ON SOME THEOREMS OF LITTLEWOOD AND SELBERG III

K. Ramachandra and A. Sankaranarayanan

1. Introduction

In paper II with the same title [2] we proved some unconditional results about \( \zeta(s) \) in a more general set up. In this paper we continue these investigations. As before we begin by stating the final result of this paper as follows.

**Theorem 1.** Let \( s = \sigma + it \) and

\[
F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}
\]

where \( p \) runs over all primes and \( \omega(p) \) are arbitrary complex numbers (independent of \( s \)) with absolute value not exceeding 1. Suppose \( \alpha \) and \( \delta \) are positive constants satisfying \( \frac{1}{2} \leq \alpha \leq 1 - \delta \) and that in \( \{ \sigma \geq \alpha - \delta, T - H \leq t \leq T + H \} \), \( F(s) \) can be continued analytically and there \( |F(s)| < T^A \). Here \( A \) is a positive constant, \( T \geq T_0 \), \( H = C \log \log T \) where \( T_0 \) and \( C \) are large positive constants. Let \( F(s) \neq 0 \) in \( \{ \sigma > \alpha, T - H \leq t \leq T + H \} \). Then for \( \alpha \leq \sigma \leq \alpha + C_1 (\log \log T)^{-1} \), and \( T - \frac{1}{2} H \leq t \leq T + \frac{1}{2} H \), we have,

(a) \( \log |F(\sigma + it)| \) lies between \( C_2 (\log T)(\log \log T)^{-1} \) and
\[
-C_3 (\log T)(\log \log T)^{-1} \log \{ C_4 ((\sigma - \alpha) \log \log T)^{-1} \}
\]
(b) \( |\arg F(\sigma + it)| \leq C_5 (\log T)(\log \log T)^{-1} \)

where \( C_1, C_2, C_3, C_4 \) and \( C_5 \) are certain positive constants.

**Corollary 1.** For \( \alpha + C_1 (\log \log T)^{-1} \leq \sigma \leq 1 - \delta, t = T \), we have,

\[
|\log F(\sigma + it)| \leq C_6 (\log T)^{\theta} (\log \log T)^{-1}
\]

where \( \theta = (1 - \sigma)/(1 - \alpha) \) and \( C_6 \) is a positive constant.

**Corollary 2.** For \( \alpha \leq \sigma \leq 1 - \delta, t = T \), we have,

\[
|F(\sigma + it)| \leq \exp(C_7 (\log T)^{\theta} (\log \log T)^{-1})
\]

where \( \theta \) is as before and \( C_7 \) is a positive constant.

Remark 1. The application of Theorem 1 to $\zeta(s)$ is clear by density results. Under the conditions of the theorem we can also prove density theorems for $F(s)$.

Remark 2. We now indicate the proof of the corollaries. Corollary 1 is already proved in [2] for the $\sigma$-range $\alpha + \delta \leq \sigma \leq 1 - \delta$. The theorem above gives an upper bound $|\log F(\sigma + it)| \leq C_8(\log T)(\log \log T)^{-1}$ for $\sigma = \sigma_1 = \alpha + (\log \log T)^{-1}$ and $T - H/3 \leq t \leq T + H/3$ (where $C_8 > 0$ is a constant). We have already an upper bound $|\log F(\sigma + it)| \leq C_8(\log T)^{\theta}(\log \log T)^{-1}$ for $\sigma = \sigma_2 = 1 - \delta$ and the same $t$-range. We now apply maximum modulus principle to the function (for suitable $X > 0$)

$$\varphi(w) = (\log F(s + w))X^w \exp \left(\left(\sin \frac{w}{100}\right)^2\right).$$

According to this its absolute value at $w = 0$, namely $|\log F(s)|$ is majorised by its maximum modulus on the boundary of the rectangle $\{\sigma_1 - \sigma \leq \Re w \leq \sigma_2 - \sigma, -H/10 \leq \Im w \leq H/10\}$. Corollary 1 follows by a proper choice of $X$ as a suitable power of $\log T$. (The bound for $|\log F(s + w)|$ namely $O((\log T)^{20})$, needed on the horizontal sides of the rectangle can be obtained by Borel–Carathéodory theorem). This completes the proof of Corollary 1. Corollary 2 follows from Corollary 1 and the part (a) of the theorem.

By a modification of our proof of Theorem 1 we can prove

**Theorem 2.** Let $1 - \delta \leq \alpha \leq 1 - 10(\log \log T)^{-1}$. Then for $\alpha \leq \sigma \leq \alpha + (\log \log T)^{-1}$ (in place of $(\alpha \leq \sigma \leq \alpha + C_1(\log \log T)^{-1})$) the assertions (a) and (b) hold, provided $F(s) \neq 0$ in the region mentioned in Theorem 1.

From Theorems 1 and 2 we can prove by the methods of [3] theorems like

**Theorem 3.** Let $\lambda_0$ ($> 7/12$) be a constant. Then for any fixed $k$, the number of lattice points $(n_1, n_2, \ldots, n_k)$ in the first quadrant of the $k$ dimensional Euclidean space such that

$$X \leq n_1 \cdots n_k \leq X + X^{\lambda_0}$$

is given by

$$X^{\lambda_0}(\log X)^{k-1} + O(X^{\lambda_0}(\log X)^{k-2}).$$

**Remark.** In fact we can prove an asymptotic formula valid uniformly for $k \leq \varepsilon(\log \log X)(\log \log \log X)^{-1}$. These and similar results will form the subject matter of another paper. Theorem 3 is due to M.N. Huxley and C. Hooley (unpublished).

In what follows we will prove only Theorem 1.
2. Notation

We use \( z = x + iy \), \( w = u + iv \) and \( s = \sigma + it \) in various contexts and we hope that this does not cause confusion. For any analytic function \( F(s) \) we write \((F'/F)(s)\) for \( F'(s)/F(s) \). The symbol \( \equiv \) denotes a definition.

3. Proof of Theorem 1

It suffices to prove the theorem for \( t = T \). Because we can consider a larger \( H \), and every point of the smaller interval \( T - \frac{1}{2}H \leq t \leq T + \frac{1}{2}H \) will be a midpoint \( \tau \) of a bigger interval of the type \( \tau - \frac{1}{2}H \leq t \leq \tau + \frac{1}{2}H \) contained in \([T-H, T+H]\). We split the proof into three parts. The first two parts deal with an upper bound for a positive quantity \( J_0 \) in the form \( O((\log T)(\log \log T)^{-1}) \). The third part deals with an application of this result to the proof of the theorem.

**Lemma 1.** Let \( z = x + iy \) be a complex variable and

\[
F(z) = \sum_{n=1}^{\infty} a_n n^{-z} = \prod_p (1 - \omega_p p^{-z})^{-1}
\]

where \( p \) runs over all the primes and \( \omega(p) \) are complex numbers independent of \( z \) with \( |\omega(p)| \leq 1 \). Let \( F(z) \) be regular in \( \{x \geq \alpha - \delta, T-H \leq y \leq T+H\} \) and there \( |F(z)| < T^A \) where \( A > 0 \) is a constant. Here \( T \geq T_0, \frac{1}{2} \leq \alpha \leq 1 - \delta \), \( H = C \log \log \log T \) where \( \delta \) is a small positive constant, \( \alpha \) is a constant and \( T_0 \) and \( C \) are large positive constants. Put \( z_0 = 2 + iy \). Then for \( z_1 = x_0 + iy_0 \) where \( T - \frac{1}{2}H \leq y_0 \leq T + \frac{1}{2}H \) with \( \alpha - \delta_1 \leq x_0 \leq 2 \), we have

\[
\left| \frac{F'(z_1)}{F(z_1)} - \sum_{\rho \in D} \frac{1}{z_1 - \rho} \right| \ll \log T,
\]

where \( \rho \) runs over all the zeros of \( F(z) \) in the disc \( D = D(z_0, 2 - \alpha + 2\delta_1) \) defined by

\[
|z - z_0| \leq 2 - \alpha + 2\delta_1.
\]

Here \( \delta_1 \) is any positive constant. We will suppose \( 11\delta_1 < \delta \).

**Proof.** This is Lemma 3 of [2].

**Lemma 2.** Under the conditions of Lemma 1, we have,

\[
\text{Re} \frac{F'(x_0 + iy_0)}{F(x_0 + iy_0)} = \sum_{\rho \in D} \frac{x_0 - \beta}{(x_0 - \beta)^2 + (y_0 - \gamma)^2} + O(\log T)
\]

where we have written \( \rho = \beta + i\gamma \). This holds in particular for \( \alpha \leq x_0 \leq 2 \).
4. Another expression for the left-hand side of (5)

Lemma 3. Let $s = \sigma + it$ where $\alpha + (\log X)^{-1} = \sigma_1 \leq \sigma \leq 1 - \delta$ and $X = (\log T)^\lambda$ with some positive constant $\lambda < 1$. Let $B(\geq 100000)$ be a constant. Then

\[
I \equiv \frac{1}{2\pi i} \int_{u=2}^{F'} (s + w) \left( \frac{X^w - X^w}{w^2 \log X} \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \, dw \\
= O((\log T)^\lambda (\log \log T)^{-1}).
\]

Proof. We have,

\[
\frac{1}{2\pi i} \int_{u=2}^{X} \left( \frac{X}{n} \right)^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w^2 \log X} = (\log X)^{-1} \log \frac{X}{n} + O\left( \frac{n}{X \log X} \right),
\]
if $n \leq X$ and

\[
\frac{1}{2\pi i} \int_{u=2}^{X} \left( \frac{X}{n} \right)^w \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w^2 \log X} = O\left( \frac{X}{n \log X} \right),
\]
if $n \geq X$.

Hence for any constant $\varepsilon$ ($0 < \varepsilon < 1$), we have,

\[
\frac{1}{2\pi i} \int_{u=2}^{X} \left( \left( \frac{X^2}{n} \right)^w - \left( \frac{X}{n} \right)^w \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w^2 \log X} = 1 + O\left( \frac{n}{X \log X} \right),
\]
if $n \leq X$ and

\[
\frac{1}{2\pi i} \int_{u=2}^{X} \left( \left( \frac{X^2}{n} \right)^w - \left( \frac{X}{n} \right)^w \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w^2 \log X} = O_{\varepsilon}\left( \left( \frac{X^2}{n} \right)^\varepsilon (\log X^{-1}) \right),
\]
if $X \leq n \leq X^2$ and

\[
\frac{1}{2\pi i} \int_{u=2}^{X} \left( \left( \frac{X^2}{n} \right)^w - \left( \frac{X}{n} \right)^w \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) \frac{dw}{w^2 \log X} = O\left( \frac{X^2}{n \log X} \right),
\]
if $n \geq X^2$.

Thus if $1 - \delta + \varepsilon < 1$ we obtain

\[
I = O\left( \sum_{n \leq X} \frac{\Lambda(n)}{n^{1+\sigma} \log X} + \sum_{X \leq n \leq X^2} \left( \frac{X^2}{n} \right)^\varepsilon \frac{\Lambda(n)}{n^{1+\sigma} \log X} + \sum_{n \geq X^2} \frac{X^2 \Lambda(n)}{n^{1+\sigma} \log X} \right)
= O(X^{2-2\sigma}(\log X)^{-1}),
\]
where we have used $\Lambda(n) = \log p$ if $n = p^m$, 0 otherwise. This proves the lemma since $2\lambda(1 - \sigma) \leq \lambda$. 
Lemma 4. Let $3V$ be asymptotic to $H$ and $|v| \leq V'$ ($V'$ will be chosen to be asymptotic to $V$). Then for (fixed $s = \sigma + it$ and all $w = u + iv$), $u + \sigma \geq \alpha - \delta_1$, we have,

\[
\left| \frac{F'}{F}(s + w) - \sum_{\rho} \frac{1}{s + w - \rho} \right| \ll \log T,
\]

where $\rho$ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2 - \alpha + 2\delta_1)$ defined by $|z - z_0| \leq 2 - \alpha + 2\delta_1$ where $z_0 = 2 + it + iv$.

Proof. The proof follows from Lemma 1.

Lemma 5. Let

\[
\mu(\rho) = \frac{2^{s+w-\rho} - 1}{(s + w - \rho)^2 \log 2}
\]

and

\[
\mu = \sum_{\rho} \mu(\rho),
\]

where $\rho$ runs over all the zeros of $F(z)$ in “the rectangle” $R$ defined by

\[
R : \{ \text{Re } z \geq \alpha - 2\delta_1, |t - y| \leq 2V \}.
\]

Then for $|v| \leq V'$ ($V'$ will be chosen asymptotic to $V$) and $u + \sigma \geq \alpha - \delta_1$, we have,

\[
\left| \frac{F'}{F}(s + w) - \mu \right| \ll \log T.
\]

Proof. This is Lemma 6 of [2].

Lemma 6. It is possible to choose $V'$ (asymptotic to $V$) such that on $v = \pm V'$ and $u + \sigma \geq \alpha - 10\delta_1$, we have,

\[
\left| \sum_{\rho \in R} \mu(\rho) \right| \ll (\log T)^2.
\]

Proof. This is Lemma 7 of [2].

Remark. From now on we assume that $F(z) \neq 0$ in $\{ x > \alpha, T - H \leq y \leq T + H \}$.
Lemma 7. We have,

\[ I = \frac{F'}{F}(s) + I_1 + I_2 + S + O((\log T)^4(\log \log T)^{-1}), \]

where

\[ I_1 = \frac{1}{2\pi i} \int_{u=\alpha-\sigma-\frac{1}{2}\delta_1} \left( \frac{F'(s+w) - \mu}{\mu \frac{X^{2w} - X^w}{w^2 \log X}} \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) dw \]

\[ I_2 = \frac{1}{2\pi i} \int_{u=\alpha-\sigma-10\delta_1} \mu \left( \frac{X^{2w} - X^w}{w^2 \log X} \right) \exp \left( \left( \sin \frac{w}{B} \right)^2 \right) dw \]

and

\[ S = \sum_{\rho \in \mathbb{R}} \frac{X^{2(\rho-s)} - X^{\rho-s}}{(\rho-s)^2 \log X} \exp \left( \left( \sin \left( \frac{\rho-s}{B} \right) \right)^2 \right). \]

The two integrals in (14) are subject to \(|v| \leq V'\).

Proof. The proof follows by Cauchy’s theorem of residues.

Lemma 8. For \(\alpha + (\log X)^{-1} = \sigma_1 \leq \sigma \leq 1 - \delta\), we have,

\[ I_1 = O\left( (X^{2\alpha-2\sigma-\delta_1} + X^{\alpha-\sigma-\delta_1/2}) \frac{\log T}{\log \log T} \right), \]

\[ I_2 = O\left( (X^{2\alpha-2\sigma-2\delta_1} + X^{\alpha-\sigma-10\delta_1}) \frac{\log T}{\log \log T} \right) \]

and

\[ S = \sum_{\rho \in \mathbb{R}} \frac{X^{2\rho-2s} - X^{\rho-s}}{(\rho-s)^2 \log X} + O\left( \frac{\log T}{\log \log T} \right). \]

Proof. The estimates (15) and (16) follow from (11) and the fact that on \(u = \alpha - \sigma - 10\delta_1\) we have \(\mu = O(\log T)\) since \(|s + w - \rho| \geq \delta_1\) on this line.

Lemma 9. For \(\alpha + (\log X)^{-1} = \sigma_1 \leq \sigma \leq 1 - \delta\), we have,

\[ I_1 = O\left( \frac{\log T}{\log \log T} \right), \quad I_2 = O\left( \frac{\log T}{\log \log T} \right) \]

and

\[ S_0 = \omega \sum_{\rho \in \mathbb{R}} \frac{X^{2\beta-2\sigma} + X^{\beta-\sigma}}{(\log X)((\gamma - \beta)^2 + (t - \gamma)^2)} \]

where \(S_0\) denotes the sum in (17) and \(\omega\) is a complex number (depending on other parameters) with \(|\omega| \leq 1\).
Proof. The proof follows from Lemma 8.

**Lemma 10.** For $\alpha + (\log X)^{-1} = \sigma_1 \leq \sigma \leq 1 - \delta$, we have,

$$
\sum_{\rho \in D} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} = \omega' \sum_{\rho \in D} \frac{(X^{2\beta - 2\sigma} + X^{\beta - \sigma})(\sigma_1 - \alpha)}{(\sigma - \beta)^2 + (t - \gamma)^2} + O(\log T)
$$

(20)

where $\omega'$ is real and $|\omega'| \leq 1$.

Proof. The proof follows from Lemmas 2, 3, 7, 8 and 9.

**Lemma 11.** We have,

$$
J_0 \equiv \sum_{\rho \in D} \frac{\sigma_1 - \beta}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} = O(\log T).
$$

(21)

Proof. Put $\sigma = \sigma_1$ in (20). We have $\beta - \sigma_1 \leq \alpha - \sigma_1 = - (\log X)^{-1}$ and also $\sigma_1 - \alpha \leq \sigma_1 - \beta$ and so

$$
J_0 = \sum_{\rho \in D} \frac{\sigma_1 - \beta}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} = \omega'' \left( \frac{1}{e^2} + \frac{1}{e} \right) J_0 + O(\log T),
$$

where $|\omega''| \leq 1$

This proves the lemma since $e^{-2} + e^{-1} < 1$ and $|\omega''| \leq 1$.

**Lemma 12.** For $S_0$ defined by (19) we have,

$$
S_0 = w_1 \sum_{\rho \in D} \left( \frac{X^{2\beta - 2\sigma} + X^{\beta - \sigma}}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} \right)(\sigma_1 - \beta),
$$

(22)

with $|w_1| \leq 1$.

Proof. The lemma follows from $(\log X)^{-1} = \sigma_1 - \alpha \leq \sigma_1 - \beta$ and also $\sigma \geq \sigma_1$.

**Lemma 13.** With $\sigma_1 = \alpha + (\log X)^{-1}$, we have,

$$
\log F(\sigma_1 + it) = O\left( \frac{\log T}{\log \log T} \right).
$$

(23)

Proof. The lemma follows from

$$
\int_{\sigma_1}^{1 - \delta_1} \frac{F'}{F}(\sigma + it)\, dt = O\left( \frac{\log T}{\log \log T} \right) - \log F(\sigma_1 + it)
$$

and the fact that here the left-hand side is (by Lemmas 2, 3, 7, 8, 9, 11 and 12) $O((\log T)(\log \log T)^{-1})$. 

Lemma 14. For $\alpha \leq \sigma \leq \sigma_1$, we have,

$$ (\sigma_1 - \sigma) \frac{F'}{F}(\sigma_1 + it) = O\left(\frac{\log T}{\log \log T}\right). $$

Proof. The proof follows from Lemmas 2, 3, 7, 8, 9, 11 and 12.

Lemma 15. For $\alpha \leq \sigma \leq \sigma_1$ the quantity

$$ \int_{\sigma}^{\sigma_1} \operatorname{Re}\left(\frac{F'}{F}(\sigma_1 + it) - \frac{F'}{F}(u + it)\right) du $$

does not exceed $C_9(\log T)(\log \log T)^{-1}$, but exceeds $C_{10}(\log T)(\log \log T)^{-1} - C_{11}(\log T)(\log \log T)^{-1} \log ((C_{12}(\log \log T)(\sigma - \alpha))^{-1}),$

where $C_9, C_{10}, C_{11}$ and $C_{12}$ are positive constants.

Proof. By Lemma 2 the quantity in question is

$$ \int_{\sigma}^{\sigma_1} (J(\sigma_1) - J(u)) du + O\left(\frac{\log T}{\log \log T}\right) $$

where

$$ J(u) \equiv \sum_{\rho \in D} \frac{u - \beta}{(u - \beta)^2 + (t - \gamma)^2}. $$

Now

$$ J(\sigma_1) - J(u) = \sum_{\rho \in D} \frac{(\sigma_1 - u)(t - \gamma)^2}{Y} + \sum_{\rho \in D} \frac{(\sigma_1 - \beta)(u - \beta)(u - \sigma_1)}{Y} $$

where $Y = ((u - \beta)^2 + (t - \gamma)^2)((\sigma_1 - \beta)^2 + (t - \gamma)^2)$. Denote the two sums in $J(\sigma_1) - J(u)$ by $\sum_1$ and $\sum_2$. We have

$$ \sum_1 \leq \sum_{\rho \in D} \frac{\sigma_1 - u}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} $$

and so

$$ \int_{\sigma}^{\sigma_1} \sum_1 du \leq \sum_{\rho \in D} \frac{\frac{1}{2}(\sigma_1 - \sigma)^2}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} $$

$$ \leq \frac{1}{2}(\log X)^{-1} \sum_{\rho \in D} \frac{\sigma_1 - \beta}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} $$

$$ = \frac{1}{2} J_0(\log X)^{-1} = O((\log T)(\log \log T)^{-1}). $$
Now $\sum_2$ is negative and
\[
- \int_\sigma^{\sigma_1} \sum_2 du = \int_\sigma^{\sigma_1} \sum_{\rho \in D} \frac{(u - \beta)(\sigma_1 - \beta)(\sigma_1 - u)}{Y} du \\
\leq \int_\sigma^{\sigma_1} \sum_{\rho \in D} \left( \frac{\sigma_1 - u}{u - \alpha} \cdot \frac{(u - \beta)^2 (\sigma_1 - \beta)}{Y} \right) du \\
\leq J_0(\log X)^{-1} \int_\sigma^{\sigma_1} \frac{du}{u - \alpha} = O\left( \frac{\log T}{\log \log T} \log \frac{\sigma_1 - \alpha}{\sigma - \alpha} \right).
\]
This proves the lemma.

**Lemma 16.** For $\alpha \leq \sigma \leq \sigma_1$, we have,
\[
(26) \quad \int_\sigma^{\sigma_1} \text{Im} \left\{ \frac{F'}{F}(\sigma + it) - \frac{F'}{F}(u + it) \right\} du = O\left( \frac{\log T}{\log \log T} \right).
\]

**Proof.** For $\alpha \leq \sigma \leq \sigma_1$ we see that the integrand is (apart from a term of the type $O(\log T)$), by Lemma 2,
\[
\sum_{\rho \in D} \left( \frac{-t - \gamma}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} - \frac{-t - \gamma}{(u - \beta)^2 + (t - \gamma)^2} \right) \\
= \sum_{\rho \in D} \frac{(t - \gamma)((\sigma_1 - \beta)^2 - (u - \beta)^2)}{Y}
\]
and hence its absolute value is
\[
\leq \sum_{\rho \in D} |t - \gamma|(\sigma_1 - u)(2\sigma_1 - 2\beta)Y^{-1}.
\]
Hence the absolute value of the integral in question is
\[
\leq \frac{1}{\log X} \sum_{\rho \in D} \left\{ \left( \int_\sigma^{\sigma_1} \frac{|t - \gamma|}{(u - \beta)^2 + (t - \gamma)^2} du \right) \cdot \left( \frac{2(\sigma_1 - \beta)}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} \right) \right\} \\
\leq \frac{2}{\log X} \sum_{\rho \in D} \left\{ \left( \int_\beta^{\infty} \frac{|t - \gamma|}{(u - \beta)^2 + (t - \gamma)^2} du \right) \cdot \left( \frac{\sigma_1 - \beta}{(\sigma_1 - \beta)^2 + (t - \gamma)^2} \right) \right\} \\
= \frac{\pi J_0}{\log X} = O\left( \frac{\log T}{\log \log T} \right).
\]
This proves the Lemma.
5. Proof of Theorem 1

From the results of Section 4 Theorem 1 follows from the identity (valid for $\alpha \leq \sigma \leq \sigma_1 = \alpha + (\log X)^{-1}$)

\begin{equation}
\log F(\sigma + it) = \log F(\sigma_1 + it) - (\sigma_1 - \sigma) \frac{F'(\sigma_1 + it)}{F(\sigma_1 + it)} + \int_{\sigma}^{\sigma_1} K(u) \, du
\end{equation}

where

\[ K(u) = \frac{F'(\sigma_1 + it)}{F(\sigma_1 + it)} - \frac{F'(u + it)}{F(u + it)}, \]

just as in [1].

APPENDIX

1. We can in this paper replace $F(s)$ by any function

\[ \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \]

where $1 = \lambda_1 < \lambda_2 < \cdots$ is any increasing sequence of real numbers and $\{a_n\}$ with $a_1 = 1$ is any sequence of complex numbers such that the series

\[ \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \]

is absolutely convergent at some point of the complex plane. The condition $\alpha \geq \frac{1}{2}$ is unimportant. Any $\alpha$ will do. The only change is in place of Lemma 3 we have (by fixing $u$ to be large instead of $u = 2$) $I = O(X^{2u}(\log X)^{-1})$ since $(F'/F)(s+w)$ can be proved to be $O(1)$ for all $u$ exceeding some suitable $u_0$. We can now choose $X = (\log T)^\lambda$, where $\lambda$ is a sufficiently small positive constant. The rest of the proof is unaltered.

2. In our paper [2], in the condition $\alpha \geq \frac{1}{2}, \frac{1}{2}$ does not play any serious role, and the condition can be relaxed to any $\alpha \leq 1 - \delta$.

3. In [2], we can (instead of the Euler product) work with the condition

\[ F(1 + it) = O((\log T)^A) \text{ for } T - H \leq t \leq T + H \text{ where} \]

\[ H = C(\log \log T)(\log \log \log T). \]

For, it follows that in $(\sigma \geq 1, T - 3H/4 \leq t \leq T + 3H/4)$ we have $\text{Re} \log F(s) \leq C \log \log T$ (not the same $C$ at all places) and by the Borel–Carathéodory theorem we can prove that in $(\sigma \geq 1 + 1/(\log \log T), T - \frac{1}{2}H \leq t \leq T + \frac{1}{2}H)$ there holds

\[ \log F(s) = O((\log \log T)^2). \]

Now we can apply convexity arguments to obtain a bound for $|\log F(s)|$ in $(\sigma \geq \alpha + C'/\log \log T, T - H/3 \leq t \leq T + H/3)$, not very different from the results proved in paper [2].
4. Without stating the most general results obtainable by the results of this paper, we can state results like this for example. Let

\[ F(s) = \zeta(s) + \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \]

where

\[ \sum_{n \leq x} a_n = O(x^{(1/2) - \delta}) \]

where \( \delta > 0 \) is a constant. Let \( T \geq T_0 \). Then \( F(s) \) has \( \geq T^{1-\varepsilon} \) zeros in \( (\sigma \geq \frac{1}{2} - C''/\log \log T, T \leq t \leq 2T) \) where \( C'' \) depends only on \( \varepsilon \) and other constant like \( \delta \). The same is also true of the function \( F(s) \) defined in \( \sigma > 0 \) by

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\lambda_n^{\sigma}}, \]

where \( 1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) and \( \lambda_{n+1} - \lambda_n \) is both \( \gg \) and \( \ll 1 \). (It may be noted that both \( a_n \) and \( \lambda_n \) can depend on \( T \)).

References


K. Ramachandra
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road
Bombay 400 005
India

A. Sankaranarayanan
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road
Bombay 400 005
India

Received 6 June 1990