THE COMPONENTS OF A JULIA SET

A.F. Beardon

Let \( R \) be a rational map of degree \( d \) of the complex sphere \( \mathbb{C}_\infty \) onto itself, and let \( J \) and \( F \) be the Julia and Fatou sets of \( R \) respectively. We assume throughout that \( d \geq 2 \); then \( J \) is the smallest compact set \( E \) which contains at least three points, and which satisfies

\[
R(E) = E = R^{-1}(E).
\]

We call this property the complete invariance of \( E \), and the fact that \( J \) is the smallest such set is referred to as the minimality of \( J \). For details of the general theory, we refer the reader to [1], [2] and [3]. It is known that \( J \) is a perfect set (so \( J \) is uncountable, and no point of \( J \) is isolated), and also that if \( J \) is disconnected, then it has infinitely many components. The following result, which seems not to have been noticed before, contains both of these results (when \( J \) is disconnected) and more.

**Theorem.** If \( J \) is disconnected, then it has uncountably many many components, and each point of \( J \) is an accumulation point of distinct components of \( J \).

In [4], McMullen gives an example in which \( J \) has a buried component (that is, a component of \( J \) which is not on the boundary of any component of \( F \)). If each component of \( F \) has finite connectivity, and if \( J \) is disconnected, then there are only countably many components of \( J \) which lie on the boundary of some component of \( F \), and our Theorem immediately yields the following general result.

**Corollary.** Suppose that \( J \) is disconnected, and that every component of \( F \) has finite connectivity. Then \( J \) has a buried component.

The major part of the proof of the Theorem is contained in the following

**Proposition.** Let \( K \) be a compact connected subset of \( \mathbb{C}_\infty \). Then \( R^{-1}(K) \) has at most \( d \) components, and each is mapped by \( R \) onto \( K \).

The proof of the Proposition is easier if we first discuss some preliminary results. The complement of a set \( A \) with respect to the plane \( \mathbb{C} \) and the sphere \( \mathbb{C}_\infty \) are denoted by \( \mathbb{C} - A \) and \( \mathbb{C}_\infty - A \) respectively. First, we quote

Lemma 1 ([5], p. 144). A compact subset $W$ of $C_{\infty}$ is connected if and only if every component of $C_{\infty} - W$ is simply connected.

Next, consider a bounded domain $D$ in $C$ which is bounded by a finite number of Jordan curves $\gamma_j$. The winding number of $\gamma_j$ about any $z$ not on $\gamma_j$ is denoted by $n(\gamma_j, z)$, and if $z \not\in \partial D$ we write

$$n(\partial D, z) = \sum_j n(\gamma_j, z).$$

Obviously,

$$D = \{ z : z \not\in \partial D, n(\partial D, z) \neq 0 \}.$$

Finally, let $A$ and $B$ be disjoint, non-empty, compact subsets of $C$. We put a rectangular grid on $C$ which is fine enough so that no square in the grid meets both $A$ and $B$, and we let $\{Q_j\}$ be the set of those (closed) squares that meet $A$. Now let $\Omega$ be the interior of $\cup Q_j$: then $\Omega$ is a bounded open set with a finite number of components $\Omega_j$, each being bounded by a finite number of Jordan curves, and (1) holds with $D$ replaced by $\Omega_j$. Further,

$$A \subset \Omega, \quad B \cap \Omega = \emptyset, \quad \partial \Omega \cap (A \cup B) = \emptyset.$$

We now give the

Proof of the Proposition. Let $D = C_{\infty} - K$, and let $D_j$ be the components of $D$; Lemma 1 shows that each $D_j$ is simply connected. Next, it is easy to see that each component of $R^{-1}(D_j)$ is mapped by $R$ onto $D_j$, and because each $D_j$ is simply connected, we see that any component of $R^{-1}(D_j)$ is either a simply connected domain, or it is a domain of finite connectivity which contains a critical point of $R$ (for if such a component, say $\Delta$, does not contain a critical point then, by the monodromy theorem, the map $R$ of $\Delta$ onto $D_j$ is a homeomorphism). As $R^{-1}(D)$ is the union of the $R^{-1}(D_j)$, it follows that $R^{-1}(D)$ is the union of a finite number of multiply (but finitely) connected domains, say $M_1, \ldots, M_t$, and a number (possibly infinite) of simply connected domains $S_j$.

When there are no multiply connected domains $M_j$ present, all of the components of $R^{-1}(D)$ are simply connected and then Lemma 1 implies that the complement of $R^{-1}(D)$, namely $R^{-1}(K)$, is connected: thus the conclusion of the Proposition holds in this case.

We now assume that at least one domain $M_j$ exists, and we consider the minimal, and necessarily finite, set of components $E_1, \ldots, E_q$ of $R^{-1}(K)$ such that

$$\bigcup \partial M_j \subset E_1 \cup \cdots \cup E_q.$$
Next, we show that $E_1, \ldots, E_q$ are all of the components of $R^{-1}(K)$. Suppose, then, that $Q$ is another component of $R^{-1}(K)$ and write $E = E_1 \cup \cdots \cup E_q$: then $E$ and $Q$ are disjoint compact subsets of $R^{-1}(K)$, so from [5] (Theorem 5.6, p. 82), there are compact subsets $A$ and $B$ of $R^{-1}(K)$ such that

$$A \cup B = R^{-1}(K), \quad A \cap B = \emptyset, \quad Q \subset A, \quad E \subset B.$$  

We may assume that $\infty \in R^{-1}(D)$; then $A$ and $B$ are disjoint, compact subsets of $C$, so we can find an open set $\Omega$ (as described above) satisfying (2), and hence from (4), also

$$\partial \Omega \subset R^{-1}(D).$$  

Now let $\Omega_Q$ be the component of $\Omega$ that contains the connected set $Q$. Using (3) and (4), we find that for each $r$,

$$\partial M_r \subset E \subset B,$$

and so we see from (2) that $\Omega_Q$ and $\partial M_r$ are disjoint. Now $\Omega_Q$ is arcwise connected, and this means that either $\Omega_Q \subset M_r$ or $\Omega_Q \cap M_r = \emptyset$. Now the first possibility cannot occur because if it does, then

$$Q \subset \Omega_Q \subset M_r \subset R^{-1}(D)$$

which violates the fact that $Q \subset R^{-1}(K)$; thus $\Omega_Q$ is disjoint from each $M_r$. As each $M_r$ is open, we deduce that the closure of $\Omega_Q$ is disjoint from $\bigcup M_r$.

As a consequence of this, each boundary component $\gamma_j$ (a Jordan curve) of $\Omega_Q$ lies in some simply connected domain $S_m$ for, by (5), it lies in $R^{-1}(D)$; thus one side of $\gamma_j$ lies in $S_m$, while the other side contains $R^{-1}(K)$ and each $M_r$. It follows that for any $z_1$ in $M_r$, and any $z_2$ in $Q$,

$$n(\gamma_j, z_1) = n(\gamma_j, z_2),$$

and hence that

$$n(\partial \Omega_Q, z_1) = n(\partial \Omega_Q, z_2) \neq 0.$$  

This shows that $z_1$ is in $\Omega_Q$, contrary to the fact that $\Omega_Q$ and $M_r$ are disjoint. It follows that no such component $Q$ exists, and so we have proved that

$$R^{-1}(K) = E_1 \cup \cdots \cup E_q.$$  

As $R^{-1}(K)$ is compact, so is each $E_j$, and hence $R(E_j)$ also: thus $R(E_j)$ is a closed subset of $K$. We shall now show that each $R(E_j)$ is relatively open.
in $K$: then, as $K$ is connected, we find that $R(E_j) = K$. Clearly, this implies that $q \leq d$ and the proof of the Proposition will then be complete.

To show that $R(E_j)$ is relatively open in $K$, we take any $\zeta$ in $R(E_j)$, say $\zeta = R(w)$, where $w \in E_j$. We find a neighbourhood $N$ of $w$ not meeting any other $E_i$ (this is possible because $R^{-1}(K)$ has only finitely many components) and observe that

$$K \cap R(N) = R(E_j \cap N) \subset R(E_j).$$

This shows that $R(E_j)$ is relatively open in $K$, and the proof of the Proposition is complete.

We end with the

**Proof of the Theorem.** Let $K$ be the set of points in $J$ at which infinitely many components of $J$ accumulate. Our first objective is to show that $J = K$ and to do this, we prove

(a) $K$ is closed;

(b) $K$ is completely invariant, and

(c) $K$ has at least three points.

With these, the minimality of $J$ shows that $J \subset K$, and hence that $K = J$.

Obviously, $K$ is closed, so (a) holds. By assumption, $J$ has infinitely many components so $K$ is not empty, and with this, (b) implies (c) (for, from the general theory of iteration, any non-empty finite completely invariant set lies in $F$). We shall now show that (b) holds.

First, take $\zeta$ in $K$, so there is a sequence $J_1, J_2, \ldots$ of distinct components of $J$ which accumulate at $\zeta$. Obviously, the components $R(J_n)$ accumulate at $R(\zeta)$, and from the Proposition we see that at most $d$ of the $J_n$ can map to any given component of $J$. We deduce that infinitely many components of $J$ accumulate at $R(\zeta)$, and hence that $R(K) \subset K$.

Next, take any $\zeta$ in $K$ and $w$ such that $R(w) = \zeta$: then find neighbourhoods $U$ of $w$ and $V$ of $\zeta$ such that for an appropriate $k$, $R$ is a $k$-fold map of $U$ onto $V$. Again, there is a sequence $J_1, J_2, \ldots$ of distinct components of $J$ which accumulate at $\zeta$, and we may assume that all of these meet $V$. It follows that some component of each $R^{-1}(J_n)$ meets $U$, and these components must be distinct as the $J_n$ are. As $U$ and $V$ can be chosen arbitrarily small, this shows that $R^{-1}(K) \subset K$, and hence that (b) holds. We have now shown that $J = K$ and so, in particular, no component of $J$ is isolated.

It only remains to prove that $J$ has uncountably many components. We argue by contradiction, so suppose that the components of $J$ are $J_1, J_2, \ldots$. Now $J$ is a compact metric space, and $J$ is the countable union of the $J_n$ so, by Baire’s category theorem not every $J_n$ is nowhere dense in $J$. We may suppose that $J_1$ is not, so the closure of $J_1$ has a non-empty interior (all in the relative topology on $J$). But $J_1$ is a component of $J$, so it is closed in $J$. We deduce that $J_1$
has a non-empty interior in $J$, and as this violates the statement at the end of the previous paragraph, we can conclude that $J$ must have uncountably many components.

References