ON SOME THEOREMS OF LITTLEWOOD AND SELBERG II

K. Ramachandra and A. Sankaranarayanan

1. Introduction

In a previous paper with the same title [1] we proved some theorems about the Riemann zeta-function under the assumption of Riemann hypothesis. In this paper we prove some unconditional results on ζ(s). Stating somewhat more generally we prove the following.

Theorem. Let s = σ + it and

\[ F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1} \quad (\sigma > 1), \]

where p runs through all primes and \( \omega(p) \) are some complex numbers (independent of s) with absolute value not exceeding 1. Suppose \( \alpha \) and \( \delta \) are positive constants satisfying \( \frac{1}{2} \leq \alpha \leq 1 - \delta \) and that in \( \{ \sigma \geq \alpha - \delta, T - H \leq t \leq T + H \} \) \( F(s) \) can be continued analytically and there \( |F(s)| < T^A \). Here A is a positive constant, \( T \geq T_0, H = C \log \log \log T \) where \( T_0 \) and \( C \) are large positive constants. Let \( F(s) \neq 0 \) in \( \{ \sigma > \alpha, T - H \leq t \leq T + H \} \). Then for \( \alpha + \delta \leq \sigma \leq 1 - \delta, t = T \), we have

\[ \frac{F'(s)}{F(s)} = O((\log T)^\Theta) \]

and

\[ \log F(s) = O((\log T)^\Theta (\log \log T)^{-1}), \]

where \( \Theta = (1 - \sigma)/(1 - \alpha) \).

Remark 1. The application to \( \zeta(s) \) is immediate by density results. By standard methods we can also prove density results for \( F(s) \) provided in, say \( \{ \sigma \geq 3/4, t \geq T_0 \} \) \( F(s) \) can be continued analytically and there \( |F(s)| < t^A \).

Remark 2. The theorem can be generalised further by allowing some growth condition for \( \omega(p) \). We can state our theorem in a slightly different way to allow \( F(s) = L(s, \chi) \) for characters \( \chi(\mod q) \), for example for \( |t| \leq q \).

Remark 3. We can state a result for $\alpha + \delta \leq \sigma \leq 1 + \delta$ analogous to the remark made by D.R. Heath-Brown in Section 14.33 of [3].

Remark 4. In a later paper with the same title we hope to obtain inequalities dealing with $|\arg F(\sigma + it)|$ for $\sigma \geq \alpha$ and $\log |F(\sigma + it)|$ for $\sigma > \alpha$, analogous to what we proved in [1].

Remark 5. The $t$-interval condition $T - H \leq t \leq T + H$ is made possible by the kernel function $\exp((\sin \omega)^2)$ used extensively by Ramachandra in his papers.

2. Notation

In Lemmas 1 and 2 we borrow results from [4] and [3] in the same notation. But in Lemma 2 we have changed the result contained in [3] to suit our needs (see Remark below Lemma 2). We use $z = x + iy$, $w = u + iv$ and $s = \sigma + it$ in various contexts and we hope this does not cause confusion. For any analytic function $F(s)$ we write $(F'/F)(s)$ for $F'(s)/F(s)$. The symbol $\equiv$ denotes a definition.

Lemma 1. Let $f(z)$ be analytic in $|z| < R$. Suppose $f(0)$ is different from zero. For $0 \leq x < R$ let $n(x)$ denote the number of zeros of $f(z)$ in $|z| \leq x$. Then for $0 \leq r < R$ we have

$$
\int_0^r n(x) \frac{dx}{x} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(re^{i\theta})}{f(0)} \right| d\theta.
$$

Remark. This result is called Jensen's theorem. For its proof see pages 124 to 126 of [4].

Lemma 2. If $f(s)$ is regular and

$$
\left| \frac{f(s)}{f(s_0)} \right| \leq e^M \quad (M > 1)
$$

in $|s - s_0| \leq r$, then for any constant $\epsilon$ (with $0 < \epsilon < \frac{1}{2}$), we have

$$
\left| \frac{f'}{f}(s) - \sum \frac{1}{s - \rho} \right| \leq \epsilon \frac{M}{r} \quad \text{in } |s - s_0| \leq (1 - 2\epsilon)r,
$$

where $\rho$ runs over all zeros of $f(s)$ such that

$$
|\rho - s_0| \leq (1 - \epsilon)r.
$$

Remark. From Lemma 1 above and the concluding remarks in [2] on (24), (25) and (26) of that paper we obtain Lemma 2 which is nearly contained as Lemma 2 of Section 3.9 of [3].
**Lemma 3.** Let \( z = x + iy \) be a complex variable and

\[
F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z} = \prod_p \left(1 - \frac{\omega(p)}{p^z}\right)^{-1} \quad (x > 1),
\]

where \( p \) runs over all primes and \( \omega(p) \) are complex numbers independent of \( z \) with \( |\omega(p)| \leq 1 \). Let \( F(z) \) be regular in \( \{ x \geq \alpha - \delta, T - H \leq y \leq T + H \} \) and there \( |F(z)| < T^A \). Here \( T \geq T_0, \frac{1}{2} \leq \alpha \leq 1 - \delta, H = C \log \log \log T \) where \( \delta \) is a small positive constant, \( \alpha \) is a positive constant, and \( T_0 \) and \( C \) are large positive constants. Put \( z_0 = 2 + iy_0 \) where \( T - \frac{1}{2} H \leq y_0 \leq T + \frac{1}{2} H \). Then for \( z_1 = x_0 + iy_0 \) with \( \alpha - \delta_1 \leq x_0 \leq 2 \), we have,

\[
\left| \frac{F''(z_1)}{F(z_1)} - \sum_{\rho \in D} \frac{1}{z_1 - \rho} \right| \ll \log T,
\]

where \( \rho \) runs over all the zeros of \( F(z) \) in the disc \( D = D(z_0, 2 - \alpha + 2\delta_1) \) defined by

\[
|z - z_0| \leq 2 - \alpha + 2\delta_1.
\]

Here \( \delta_1 \) is any positive constant such that \( 2\delta_1 < \delta \). In particular the lemma holds for \( z_1 = \alpha + iy_0 \).

**Remark.** The lemma is trivially true for \( z_1 = x_0 + iy_0 \) with \( x_0 \geq 1 + \delta \).

**4. Proof of the theorem**

**Lemma 4.** Let \( s = \sigma + it \) where \( \alpha + \delta_1 \leq \sigma \leq 1 - \delta_1, w = u + iv, 2 \leq X \leq \exp(10(\log \log T)/(1 - \alpha)), B \geq 10000. \) Then

\[
I \equiv \frac{1}{2\pi i} \int_{u=2}^{} \frac{F'}{F}(s + w)X^w \exp \left( \left( \frac{w}{B} \right)^2 \right) \frac{dw}{w}
\]

\[
= O(X^{1-\sigma}).
\]

**Proof.** The proof follows from

\[
\frac{1}{2\pi i} \int_{u=2}^{} \left( \frac{X}{n} \right)^w \exp \left( \left( \frac{w}{B} \right)^2 \right) \frac{dw}{w} = 1 + O\left( \frac{n}{X} \right) \text{ or } O\left( \frac{X}{n} \right),
\]

according as \( n \leq X \) or \( n > X \).
Lemma 5. Let $3V \sim H$ and $|v| \leq V'$ ($V'$ will be chosen to be asymptotic to $V$). Then for (fixed $s = \sigma + it$ and all $w = u + iv$), $u + \sigma \geq \alpha - \delta_1$, we have,

$$\left| \frac{F'}{F}(s+w) - \sum_{\rho} \frac{1}{s+w-\rho} \right| \ll \log T$$

where $\rho$ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2 - \alpha + 2\delta_1)$ defined by $|z - z_0| \leq 2 - \alpha + 2\delta_1$ where $z_0 = 2 + it + iv$.

**Proof.** The proof follows from Lemma 3.

Lemma 6. Let

$$\mu(\rho) = \frac{2^{s+w-\rho} - 1}{(s+w-\rho)^2 \log 2}$$

and

$$\mu = \sum_{\rho} \mu(\rho),$$

where $\rho$ runs over all the zeros of $F(z)$ in the rectangle $R$ defined by

$$R: \{ \text{Re} z \geq \alpha - 2\delta_1, |t - y| \leq 2V \}.$$

Then for $|v| \leq V'$ ($V'$ will be chosen asymptotic to $V$) and $u + \sigma \geq \alpha - \delta_1$ we have,

$$\left| \frac{F'}{F}(s+w) - \mu \right| \ll \log T.$$  

**Proof.** For $D$ as in Lemma 5, we have

$$\sum_{\rho \notin D} \frac{1}{s+w-\rho} - \sum_{\rho \in D} \mu(\rho) = O(\log T),$$

since (by Jensen’s theorem) there are $O(\log T)$ zeros involved and for any fixed $\rho$

$$\left| \frac{1}{s+w-\rho} - \mu(\rho) \right| \ll 1$$

since it is so on $|s + w - \rho| = 10$. Again

$$\left| \sum_{\rho \notin D, \rho \in R} \mu(\rho) \right| \ll \log T$$

since for $\rho \notin D$, we have

$$|s + w - \rho| \geq |z_0 - \rho| - |s + w - z_0| \geq 2 - \alpha + 2\delta_1 - (2 - \alpha + \delta_1) = \delta_1$$

**Lemma 7.** It is possible to choose $V' \sim V$ such that on $v = \pm V'$ and $u + \sigma \geq \alpha - 10\delta_1$ we have,

$$\left| \sum_{\rho \in R} \mu(\rho) \right| \ll (\log T)^2.$$

Here $10\delta_1 < \delta$. 
Proof. By Jensen’s theorem the number of zeros of $F(z)$ in $\{x \geq \alpha - 11\delta_1, Y \leq y \leq Y + 1\}$ with $11\delta_1 < \delta$ is $O(\log T)$ provided $T - 2V \leq Y \leq T + 2V$. Hence the number of zeros of $F(s + w)$ in $\{u + \sigma \geq \alpha - 11\delta_1, V - 1 \leq v \leq V + 1\}$ is $O(\log T)$ and so there exists a line $v = V'$ such that on this line $|s + w - \rho| \gg 1/\log T$. This proves Lemma 7 since the number of zeros in $2 \geq |s + w - \rho| \gg 1/\log T$ is $O(\log T)$ and also the zeros $\rho$ with $|s + w - \rho| \geq 2$ contribute $O(\log T)$. The total contribution to $\mu$ is therefore $O((\log T)^2)$ and this proves Lemma 7 completely.

Lemma 8. We have (if there are no zeros of $F(z)$ in $x > \alpha$ and $T - H \leq y \leq T + H$)

\begin{equation}
I = \frac{F'}{F}(s) + \frac{1}{2\pi i} \int_{u = \alpha - \sigma} \left\{ \left( \frac{F'}{F}(s + w) - \mu \right) + \mu \right\} X^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w} + o(1)
\end{equation}

where the integration is restricted to $|v| \leq V'$ and we take the integral to mean the limit as we move from $u \leq 2$ to $u = \alpha - \sigma$.

Proof. First, the contribution to $I$ of Lemma 4 from $|v| \geq V'$ is $o(1)$. The lemma now follows on moving the line of integration to $u = \alpha - \sigma$ since by Lemmas 6 and 7 the horizontal bits contribute $o(1)$. Note that $\exp(\frac{(\sin w)^2}{B})$ decays like $(\exp \exp (|v|/10))^{-1}$ uniformly in $|u| \leq 1/10$.

Lemma 9. We have,

\begin{equation}
\int_{u = \alpha - \sigma, |v| \leq V'} \left( \frac{F'}{F}(s + w) - \mu \right) X^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w} = O(X^{\alpha - \sigma} \log T).
\end{equation}

Proof. The proof follows by Lemma 6.

Lemma 10. We have

\begin{equation}
J \equiv \frac{1}{2\pi i} \int_{u = \alpha - \sigma, |v| \leq V'} \mu X^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w} = O(X^{\alpha - \sigma} \log T).
\end{equation}

Proof. Let as before $11\delta_1 < \delta$. We move the line of integration to $u = \alpha - \sigma - 10\delta_1$. We obtain

\[ J = \sum_{\rho \in \mathbb{R}} \frac{X^{\rho - s}}{\rho - s} \exp \left( \left( \sin \frac{\rho - s}{B} \right)^2 \right) + \frac{1}{2\pi i} \int_{u = \alpha - \sigma - 10\delta_1} \mu X^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w} \]

the last integration being subject to $|v| \leq V'$. Now since $\text{Re} \rho \leq \alpha$ and $\sigma \geq \alpha + \delta$ and $\exp((\sin w)^2)$ tapers in $|u| \leq 1/10$ uniformly as fast as $(\exp \exp (|v|/10))^{-1}$, the lemma follows.
Lemma 11. We have, uniformly for \( \{\alpha + \delta \leq \sigma \leq 1 - \delta, t = T\} \)

\[
\frac{F'}{F}(s) = O((\log T)^\Theta)
\]

provided \( F(z) \neq 0 \) in \( \{x > \alpha, T - H \leq y \leq T + H\} \). Here \( \Theta \) is as stated in the theorem.

Proof. The proof follows from Lemmas 4, 8, 9 and 10 on choosing \( X \) by \( X^{1-\alpha} = \log T \).

Lemma 12. Subject to the conditions of Lemma 11,

\[
\log F(s) = O\left((\log T)^\Theta (\log \log T)^{-1}\right).
\]

Proof. The proof follows by integrating (22) with respect to \( \sigma \) from \( \sigma \) to \( \sigma' \equiv \frac{1}{2}(1 + \sigma) \), since (as will be proved in the next lemma) \( \log F(\sigma' + it) = O((\log T)^{\Theta - \epsilon}) \), for some fixed \( \epsilon > 0 \).

Lemma 13. We have, with \( \sigma' = \frac{1}{2}(1 + \sigma) \),

\[
\log F(\sigma' + it) = O((\log T)^{\Theta - \epsilon})
\]

for some fixed \( \epsilon = \epsilon(\sigma) > 0 \).

Proof. By a simple application of the Borel–Carathéodory theorem we have

\[
\log F(\sigma + \epsilon + it) = O(\log T) \quad \text{for} \quad \{\sigma \geq \alpha, T - \frac{1}{2}H \leq t \leq T + \frac{1}{2}H\}. \quad \text{Put} \quad s' = \sigma' + it.
\]

We proceed by considering (as in Lemma 4) the integral

\[
\frac{1}{2\pi i} \int_{u=2} F(s' + w)X^w \exp \left( \left( \frac{\sin \frac{w}{B}}{B} \right)^2 \right) \frac{dw}{w}
\]

and moving the portion \( |v| \leq H/3 \) of the line of integration to \( u + \sigma' = \alpha + \epsilon \) i.e. \( u = \alpha - \sigma' + \epsilon \). This leads to the Lemma. With Lemmas 11, 12 and 13 the theorem stated in the introduction is completely proved.

Remark. If \( F(s) \neq 0 \) in \( \{\sigma > \alpha, T - C \leq t \leq T + C\} \) where \( \frac{1}{2} \leq \alpha \leq 1 - \delta \) and here \( |F(s)| < T^A \), \( (T \geq 10) \), it follows by the proof of Theorem 14.2 of [3] that if \( C = C(\delta, \epsilon, \sigma_0) \) then uniformly in \( \alpha < \sigma_0 \leq \sigma \leq 1 \) and \( t = T \), we have \( F(s) = O((\log T)^{\Theta + \epsilon}) \), where the \( O \)-constant depends only on \( A, \sigma_0, \delta \) and \( \epsilon \).
References


K. Ramachandra
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road
Bombay 400 005
India

A. Sankaranarayanan
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road
Bombay 400 005
India

Received 6 June 1990