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MEROMORPHIC FUNCTIONS OF BOUNDED VALENCE ON AN OPEN RIEMANN SURFACE

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Introduction

In the theory of the boundary behaviour of canonical conformal mappings on open Riemann surfaces, it is a well known fact that every canonical conformal mapping on an open Riemann surface of finite genus is a vertical slit mapping ([4], [6], [9]). This theorem was first proved by M. Mori ([7]). For the proof of the theorem, she prepared a lemma (Lemma 2, p. 177) which asserts that every meromorphic function f of bounded valence on an open Riemann surface R of finite genus has a limit at each weak boundary point of the Kerékjártó–Stoïlow boundary of R.

However, P. Järvi ([3]) showed that the assertion of the lemma contains invalid arguments. Although he could not restore the assertion of Mori, he proved that the lemma holds true if R is an open Riemann surface with absolutely disconnected boundary (see [8]) and if no cluster set of f at Kerékjártó–Stoïlow boundary point of R separates the extended complex plane. He further pointed out an error in the example introduced by Kusunoki and Taniguchi ([5]) who intended to show that $\cap_{n=1}^{\infty} O_{AD,n} - O_{AD} \neq \varnothing$. Moreover, he remarked that one can obtain a planar Riemann surface of the class $\cap_{n=1}^{\infty} O_{AD,n} - O_{AD}$ provided that Mori's assertion holds true for planar Riemann surfaces.

In this note, we shall first give a simple example of a Riemann surface to show that Mori's assertion does not hold even for planar Riemann surfaces.

Secondly, we shall prove that every meromorphic function of bounded valence on an open Riemann surface R with absolutely disconnected boundary admits a continuous extension to the Kerékjártó–Stoïlow boundary of R. Consequently, according to the proof of Järvi ([3]), we can conclude that there exists a planar Riemann surface which belongs to the class $\cap_{n=1}^{\infty} O_{AD,n} - O_{AD}$.

1. The boundary behaviour of meromorphic functions of bounded valence at weak boundary points

Let R be an open Riemann surface and Γ be a family of locally rectifiable curves γ in R. Consider the class of Borel measurable linear densities $\varrho|dz|$ on R for which the quantities

$$L(\Gamma,\varrho) = \inf\left\{ \int_{\gamma} \varrho\,|dz|\,;\ \gamma\in\Gamma\right\}, \qquad A(\varrho,R) = \iint_R \varrho^2\,dx\,dy$$

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are well-defined and not simultaneously 0 or ∞. Then the quantity

$$\lambda(\Gamma) = \sup\left\{\frac{L(\Gamma, \varrho)^2}{A(\varrho, R)}; \varrho\,|dz|\right\}$$

is called the extremal length of Γ. Let R^* be the Kerékjártó–Stoïlow compactification of R and β be the Kerékjártó–Stoïlow boundary of R. Let p be a point in β and V be a subregion of R whose relative boundary ∂V is compact and regular. We say that V is an end of p if the closure of V in R^* contains p. Let $\Gamma(p, V)$ be the family of cycles γ in V separating p and ∂V. We say that a point $p \in \beta$ is weak or a weak boundary point if, for any end V of p, the extremal length $\lambda(\Gamma(p, V))$ of $\Gamma(p, V)$ equals 0.

For a meromorphic function f on R and $p \in \beta$ we call

$$\mathrm{Cl}(f, p) = \cap\{\overline{f(V)};\ V \text{ is an end of } p\}$$

the cluster set of f at p. We say that f has a limit at p if $\mathrm{Cl}(f,p)$ consists of one point.

Mori ([7]) asserts in Lemma 2 that every meromorphic function f of bounded valence on an open Riemann surface R of finite genus has a limit at each weak boundary point of the Kerékjártó–Stoïlow boundary of R. However, P. Järvi ([3]) pointed out that the proof contains incorrect arguments. In fact, we can give a counter-example.

Take a decreasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} \log(a_{2n-1}/a_{2n}) = \infty$. Take another sequence $\{b_n\}_{n=2}^{\infty}$ such that $a_{2n+1} < b_{2n+1} < b_{2n} < a_{2n}$. Delete a countable number of closed intervals $[b_{2n+1}, b_{2n}]$ $(n = 1, 2, \dots)$ on the real axis from the punctured extended complex plane $\{0 < |z| \le \infty\}$. Denote the remaining region by R_1. Take a countable number of rectangles without closed intervals on the real axis and denote them by

$$B_n = \{z\,;\ a_{2n+1} < \mathrm{Re}(z) < a_{2n}, -1 < \mathrm{Im}(z) < 1\} - [b_{2n+1}, b_{2n}]$$

$(n = 1, 2, \dots)$. For each n $(n = 1, 2, \dots)$, join R_1 with B_n crosswise along the slit $[b_{2n+1}, b_{2n}]$. By this construction we obtain a two-sheeted covering surface R over the extended complex plane. It is easy to see that R is a planar Riemann surface. Let f be the projection map from R to the extended complex plane. Then f is a meromorphic function of 2-valence on R. Let p be the Kerékjártó–Stoïlow boundary point over 0. Then f has not a limit at p. In fact, for any end V of p, there is an n such that $V \supset \cup_{k=n}^{\infty} B_k$. Hence the cluster set of f at p contains a proper continuum $\{iy\,;\ -1 \le y \le 1\}$.

On the other hand, we can prove that p is a weak boundary point as follows. Let V be an arbitrary end of p and A_n be the annulus in R_1 lying over the annulus $\{z\ ;\ a_{2n} < |z| < a_{2n-1}\}$ $(n = 1, 2, \dots)$. Let Γ_n be the family of curves

γ_r in A_n lying over the circles $\{z\,;\,|z|=r\}$, where $a_{2n}<r<a_{2n-1}$. Then there exists an n such that $\cup_{k=n}^{\infty}\Gamma_k\subset\Gamma(p,V)$. Since

$$\frac{1}{\lambda(\Gamma(p,V))}\geq\sum_{k=n}^{\infty}\frac{1}{\lambda(\Gamma_k)}=\frac{1}{2\pi}\sum_{k=n}^{\infty}\log(a_{2k-1}/a_{2k})=\infty,$$

$\lambda(\Gamma(p,V))=0$. Thus p is a weak boundary point.

2. Cluster sets of meromorphic functions of bounded valence

Let R be an open Riemann surface and f a meromorphic function of bounded valence on R. We call

$$\mathrm{Cl}(f,\beta)=\cap\{\overline{f(R-K)}\,;\,K\text{ is a compact set in }R\}$$

the cluster set of f at the Kerékjártó–Stoïlow boundary β of R.

Lemma 1. *Let f be a meromorphic function of bounded valence on R. Then $\mathrm{Cl}(f,\beta)$ is nowhere dense in the extended complex plane.*

Proof. Suppose that $\mathrm{Cl}(f,\beta)$ includes an open disk D_0. There exist a point $z_1\in R$ and a relatively compact neighbourhood U_1 of z_1 such that $f(U_1)\subset D_0$. Inductively, for each integer m $(m=2,3,\dots)$ we have points $z_m\in R$ and neighbourhoods U_m of z_m such that $f(U_m)\subset f(U_{m-1})$ and $U_m\cap\left(\cup_{i=1}^{m-1}U_i\right)=\varnothing$. This is contradictory to f being of bounded valence.

Lemma 2. *Let f be a meromorphic function of bounded valence on R and γ' be a component of $\mathrm{Cl}(f,\beta)$ which is a proper continuum. Then, for any open disk D with $D\cap\gamma'\neq\varnothing$, there exists a point $p\in\beta$ such that $\mathrm{Cl}(f,p)$ is a proper continuum and $\mathrm{Cl}(f,p)\cap\gamma'\cap D\neq\varnothing$.*

Proof. Contrary to the assertion, suppose that $\mathrm{Cl}(f,p)$ reduces to a point for any $p\in\beta$ satisfying $\mathrm{Cl}(f,p)\cap\gamma'\cap D\neq\varnothing$. There exists a point $p_1\in\beta$ such that $\mathrm{Cl}(f,p_1)\cap\gamma'\cap D\neq\varnothing$. Since $\mathrm{Cl}(f,p_1)$ is a singleton, denoted by w_1, we can take an end V_1 of p_1 such that $f(V_1)\subset D$ and the unbounded component of $\hat{\mathbf{C}}-f(V_1)$ meets γ'.

Now suppose $f(V_1)\cap\gamma'=\varnothing$. There is an end V_1' of p_1 such that V_1' is properly contained in V_1 and $f(\partial V_1')\cap\gamma'=\varnothing$. Then the unbounded component E of $\hat{\mathbf{C}}-f(V_1')$ contains γ'. On the other hand, $f(V_1')$ contains a sequence of points converging to w_1. Hence the outer boundary ∂E of $f(V_1')$ is contained in γ'. Similarly we should have the other end V_1'' of p_1 such that E is properly included in the unbounded component of $\hat{\mathbf{C}}-f(V_1'')$ and the outer boundary of $f(V_1'')$ is contained in γ'. This is contradictory to $f(V_1)$ being connected. Hence, $f(V_1)\cap\gamma'\neq\varnothing$.

Let w_2 $(\neq w_1) \in f(V_1) \cap \gamma'$. Then there exist a relatively compact subregion U_1 of V_1 such that $w_2 \in f(U_1) \subset f(V_1) \subset D$. On the other hand, there is a point $p_2 \in \beta$ such that $\mathrm{Cl}(f, p_2) = \{w_2\} \subset \gamma' \cap D$. Hence there is an end V_2 such that $f(V_2) \subset f(U_1)$. Therefore there exist a point $w_3 \in \gamma'$ and a relatively compact subregion U_2 of V_2 such that $w_3 \in f(U_2)$ and $U_1 \cap U_2 = \varnothing$. Inductively, we have ends V_m and relatively compact subregions U_m of V_m such that $f(U_m) \subset f(U_{m-1})$ and $U_m \cap \left(\cup_{i=1}^{m-1} U_i\right) = \varnothing$. This is contradictory to f being of bounded valence.

3. Meromorphic function of bounded valence and absolutely disconnected boundary

Let R be an open Riemann surface and β be the Kerékjártó–Stoïlow boundary of R. Let f be a meromorphic function of bounded valence. We denote the totality of components of $\hat{\mathbf{C}} - \mathrm{Cl}(f, \beta)$ by $\{G_i\}$. Note that $f(R) \cap G_i \neq \varnothing$ implies $f(R) \supset G_i$.

Lemma 3. *Let f be a meromorphic function of bounded valence on R. Suppose that p is a weak boundary point of β such that $\mathrm{Cl}(f,p)$ is a proper continuum. Then, for any disk D with $D \cap \mathrm{Cl}(f,p) \neq \varnothing$ there is an infinite number of members $\{G_{i_n}\}$ of $\{G_i\}$ such that $D \cap G_{i_n} \neq \varnothing$ and $f(R) \supset G_{i_n}$.*

Proof. Denote $\mathrm{Cl}(f,p)$ by γ. We may assume that γ is a bounded closed set in $\mathbf{C}$. Take an end V_0 of p such that $\overline{f(V_0)}$ is a bounded closed set in $\mathbf{C}$. Let $\varrho\,|dz|$ be the linear density on V_0 which is the pull-back of the Lebesgue measure on $\mathbf{C}$ by f. Since f is of bounded valence on $f(V_0)$, $A(\varrho, V_0) < \infty$ and $L(\Gamma(p, V_0), \varrho) = 0$. Take a point $w_0 \in \gamma$ and a disk $D(w_0, \varepsilon) = \{w\,;\,|w - w_0| < \varepsilon\}$ whose boundary meets γ. By Lemma 1, there is a member of $\{G_i\}$, say G_1, such that $D(w_0,\varepsilon) \cap G_1 \neq \varnothing$ and $f(R) \supset G_1$.

Contrary to the assertion, suppose that there is only a finite number of members, say $\{G_i\}_{i=1,\dots n}$, such that $D(w_0,\varepsilon) \cap G_i \neq \varnothing$ and $f(R) \supset G_i$. We can choose ends $\{V_k\}$ of p such that $\partial V_k \in \Gamma(p, V_0)$ and $\int_{\partial V_k} \varrho\,|dz| < \varepsilon/2^k$. Let J_k be the unbounded component (that is, the component including ∞) of $\hat{\mathbf{C}} - f(\partial V_k)$. Since $f(\partial V_k)$ cannot enclose $\gamma \cap D(w_0, \varepsilon)$, J_k contains a point in $\gamma \cap D(w_0,\varepsilon)$. There exist a G_i $(1 \leq i \leq n)$ and a component K_i of $J_k \cap G_i$ such that $f(V_k) \supset K_i$. We assume that ∂V_k is oriented so that V_k lies to the left of ∂V_k. Then $f(\partial V_k)$ is a finite union of oriented closed curves. If the boundary of a component of $J_k \cap G_i$ contains an oriented subarc C of $f(\partial V_k)$ and the component lies to the left of C, then the component is contained in $f(V_k)$. Generally the sheet number of $f(V_k)$ over a component of $\{f(V_k) - J_k - f(\partial V_k)\} \cap G_i$ is higher than the winding number of $f(\partial V_k)$ about a point in the component. In other words we can say what follows. Let K_i' be any other component of $J_k \cap G_i$. Take $\zeta_1 \in K_i$ and $\zeta_2 \in K_i'$. Then we can take an oriented piecewise analytic arc l connecting ζ_1 to ζ_2 in G_i. When l crosses $f(\partial V_k)$ once transversally from right to left (or left to right), then the sheet number of $f(V_k)$ increases (or decreases) by one. Since ζ_1 and ζ_2 are not enclosed by $f(\partial V_k)$, the number of times l crosses $f(\partial V_k)$ from right to left is equal to the number of times l crosses from left to right. Hence we can conclude that $\zeta_2 \in f(V_k)$. Therefore $J_k \cap G_i$ is contained in $f(V_k)$. There exist G_{i_0} $(1 \le i_0 \le n)$ and a sequence of ends V_{k_m} of p such that $J_{k_m} \cap G_{i_0} \subset f(V_{k_m})$. Let W_0 be a closed disk contained in G_{i_0}. Since $\cap \overline{f(V_{k_m})} = \gamma$, for every sufficiently large number k_m, $(J_{k_m} \cap G_{i_0}) \cap W_0 = \varnothing$; hence $J_{k_m} \cap G_{i_0} \subset G_{i_0} - W_0$. Since the length of ∂J_k is less than $\varepsilon/2^k$,

$$\{\text{area of } (G_{i_0} - W_0)\} \ge \{\text{area of } J_{k_m} \cap G_{i_0}\} \ge \{\text{area of } G_{i_0}\} - \frac{\varepsilon^2}{\pi 2^{2k_m+2}}.$$

This is a contradiction for a sufficiently large number k_m.

We say that the Kerékjártó–Stoïlow boundary β of R is absolutely disconnected if every point $p \in \beta$ is weak.

Theorem 1. *Let f be a meromorphic function of bounded valence on R and β be absolutely disconnected. Then $\mathrm{Cl}(f,\beta)$ is totally disconnected. Especially, f admits a continuous extension to β.*

Proof. Suppose that $\mathrm{Cl}(f,\beta)$ is not totally disconnected. We may assume that $\mathrm{Cl}(f,\beta)$ is a bounded closed set in $\mathbf{C}$. Let $\varrho\,|dz|$ be the linear density on R which is the pull-back of the Lebesgue measure on $\mathbf{C}$ by f on $R - f^{-1}(\infty)$ and 0 on $f^{-1}(\infty)$. There is a component G_i of $\hat{\mathbf{C}} - \mathrm{Cl}(f,\beta)$ such that $f(R) \supset G_i$ and there is a boundary component γ_i of G_i which is a proper continuum. Take a disk $D(w_0,\varepsilon) = \{w\,;\, |w - w_0| < \varepsilon\}$ such that $w_0 \in \gamma_i$ and $\partial D(w_0,\varepsilon) \cap \gamma_i \ne \varnothing$.

We prove that there exist $w_1 \in \gamma_i \cap D(w_0,\varepsilon)$ and $z_1 \in R$ such that $f(z_1) = w_1$. Contrary to the assertion, suppose that $f^{-1}(\gamma_i \cap D(w_0,\varepsilon)) = \varnothing$. There is a sequence $\{a_n\}$ in G_i which converges to w_0. Then $\{f^{-1}(a_n)\}$ does not cluster in R. Let $q \in \beta$ be one of the accumulation points of $\{f^{-1}(a_n)\}$. Then $\mathrm{Cl}(f,q) \ni w_0$. Take an end V_0 of q such that $\overline{f(V_0)}$ is a bounded closed set in $\mathbf{C}$. Since q is weak, there is a sequence of ends $\{V_k\}$ of q such that $\partial V_k \in \Gamma(q,V_0)$ and $\int_{\partial V_k} \varrho\,|dz| < \varepsilon/2^k$. By our supposition $f(\partial V_k)$ does not meet $\gamma_i \cap D(w_0,\varepsilon)$. Since $f(V_k) \cap G_i \ne \varnothing$, by a similar argument in the proof of Lemma 3, $f(V_k) \supset G_i \cap$ $\{$ the unbounded component of $\hat{\mathbf{C}} - f(\partial V_k)\}$. It follows that $\mathrm{Cl}(f,q)$ contains G_i, which contradicts Lemma 1. Hence there exist $w_1 \in \gamma_i \cap D(w_0,\varepsilon)$ and $z_1 \in R$ such that $f(z_1) = w_1$. Therefore we have a relatively compact neighbourhood U_1 of z_1 and a disk $D(w_1,\varepsilon_1)$ such that $f(U_1) = D(w_1,\varepsilon_1) \subset D(w_0,\varepsilon)$.

Since $D(w_1,\varepsilon_1) \cap \gamma_i \ne \varnothing$, by Lemmas 2 and 3, there is an infinite number of members $\{G_{i_n}\}$ of $\{G_k\}$ such that $D(w_1,\varepsilon_1) \cap G_{i_n} \ne \varnothing$ and $\overline{f(R)} \supset G_{i_n}$. Since $\overline{f(R - U_1)} \cap D(w_1,\varepsilon_1) \ne \varnothing$, we can take G_j $(j \ne 1)$ such that $\overline{f(R - U_1)} \supset G_j \cap D(w_1,\varepsilon_1)$ and $D(w_1,\varepsilon_1)$ meets a component γ_j of ∂G_j which is not a point. In the same way we can take $w_2 \in \gamma_j \cap D(w_1, \varepsilon_1)$ and a relatively compact neighbourhood U_2 of w_2 such that $f(U_2) = D(w_2, \varepsilon_2) \subset D(w_1, \varepsilon_1)$ and $U_2 \cap U_1 = \varnothing$. We can repeat this procedure infinitely, which is contradictory to f being of bounded valence.

Remark. We denote by O_{SB} the class of Riemann surfaces which tolerates no univalent bounded analytic function. Let AD be the family of analytic functions with finite Dirichlet integral. We denote by O_{AD} the class of Riemann surfaces on which there are no nonconstant AD-functions and by $O_{AD,n}$ the class of Riemann surfaces on which there are no AD-functions of at most n-valence. Järvi ([3]) asserts that a plane region G in the class $O_{SB} - O_{AD}$ belongs to $\bigcap_{n=1}^{\infty} O_{AD,n} - O_{AD}$, provided that the assertion of Lemma 2 in Mori [7] holds for planar Riemann surfaces. We have proved in Section 1 that the assertion of the lemma does not hold even for planar Riemann surfaces. However, we know that the Kerékjártó–Stoïlow boundary of a plane region in the class O_{SB} is absolutely disconnected. Hence, we can apply Theorem 1 to restore the proof of Järvi ([3, p. 179]). Accordingly, we conclude that there exists a planar Riemann surface which belongs to the class $\bigcap_{n=1}^{\infty} O_{AD,n} - O_{AD}$.

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