5). Since \( v, w \) and \( \left\| \nabla_x w \right\| \) are bounded, this expression is dominated by a multiple of

\[
(14) \frac{\left\| x_0 - x \right\|^2}{(t_0 - t)^2} + \frac{2n}{(t_0 - t)},
\]

which is obviously integrable on \( V(t_0 - c_1 e^{-1}) \). Furthermore, in \( A(p_0, c_1, c) \),

\[
\left\| x_0 - x \right\|^2 \geq 2n(t_0 - t) \log \left( c_1 / (t_0 - t) \right),
\]

so that on \( A(p_0, c_1, c) \setminus V(t_0 - c_1 e^{-1}) \) we have

\[
\frac{\left\| x_0 - x \right\|^2}{(t_0 - t)^2} \geq \frac{2n}{(t_0 - t)},
\]

and therefore the expression (14) is dominated by \( \left\| x_0 - x \right\|^2 (t_0 - t)^{-2} \), which is integrable by [4, Lemma 4]. It follows that we can make \( t \to t_0 \)- in (12); in view of (13), we thus obtain (9) with \( A = A(p_0, c_1, c) \) and \( v = W_0^{-2/n}/4 \pi \). Next,

\[
\int \int _{A} f dx dt = \int_{c_1}^{c_0} d\gamma \int_{\partial \Omega(\gamma)} fJ_0 d\sigma
\]
for any function $f$ such that either side exists [4, p. 388]. It therefore follows from (9) that

$$\left( \int_{\partial \Omega(c)} - \int_{\partial \Omega(c_1)} \right) w(\nabla_z v, \nu_z) d\sigma = \int_{c_1}^{c} d\gamma \int_{\partial \Omega(\gamma)} \left( w \theta^* v + \langle \nabla_z w, \nabla_z v \rangle - w D_t v \right) J_0 d\sigma,$$

so that

$$(15) \quad \left( \int_{\partial \Omega(c)} w(\nabla_z v, \nu_z) d\sigma \right)_c = \int_{\partial \Omega(c)} \left( w \theta^* v + \langle \nabla_z w, \nabla_z v \rangle - w D_t v \right) J_0 d\sigma.$$

In (15), we take $w = u$ and $v = W_0^{-2/n}/4\pi$. Then, by (7),

$$\int_{\partial \Omega(c)} w(\nabla_z v, \nu_z) d\sigma = \frac{1}{4\pi} \int_{\partial \Omega(c)} u \left( - \frac{2}{n} \right) W_0^{-(2/n)-1} \langle \nabla_z W_0, \nu_z \rangle d\sigma = \kappa_n c^{(n/2)+1} M(c),$$

so that the left side of (15) is

$$\kappa_n \left( c^{(n/2)+1} M(c) \right)_c.$$

Next, by (6), (4), and (1),

$$\int_{\partial \Omega(c)} w(\theta^* v) J_0 d\sigma = \frac{1}{4\pi} \int_{\partial \Omega(c)} u \left( \frac{2}{n} + 1 \right) \left| \nabla_z W_0 \right|^2 W_0^{-2/n} J_0 d\sigma$$

$$= \left( \frac{2}{n} + 1 \right) \int_{\partial \Omega(c)} \left( c \left| x_0 - x \right|^2 J_0 \right) d\sigma$$

$$= \left( \frac{2}{n} + 1 \right) \int_{\partial \Omega(c)} Q_0 u d\sigma = \kappa_n \left( 1 + \frac{n}{2} \right) c^{n/2} M(c).$$

By (2), (3), (4) and (5), the remainder of the right side of (15) is

$$\frac{1}{4\pi} \int_{\partial \Omega(c)} \left( - \frac{2}{n} W_0^{-(2/n)-1} \langle \nabla_z u, \nabla_z W_0 \rangle + \frac{2}{n} W_0^{-(2/n)-1} u D_t W_0 \right) J_0 d\sigma$$

$$= \int_{\partial \Omega(c)} \left( \langle \nabla_z u, \nu_z \rangle - u \nu_t \right) d\sigma.$$

Hence (15) yields

$$\kappa_n \left( c^{(n/2)+1} M(c) \right)_c = \kappa_n \left( 1 + \frac{n}{2} \right) c^{n/2} M(c) + \int_{\partial \Omega(c)} \left( \langle \nabla_z u, \nu_z \rangle - u \nu_t \right) d\sigma,$$
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which implies the identity (10).

Taking \( A = A(p_0, c_1, c) \) in (8), and using an argument similar to the one that gave us (15) from (9), we obtain

\[
(16) \quad \left( \int_{\partial \Omega(c)} \left( (v \nabla_x w, \nu_x) - vw \nu_t \right) d\sigma \right)_c = \int_{\partial \Omega(c)} (v \theta w + (\nabla_x v, \nabla_x w) - w D_t v) J_0 \, d\sigma.
\]

In (16), we take \( v = 1 \) and \( w = u \). Then the left side becomes

\[
\left( \int_{\partial \Omega(c)} \left( (\nabla_x u, \nu_x) - u \nu_t \right) d\sigma \right)_c = \kappa_n \left( c^{(n/2)+1} M_c(c) \right)_c
\]

in view of (10), and the right side becomes

\[
\int_{\partial \Omega(c)} J_0 \theta u \, d\sigma;
\]

thus (11) is established.

Lemma 1 provides the following elementary proof of [4, Theorem 12].

**Corollary.** Let \( w \) be a subtemperature on an open set \( D \), and let \( p_0 \in D \). Then \( M(w, p_0, \cdot) \) is increasing on the set of \( c \) such that \( \Omega(p_0, c) \subseteq D \).

**Proof.** If \( w \) is smooth on \( D \), we can take \( A = \Omega(p_0, c) \) and \( v = 1 \) in (8), to obtain

\[
\int_{\Omega(c)} \theta w \, dx \, dt = \int_{\partial \Omega(c)} \left( (\nabla_x w, \nu_x) - w \nu_t \right) d\sigma.
\]

Therefore, taking \( u = w \) in (10), we obtain

\[
M_c(c) = \kappa_n^{-1} c^{-(n/2)-1} \int_{\Omega(c)} \theta w \, dx \, dt.
\]

Since \( \theta w \geq 0 \), this formula immediately implies that \( M \) is increasing. (A similar argument was given by Pini [3] for the case \( n = 1 \).) If \( w \) is an arbitrary subtemperature, we can take a decreasing sequence \( \{w_j\} \) of smooth subtemperatures, with limit \( w \) on a neighbourhood of \( \Omega(p_0, c) \) [2, p. 281]. Then \( M(w_j, p_0, \cdot) \) is increasing for every \( j \), so that the same is true of its limit, which is \( M(w, p_0, \cdot) \) by the monotone convergence theorem.

We can now give a proof of [6, Theorem 2] that does not rely upon knowledge of the Dirichlet problem for \( A(p_0, c_1, c_2) \).

**Theorem 1.** Let \( w \) be a subtemperature on an open superset of \( \bar{A}(p_0, c_1, c_2) \). Then there is a real-valued, convex function \( \phi \) such that \( M(w, p_0, c) = \phi(c^{-n/2}) \) for all \( c \in [c_1, c_2] \).
Proof. We require the fact that $\mathcal{M}(w, p_0, \cdot)$ is real-valued, which was proved in [6]. That proof requires the special case of [6, Theorem 1] in which $u$ is a temperature on an open superset of $\tilde{A}(p_0, c_1, c_2)$, which was proved by elementary techniques (and could alternatively be deduced from (11)). It also requires the result given as an example in [6], which depends only upon the aforementioned special case of [6, Theorem 1] and the fact that $\mathcal{M}(w, p_0, \cdot)$ is increasing (which we have just given an elementary proof of). Suppose that $w$ is smooth. Then $\theta w \geq 0$, so that

$$
(c^{(n/2)+1} \mathcal{M}_c(c))_c \geq 0
$$

(17)

by (11). Suppose also that $w > 0$, so that $\mathcal{M} > 0$. Then we can rearrange (17) to obtain

$$
\frac{\mathcal{M}_{cc}}{\mathcal{M}} + \left( \frac{n}{2c} + \frac{1}{c} \right) \frac{\mathcal{M}_c}{\mathcal{M}} \geq 0.
$$

(18)

Put $\lambda(c) = c^{n/2} \mathcal{M}(c)$. Then

$$
\frac{\mathcal{M}_c}{\mathcal{M}} = \left( \frac{\lambda_c}{\lambda} - \frac{n}{2c} \right)
$$

and

$$
\frac{\mathcal{M}_{cc}}{\mathcal{M}} = \frac{\lambda_{cc}}{\lambda} - \frac{n \lambda_c}{c \lambda} + \frac{n^2}{4c^2} + \frac{n}{2c^2},
$$

so that (18) becomes

$$
\frac{\lambda_{cc}}{\lambda} + \left( 1 - \frac{n}{2} \right) \frac{\lambda_c}{c \lambda} \geq 0.
$$

Putting $\xi = c^{n/2}$, we obtain

$$
\lambda_{\xi \xi} = \frac{4c^2 - n}{n^2} \left( \lambda_{cc} + \left( 1 - \frac{n}{2} \right) \frac{\lambda_c}{c} \right) \geq 0,
$$

so that $\lambda$ is a convex function of $\xi$. Hence $c^{n/2} \mathcal{M}(c)$ is a convex function of $c^{n/2}$, which implies that $\mathcal{M}(c)$ is a convex function of $c^{-n/2}$.

If $w$ is smooth but not positive, we can find an open superset $S$ of $\tilde{A}(p_0, c_1, c_2)$ and a constant $K$ such that $w - K > 0$ on $S$, so that

$$
\mathcal{M}(w, p_0, c) = \mathcal{M}(w - K, p_0, c) + K
$$

is a convex function of $c^{-n/2}$. If $w$ is not smooth, take a decreasing sequence $\{w_j\}$ of smooth subtemperatures that converges to $w$ on an open superset of $\tilde{A}(p_0, c_1, c_2)$ [2, p. 281]. Then $\{\mathcal{M}(w_j, p_0, c)\}$ is a decreasing sequence with limit $\mathcal{M}(w, p_0, c) \in \mathbb{R}$, so that $\mathcal{M}(w, p_0, c)$ is also a convex function of $c^{-n/2}$. 
We now present a simple consequence of Theorem 1 that was not considered in [6]. The subharmonic analogue can be found in [2, p. 24].

**Corollary.** Let \( w \) be a subtemperature on an open superset of \( \mathcal{A}(p_0, c_1, c_2) \). If
\[
v(x, t) = M \left( w, p_0, (t_0 - t) \exp \left( \|x_0 - x\|^2 / 2n(t_0 - t) \right) \right)
\]
for all \((x, t) \in \mathcal{A}(p_0, c_1, c_2)\), then \( v \) is a \( \theta^* \)-subtemperature (that is, a subtemperature relative to the adjoint equation).

**Proof.** By Theorem 1, there is a finite, convex function \( \phi \) on \( ]c_2^{-n/2}, c_1^{-n/2} [ \) such that \( v(x, t) = \phi \left( (4\pi)^{n/2} W(x_0 - x, t_0 - t) \right) \). By Lemma 1, Corollary, \( \phi \) is decreasing, so that if \( \psi(s) = \phi \left( c_2^{-n/2} + c_1^{-n/2} - s \right) \) then \( \psi \) is increasing on \( ]c_2^{-n/2}, c_1^{-n/2} [ \) and \( v(x, t) = \psi \left( c_2^{-n/2} + c_1^{-n/2} - (4\pi)^{n/2} W(x_0 - x, t_0 - t) \right) \). Since the function of \((x, t)\) with which \( \psi \) is composed to get \( v \), is a solution of the adjoint equation, the dual of [4, Theorem 2] implies that \( v \) is a \( \theta^* \)-subtemperature.

### 3. Thermic majorization

Let \( w \) be a subharmonic function on \( \mathbb{R}^n \), and for each \( r > 0 \) let \( \mathcal{L}(w, 0, r) \) denote its mean over the sphere of radius \( r \) centred at the origin. It is a well-known, elementary result that \( w \) has a harmonic majorant on \( \mathbb{R}^n \) if and only if \( \mathcal{L}(w, 0, \cdot) \) is bounded above on \( ]0, \infty[; \) and that, if \( w \) has such a majorant and \( u \) is the least one, then
\[
u(0) = \sup_{r > 0} \mathcal{L}(w, 0, r) = \lim_{r \to \infty} \mathcal{L}(w, 0, r).
\]

We seek an analogous result for subtemperatures. First, we must replace the whole space (\( \mathbb{R}^{n+1} \) in this case) by a lower half-space \( \mathbb{R}^n \times ] - \infty, a[ \), because
\[
\bigcup_{c > 0} \Omega(p_0, c) = \mathbb{R}^n \times ] - \infty, t_0[.
\]
Note that a subtemperature on \( \mathbb{R}^{n+1} \) can have a thermic majorant on a half-space \( \mathbb{R}^n \times ] - \infty, 0[ \) without having one on \( \mathbb{R}^{n+1} \). For example, it is well-known that there is a temperature \( u \) on \( \mathbb{R}^2 \) that is identically zero on \( \mathbb{R}^n \) but not on any open superset thereof [7, p. 86]. The subtemperature \( u^+ \) cannot have a thermic majorant on \( \mathbb{R} \times ]0, \infty[ \), since that would imply that
\[
u(x, t) = \int_{\mathbb{R}^n} W(x - y, t) u(y, 0) dy = 0
\]
whenever \( t > 0 \) [7, pp. 100–102].
We therefore seek a necessary and sufficient condition, in terms of the surface means $\mathcal{M}$, for a subtemperature $w$ on a half-space $H_a = \mathbb{R}^n \times ]-\infty, a[ \to \infty, a[ \to \infty$ to have a thermic majorant there. There are two essential elements for this. The first is that, if $q = (y, s) \in H_a$ and $\mathcal{M}(w, q, \cdot)$ is bounded above on $]0, \infty[$, then $w$ has a thermic majorant on $H_s$. This is far less elementary than the subharmonic theorem, and depends on the continuity of $\mathcal{M}(w, q, \cdot)$ that is implied by Theorem 1. The second is that, if $t_j \to a-$ and $w$ has a thermic majorant on $H_{t_j}$ for every $j$, then $w$ has a thermic majorant on $H_a$. This result can be generalized to arbitrary open sets, and is given in Theorem 2 below.

**Lemma 2.** Let $w$ be a subtemperature on an open superset $D$ of $H_a \cup \{p_a\}$ for some $p_a \in \partial H_a$. If $\mathcal{M}(w, p_a, \cdot)$ is bounded above on $]0, \infty[$, then there is an increasing family $\{w_c : c > 0\}$ of subtemperatures on $D$ such that $w_\infty = \lim_{c \to \infty} w_c$ is the least thermic majorant of $w$ on $H_a$. Furthermore,

$$w_\infty(p_a) = \lim_{c \to \infty} \mathcal{M}(w, p_a, c).$$

**Proof.** If $c > 0$, then there is a unique subtemperature $w_c$ on $D$ such that $w_c$ is a temperature on $\Omega(p_a, c)$, $w_c = w$ on $D \setminus (\Omega(p_a, c) \cup \{p_a\})$, and $w_c \geq w$ on $D$ [6, Theorem 5]. If $d > c$, then the same theorem yields not only $w_d$, but also a unique subtemperature $u_d$ on $D$ such that $u_d$ is a temperature on $\Omega(p_a, d)$, $u_d = w_c$ on $D \setminus (\Omega(p_a, d) \cup \{p_a\})$, and $u_d \geq w_c$ on $D$. Since $w_c = w$ on a superset of $D \setminus (\Omega(p_a, d) \cup \{p_a\})$ and $w_c \geq w$ on $D$, we see that $w_d = u_d \geq w_c$ on $D$, so that $w_\infty = \lim_{c \to \infty} w_c$ exists on $D$. By hypothesis, there is $\alpha \in \mathbb{R}$ such that $\mathcal{M}(w, p_a, c) \leq \alpha$ for all $c > 0$. Therefore

$$w_c(p_a) \leq \mathcal{M}(w_c, p_a, c) = \mathcal{M}(w, p_a, c) \leq \alpha$$

for every $c > 0$, so that $w_\infty(p_a) \leq \alpha$. If $0 < c \leq d$ then, since $w_d$ is a subtemperature on $D$ and a temperature on $\Omega(p_a, d)$, we have $w_d(p_a) = \mathcal{M}(w_d, p_a, c)$ by [6, Theorem 4]. Since the mean values of subtemperatures are real-valued (by Theorem 1), we can use the monotone convergence theorem to deduce that

$$\mathcal{M}(w_\infty, p_a, c) = \lim_{d \to \infty} \mathcal{M}(w_d, p_a, c) = w_\infty(p_a) \leq \alpha$$

for every $c > 0$. Hence $w_\infty$ is finite $\sigma$-a.e. on $\partial \Omega(p_a, c)$, and therefore the Harnack convergence theorem [2, p. 276] implies that $w_\infty$ is a thermic majorant of $w$ on $H_a$. Since any thermic majorant of $w$ on $H_a$ will also majorize $w_c$ for every $c > 0$, the function $w_\infty$ is the least such majorant. Finally, if $c > 0$ we have

$$\mathcal{M}(w, p_a, c) = \mathcal{M}(w_c, p_a, c) = w_c(p_a)$$

(by [6, Theorem 4]), from which (19) follows immediately.
Theorem 2. Let \( w \) be a subtemperature on an open set \( E \), and let \( \{ p_j \} \) be a sequence in \( E \) such that

\[
E = \bigcup_{j=1}^{\infty} \Lambda(p_j).
\]

If \( w \) has a thermic majorant on \( \Lambda(p_j) \) for every \( j \), then \( w \) has a thermic majorant on \( E \); and if \( u \) is the least thermic majorant of \( w \) on \( E \), then for any \( j \) the restriction of \( u \) to \( \Lambda(p_j) \) is the least thermic majorant of \( w \) on \( \Lambda(p_j) \).

Proof. Let \( u_j \) denote the least thermic majorant of \( w \) on \( \Lambda(p_j) \). Then \( u_j - w \) is a potential on \( \Lambda(p_j) \), and since the Green function for \( \Lambda(p_j) \) is the restriction to \( \Lambda(p_j) \times \Lambda(p_j) \) of the Green function \( G_E \) for \( E \) [5; 2, p. 300], we have

\[
\mu_j(q) = \int_{\Lambda(p_j)} G_E(q, q') d\mu_j(q)
\]

on \( \Lambda(p_j) \), for some positive Borel measure \( \mu_j \). Next, \( G_E(p, q) > 0 \) if and only if \( q \in \Lambda(p) \) [5; 2, p. 300], so that

\[
u_j(p) - w(p) = \int_{\Lambda(p)} G_E(p, q) d\mu_j(q)
\]

for all \( p \in \Lambda(p_j) \). Next, by the form of the Riesz decomposition theorem given in [5] and the uniqueness of representing measures, \( \mu_j \) is the measure given by the distribution \( -\theta(u_j - w) = \theta w \) on \( \Lambda(p_j) \). Therefore, whenever \( \Lambda(p_j) \cap \Lambda(p_k) \neq \emptyset \), the measures \( \mu_j \) and \( \mu_k \) coincide there. In view of (20), we can therefore define a measure \( \mu \) on \( E \) by putting \( \mu = \mu_j \) on every \( \Lambda(p_j) \). This yields the representation

\[
u_j(p) = w(p) + \int_{\Lambda(p)} G_E(p, q) d\mu(q)
\]

for quasi every \( p \in \Lambda(p_j) \). Since the right-hand side is independent of \( j \), whenever \( \Lambda(p_j) \cap \Lambda(p_k) \neq \emptyset \) we have \( u_j = u_k \) q.e., and hence everywhere, on that intersection. We can therefore define a temperature \( u \) on \( E \) by putting \( u = u_j \) on \( \Lambda(p_j) \). Obviously \( u \geq w \) on \( E \), and if \( v \) is a temperature on \( E \) such that \( v(q) < u(q) \) for some \( q \in E \), then there is \( i \) such that \( v(q) < u_i(q) \) and so \( v \) does not majorize \( w \) on \( \Lambda(p_i) \).

Theorem 3. Let \( w \) be a subtemperature on \( H_a = \mathbb{R}^n \times ]-\infty, a[ \). Then \( w \) has a thermic majorant on \( H_a \) if and only if there is a sequence \( \{ p_j \} \) in \( H_a \) such that

\[
H_a = \bigcup_{j=1}^{\infty} \Lambda(p_j)
\]
and $M(w,p_j,\cdot)$ is bounded above on $]0,\infty[$ for every $j$. If $w$ has a thermic majorant on $H_a$ and $u$ is the least one, then

$$u(p) = \sup_{c>0} M(w,p,c) = \lim_{c \to \infty} M(w,p,c)$$

for every $p \in H_a$.

**Proof.** If there is a sequence $\{p_j\}$ as described, then $w$ has a thermic majorant on every $\Lambda(p_j)$, by Lemma 2, so that $w$ has a thermic majorant on $H_a$, by Theorem 2.

Conversely, if $w$ has a thermic majorant $v$ on $H_a$, then for any $p \in H_a$ and $c > 0$ we have

$$M(w,p,c) \leq M(v,p,c) = v(p) < \infty.$$

Finally, if $w$ has a least thermic majorant $u$ on $H_a$, and $p = (x,t) \in H_a$, then Theorem 2 shows that the restriction of $u$ to $H_t$ is the least thermic majorant of $w$ on $H_t$. Therefore, by Lemma 2,

$$u(p) = \lim_{c \to \infty} M(w,p,c),$$

and (21) follows because $M(w,p,\cdot)$ is increasing.

**References**


