GENERALIZED CONFORMAL WELDING

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Abstract. Let $\Phi$ be a homeomorphism of the unit circle $T$ onto itself. Suppose that for any Borel set $G \subset T$ with $\dim G = 0$ we have measure $m_1(\Phi(G)) = m_1(\Phi^{-1}(G)) = 0$. We prove that the unit disk $D$ and its exterior $D^*$ are mapped by conformal mappings $f$ and $f^*$ onto disjoint domains $\Omega$ and $\Omega^*$, respectively, so that the radial boundary values satisfy

$$(f \circ \Phi)(z) = f^*(z)$$

for $z \in T - E$, where $m_1(E) = 0$.

1. Introduction

A homeomorphism $\Phi$ of the unit circle $T = \{ |z| = 1 \}$ onto itself is conformally welded in the classical sense if the unit circle $D$ and its exterior $D^*$ may be mapped by some conformal mappings $f$ and $f^*$ into disjoint Jordan domains $\Omega$ and $\Omega^*$, such that

$$f^* = f \circ \Phi$$

holds on $T$. The present paper generalizes the concept of conformal welding to the case where $\Omega$ and $\Omega^*$ are not necessarily Jordan domains. It follows from a result of Beurling and Ahlfors, see [4], that any quasi-symmetric $\Phi$ is conformally welded, see Lehto and Virtanen [10], Pfüger [14]. Lehto [9] and David [5] prove conformal welding for other classes of homeomorphisms. One problem is that there exists $\Phi$ for which there is no Jordan curve $\alpha$, e.g.

$$\Phi(e^{i\theta}) = \begin{cases} e^{i\pi(\theta/\pi)} & 0 < \theta < \pi, \\ e^{-i\pi(-\theta/\pi)} & -\pi < \theta < 0, \end{cases}$$

and $0 < a < b < 1$. For counterexamples see Oikawa [13], Huber [8], Semmes [16] and Bishop [2], [3].

Nevertheless Bers has asked when is some form of conformal welding possible. This is important for the uniformization of Riemann surfaces. In a companion paper [6] we introduced a generalized conformal welding. The use of Fuchsian groups made it more specialized than the general results we now develop.

We measure sets $E$ by using $p$-Hausdorff measures $m_p(E)$, $0 \leq p \leq 1$. A set $E$ has dimension $0$ if $m_p(E) = 0$ for all $p > 0$.

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Definition 1. A homeomorphism $\Phi: T \to T$ is regular if for every $E \subset T$ with $\dim E = 0$ we have $m_1(\Phi(E)) = m_1(\Phi^{-1}(E)) = 0$.

Examples. It is easy to see that any homeomorphism $\Phi$ of the Lusin class, i.e., for each

$$G \subset T, \quad m_1(G) = 0 \quad \text{implies} \quad m_1(\Phi(G)) = m_1(\Phi^{-1}(G)) = 0,$$

is regular. Another example is “Bihölder” homeomorphisms, i.e., there exist positive constants $k$, $\alpha$:

$$k^{-1}|z_1 - z_2|^{-\alpha} < |\Phi(z_1) - \Phi(z_2)| \leq k|z_1 - z_2|^\alpha,$$

for all $z_1, z_2 \in T$. This includes all quasisymmetric maps.

Beurling, see [15], proved that any conformal map $f$ on $D$ has radial boundary values

$$f(e^{i\theta}) = \lim_{r \to 1} f(e^{i\theta})$$

except for a set of dimension zero. Thus for any regular homeomorphism $\Phi$, $f \circ \Phi$ is defined on $T$ (a.e). Our conformal maps will always be bounded on $T$. Thus one can regard $f \circ \Phi = f^*$ as an identity in $L^\infty(T)$. We prove

Theorem 1. Let $\Phi: T \to T$ be a regular homeomorphism. Then there exists conformal maps $f, f^*$ on $D, D^*$ respectively so that

(i) $f(D) \cap f^*(D^*) = \emptyset$,

(ii) $(f \circ \Phi)(z) = f^*(z),

for all $z \in T - E$, where $m_1(E) = 0$.

2. Background results

Our results depend on the theory of the “fractional-derivative capacity” as well as conformal and quasiconformal mapping.

2.1. Capacity. Maz’ya (Chapter 7 [12]) is the reference for this section. For $u \in C_0^\infty(R^2)$ and vanishing in $\{|z| > 2\}$, let $\hat{u}$ denote the Fourier transform. For $0 \leq p \leq 1$ define a norm

$$\|u\|_p = \left[ \iint |z|^{2p} |\hat{u}|^2 \, dm_2 \right]^{1/2},$$

i.e., the $L^2$ norm of the $p^{th}$ order fractional derivative. The space $D_p$ is the closure of $C_0^\infty(R^2)$ (vanishing in $\{|z| \geq 2\}$) in the $\|\cdot\|_p$ norm. The capacity $C_p$ is defined for compact sets $E \subset \{|z| \leq 1\}$ by

$$C_p(E) = \inf \{ \|u\|_p : u \in D_p, u \geq 1 \text{ on } E \}. $$
This is indeed a capacity in the sense of Choquet, essentially coinciding with the logarithmic capacity for \( p = 1 \).

The first property we need is:

(i) For \( p < q \), \( D_q \) is compactly embedded in \( D_p \), i.e., suppose we have a sequence \( u_n \in D_q \) with \( \| u_n \|_q \leq 1 \). Then there is a subsequence \( u_{n_k} \) and \( u \in D_q \) so that

\[
\| u_{n_k} - u \|_p \to 0.
\]

It follows from the definition and property (i) that: For any sequence \( u_n \in D_q \) with \( \| u_n \|_q \leq 1 \), any \( p < q \), there is a set \( E \), \( C_p(E) < \varepsilon \),

\[
u_{n_k} \to u(z)
\]

uniformly on \( \mathbb{R}^2 - E \).

We also make use of the relation between \( p \)-capacity and Hausdorff dimension.

(ii) For any \( 0 < p \leq 1 \), \( E \subset \mathbb{R}^2 \)

\[
\dim E > 2 - 2p \quad \text{implies} \quad C_p(E) > 0.
\]

We apply these results to functions \( h(z) = \sum_{k=1}^{\infty} b_k z^k \) analytic on \( D \) with

\[
\iint_D |h'|^2 dx \, dy = \pi \sum_{k=1}^{\infty} k |b_k|^2 < \infty.
\]

By Beurling, see [15],

\[
\lim_{r \to 1} h(re^{i\theta})
\]

exists and is finite except for \( e^{i\theta} \in E \), with \( C_1(E) = 0 \). We refer to this limit as \( h(e^{i\theta}) \) (when it exists). (In general, \( h(z) \) may be extended to \( u \in D_1 \).)

Applying (i), (ii) immediately yields:

**Theorem 2.** Let \( h(z) = \sum_{k=1}^{\infty} a_{k,n} z^k \) be analytic on \( D \) with \( \sum_{k=1}^{\infty} k |a_{k,n}|^2 \leq 1 \). Then there exists a subsequence \( h_{n_k} \) and a limit \( h(z) \) so that for every \( p < 1 \) and \( \varepsilon > 0 \) there is a set \( E \subset \mathbb{T} \) with \( C_p(E) \leq \varepsilon \) and

\[
h_{n_k}(z) \to h(z)
\]

uniformly on \( \mathbb{T} - E \).

2.2. **Conformal mapping.** We shall be considering pairs \( \{f, f^*\} \) where \( f \) is conformal on \( D \) and \( f^* \) is conformal on \( D^* \). It is important that \( f(D) \cap
$f^*(D^*) = \emptyset$. However, as we can just as well consider \( \{ j \circ f, j \circ f^* \} \) for any Möbius transformation $j$, we shall assume the following normalization:

$$f^*(z) = z + \sum_{k=1}^{\infty} a_k z^{-k}, \quad |z| > 1,$$

and

$$f(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |z| < 1.$$

This implicitly assumes $b_0 \in f(D)$, $b_0 \not\in f^*(D^*)$. The classical theory shows that

$$\frac{\partial f^*(D^*)}{f(D)} \subset \{|w| \leq 2\},$$

see Pommerenke [15]. We shall call this the $\mathcal{S}$-normalization. Now as $f$ and $f^* - z$ may be extended to $D_1$, from Section 2.1 we immediately deduce

**Lemma 1.** Let $\{f_n, f_n^*\}$ be a sequence of $\mathcal{S}$-normalized pairs. Then there is an analytic function $f$ on $D$, $f^*$ conformal on $D^*$, and a subsequence $n_k$ so that for any $p < 1$, $\varepsilon > 0$

$$f_n^*(e^{i\theta}) \rightarrow f^*(e^{i\theta}), \quad f_{n_k}(e^{i\theta}) \rightarrow f(e^{i\theta})$$

uniformly for $e^{i\theta} \in T - E$, $C_p(E) \leq \varepsilon$.

**Remarks.**

1. Thus $f^*(e^{i\theta}) \rightarrow f^*(e^{i\theta})$ pointwise except for a set of dimension zero.

2. The exceptional set cannot be replaced by one with logarithmic capacity zero.

3. The function $f$ may be identically constant. We make use of a lemma of Beurling (see [15]), to ensure that $f_{n_k}$ does not collapse.

**Lemma 2.** Let $f^*$ be conformal on $D^*$. Then for any constant $b$

$$C_1\{e^{i\theta} : f^*(e^{i\theta}) = b\} = 0.$$

**Remarks.** This fails in $D_1$.

2.3. **Quasiconformal mappings.** References may be found in [11]. Let a homeomorphism $\Psi : T \rightarrow T$ be quasisymmetric on $T$. Beurling and Ahlfors, see [11], proved that $\Psi$ extends to a quasiconformal mapping of $C$, i.e., $\partial \Phi$ and $\tilde{\partial} \Phi \in L^2(C)$ and $\|\partial \Phi / \partial \Phi\|_\infty < 1$. Let the quasiconformal map $\Phi^{-1}$ have complex dilatation

$$\mu(z) = \frac{\partial \Phi^{-1}}{\partial \Phi^{-1}}, \quad \text{for } z \in C.$$
Bojarski's theorem says that for any measureable \( \lambda \) with \( \| \lambda \|_\infty < 1 \) there is a quasiconformal homeomorphism \( \psi: \mathbb{C} \to \mathbb{C} \) so that \( \partial \psi = \lambda \partial \psi \) (a.e. with respect to area). Applying this to

\[
\lambda(z) = \begin{cases} 
\mu(z) & z \in D^* \\
0 & z \in D 
\end{cases}
\]

yields a quasiconformal map \( f \) with dilatation \( 0 \) on \( D \). The composition formulae shows that \( f^* \equiv f \circ \Phi \) has dilatation \( 0 \) on \( D^* \). Thus \( f \) is conformal on \( D \) and \( f^* \) is conformal on \( D^* \). This is the argument of Lehto and Virtanen [10] or Pfluger [14] for

**Lemma 3.** For quasisymmetric \( \Phi: \mathbb{T} \to \mathbb{T} \) there are complementary Jordan domains \( \Omega, \Omega^* \) and conformal maps

\[
f: D \to \Omega, \quad f^*: D^* \to \Omega^*
\]

so that the boundary values (on \( \mathbb{T} \)) satisfy

\[
f^* = f \circ \Phi.
\]

**2.4. Regular homeomorphisms.** We now give some basic properties of regular homeomorphisms. First we observe that a regular homeomorphism is absolutely continuous in the sense of:

**Definition 2.** A homeomorphism \( \Phi: \mathbb{T} \to \mathbb{T} \) is RC ("regularly continuous") if for every \( p < 1 \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for any \( E \subset \mathbb{T} \) with \( C_p(E) < \delta \), we have

\[
m_1(\Phi(E)), m_1(\Phi^{-1}(E)) < \varepsilon.
\]

The proof is given in

**Lemma 4.** A homeomorphism \( \Phi: \mathbb{T} \to \mathbb{T} \) is RC if and only if it is regular.

We only show that every regular \( \Phi: \mathbb{T} \to \mathbb{T} \) is RC. Otherwise there is a sequence of \( E_n \subset \mathbb{T}, C_p(E_n) \leq 1/2^n \), so that \( m_1 \Phi E_n \geq \varepsilon > 0 \), say. Setting \( F_n = \sum_{k=n}^{\infty} E_k \) and as \( \dim F_n \leq 1/2^{n-1} \) we obtain \( F_n \supset F_{n+1} \supset \cdots \) with \( m_1 \Phi F_n \geq \varepsilon \). Thus \( F_n \downarrow \lim F_n \equiv F \) with \( \dim F = 0 \) but \( m_1 \Phi(F) \geq \varepsilon \) by monotone convergence.

We cannot approximate a regular homeomorphism \( \Phi: \mathbb{T} \to \mathbb{T} \) by smooth \( \Phi_n \) which are "uniformly RC". However a one sided approximation is possible.

**Lemma 5.** Let \( \Phi_n: \mathbb{T} \to \mathbb{T} \) be a regular homeomorphism with

\[
\omega(\delta, p) = \sup \{ m_1 \Phi^{-1}(E) : C_p(E) \leq \delta \}.
\]

Then there exists a sequence of smooth homeomorphisms \( \Phi_n: \mathbb{T} \to \mathbb{T} \):

(i) \( \sup \{ m \Phi^{-1}_n(E) : C_p(E) \leq \delta \} \leq \omega(\delta, p) \),

(ii) \( k^{-1}_n|z - w| \leq |\Phi_n(z) - \Phi_n(w)| \leq k_n|z - w| \) for all \( z, w \in \mathbb{T} \), with constants \( k_n \),

(iii) \( \Phi_n^{-1}(z) \to \Phi^{-1}(z), \Phi_n(z) \to \Phi(z) \) uniformly on \( \mathbb{T} \).
This result is only about \( \Phi^{-1} \) which we write as
\[
\Phi^{-1}(e^{i\theta}) = e^{i\varphi(\theta)}
\]
where \( \varphi: \mathbb{R} \to \mathbb{R} \) is a homeomorphism so that for all integers \( k \)
\[
\varphi(\theta + 2\pi k) = \varphi(\theta) + 2\pi k.
\]

Up to a constant the \( p \)-capacity on \( T \) is equivalent to the obvious capacity for \( \mathbb{R} \). Clearly \( \varphi \) is “RC”, i.e. for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for any
\[
E \subset [0,2\pi], \ C_p(E) < \delta,
\]
\[
m_1(\varphi(E)) < \varepsilon.
\]
Let \( \tau_n \in C^\infty \) be an approximation to the identity
(i) \( \tau_n(x) > 0 \) on \([-1/n,1/n]\),
(ii) \( \tau_n(x) = 0 \), \( |x| \geq 1/n \),
(iii) \( \int_{-\infty}^{\infty} \tau_n(x) \, dx = 1 \).
We set
\[
\varphi_n(x) = \int_{x-1/n}^{x+1/n} \tau_n(y) \varphi(x + y) \, dy.
\]

Clearly \( \varphi_n \) is a smooth homeomorphism with \( \varphi_n(x + 2\pi k) = \varphi_n(x) + 2\pi k \). Also for any \( E \subset [0,2\pi), \ C_p(E) \leq \delta \),
\[
m_1(\varphi_n(E)) = \int_E d\varphi_n = \int \int_{\mathbb{R}} \tau_n(y) \, dy \, dx
= \int_{\mathbb{R}} \tau_n(y) \int_E d\varphi(x + y) \, dx \, dy
= \int_{\mathbb{R}} \tau_n(y) m_1(\varphi(E + y)) \, dy \leq \omega_p(\delta).
\]
Thus setting \( \Phi_n^{-1}(e^{i\theta}) = e^{i\varphi_n(\theta)} \) we obtain a homeomorphism with the required properties.

3. Proof of Theorem 1

Let \( \Phi: T \to T \) be a regular homeomorphism. We use the approximating homeomorphisms \( \Phi_n \) constructed in Lemma 5. These quasisymmetric maps are extended to \( C \) and as in 2.2 we obtain maps \( f_n, f_n^* \) conformal on \( D, D^* \) respectively so that
\[
f_n^*(z) = f_n \circ \Phi_n(z)
\]
for \( z \in T \). We assume that \( \{f_n, f_n^*\} \) are \( S \)-normalized and thus that \( f_n^*, f_n \) converge normally on \( D^* \cup D \) to \( f^*, f \) (which may be constant). Observe that
$f^*, f \circ \Phi$ are well defined functions of $L^\infty(\mathbb{T})$. Let $p$ be any polynomial in $z, \bar{z}$. Now

$$\int_{\mathbb{T}} p(z)f^*_n(z) |dz| \to \int_{\mathbb{T}} p(z)f^*(z) |dz|$$

by normal convergence. The left hand side is equal to

$$\int_{\mathbb{T}} p(z)f_n \circ \Phi_n(z) |dz| = \int_{\mathbb{T}} p(\Phi_n^{-1}(w)) f_n(w) |d\Phi_n^{-1}|$$

where $w = \Phi_n(z)$. Now for any $\delta > 0$ and $p < 1$ there is a set $E$ with $C_p(E) < \delta$ and $f_n(w) \to f(w)$ uniformly on $\mathbb{T} - E$. For any $\varepsilon > 0$, for small enough $\delta$,

$$\int_E |d\Phi_n^{-1}| < \varepsilon,$$

by the results of Sections 2.1–2.4. Thus as $p \circ \Phi_n^{-1} \to p \circ \Phi^{-1}$ uniformly on $\mathbb{T} - E$:

$$\int_{\mathbb{T} - E} p(\Phi_n^{-1}(w)) f_n(w) |d\Phi_n^{-1}| \to \int_{\mathbb{T} - E} (p \circ \Phi^{-1}) f(w) |d\Phi^{-1}|.$$

Now as $p, f_n$ are bounded on $E$

$$\left| \int_E p(\Phi_n^{-1}(w)) f_n(w) |d\Phi_n^{-1}| \right| < C\varepsilon.$$

Combining these results implies

$$\int_{\mathbb{T}} p(z)f_n \circ \Phi_n(z) |dz| \to \int_{\mathbb{T}} (p \circ \Phi^{-1}) f(w) |d\Phi^{-1}|.$$

Now as $\Phi$ is regular and $f$ may be approximated uniformly (except for a set of small capacity) by continuous functions, we change variable of integration to obtain

$$\int_{\mathbb{T}} p(z)(f \circ \Phi)(z) |dz| = \int_{\mathbb{T}} p(z)f^*(z) |dz|.$$

Therefore $f \circ \Phi = f^*$ in $L^\infty(\mathbb{T})$.

Finally observe that $f$ cannot be constant, as otherwise so is the conformal mapping $f^*$, by Lemma 2.
4. Uniqueness of representation

Let $q: \mathbb{C} \to \mathbb{C}$ be any homeomorphism which is conformal on $\Omega \cup \Omega^*$, so that $q(z) = z + \sum_{k=1}^{\infty} c_k z^{-k}$ near $\infty$. Then if $\{f, f^*\}$ is a $(\Sigma)$-normalized) conformal welding for a homeomorphism $\Phi: \mathbb{T} \to \mathbb{T}$ we obtain a second $(\Sigma)$-normalized) conformal welding $\{q \circ f, q \circ f^*\}$.

Thus even in the classical case that $\Omega, \Omega^*$ are the inner and outer domain of a closed Jordan curve $\alpha$ there may exist such a $q$. This is obvious if $\text{Area} \alpha > 0$ for then we define dilatation $\mu$ on $\alpha$ and let $q$ be the normalized quasiconformal solution of the Beltrami equation

$$\bar{\partial} q = \mu \partial q.$$  

However there exists examples of nonuniqueness even when $\text{Area} \alpha = 0$ (but $\dim \alpha > 1$), see Bishop [2]. Our example is constructed by classification results of Ahlfors and Beurling.

Let $\Omega$ be any domain containing $\infty$ with complement $E$. The Dirichlet class $\mathcal{D}(\Omega)$ is the set of functions

$$h(z) = \sum_{k=1}^{\infty} a_k z^{-k}, \quad (|z| > R)$$

analytic on $\Omega$ with

$$||h|| = \sqrt{\int_\Omega |h'|^2 dx \, dy} < \infty.$$  

The capacity with respect to $\mathcal{D}(\Omega)$ is

$$c(E) = \sup \{|a_1|: h \in \mathcal{D}(\Omega), ||h|| \leq 1\}.$$  

Then there are nonmöbius conformal maps defined on $\Omega$ if and only if $c(E) > 0$.

Now let $E$ be a totally disconnected compact set. Let $\Omega, \Omega^*$ be disjoint simply connected domains with $\infty \in \Omega^*$ and constructed so that

$$\partial \Omega = \partial \Omega^* \supset E.$$  

By scaling and translation we may assume that the conformal maps

$$f: D \to \Omega, \quad f^*: D^* \to \Omega^*$$

are $S$ normalized.

We can construct $\Omega$ and $\Omega^*$ by a family of Jordan arcs connecting components of $E$. Thus we may define a map

$$\Phi = f^{-1} \circ f^*$$
defined at first only on $T - F$, where $f^*(F) = E$. By suitable construction of $\Omega$, $\Omega^*$ one can ensure that $\dim F > 0$ and $\Phi$ extends to a regular homeomorphism of $T$. Thus $\{f, f^*\}$ is a generalized conformal welding of $\Phi$.

Now provided $c(E) > 0$ there is a nonmöbius conformal map $q$ of $C - E$ (not necessarily a homeomorphism of $C$). We may assume that $q$ is normalized. Thus we obtain a pair $\{q \circ f, q \circ f^*\}$ which forms a conformal welding in our generalized sense.

5. Non Jordan case

Now we discuss the case that $\partial \Omega$ (or $\partial \Omega^*$) is not a Jordan curve. The example in the introduction cannot be conformally welded in the classical sense. Our proof of Theorem 1 shows that there is a dense subset $F \subset \partial \Omega = \partial \Omega^*$ so that every $z \in F$ is the endpoint of open Jordan arcs $\beta \subset \Omega$, $\beta^* \subset \Omega^*$. Furthermore $f^{-1}(F)$ is a dense subset of $T$ (of positive dimension). Thus if $\lim_{r \to 1} f(re^{i\theta})$ does not exist there is then a nontrivial cluster set $\chi$ of $f$ at $e^{i\theta}$. In prime-end theory (see [15]) $\chi$ is the impression associated with $e^{i\theta}$. The arcs $\beta, \beta^*$ separate these continua so that each $e^{i\theta}$ maps to a unique impression $I(e^{i\theta})$. Thus for a regular homeomorphism $\Phi$

$$\Phi = f^* \circ f^{-1}$$

in the sense of prime-ends.

As soon as there is a single nontrivial continua $\chi = I(e^{i\theta})$ then the conformal welding of $\Phi$ by $\{f, f^*\}$ is not unique. Any normalized conformal map $q$ of $C - \chi$ gives a conformal welding of $\Phi$ by $\{q \circ f, q \circ f^*\}$.

Note that we can always find a conformal mapping $q$ so that $q(\chi)$ is a horizontal line segment. In the next section we observe that there is always a conformal welding so that all the impressions $\chi$ are horizontal line segments.

6. Class of representations

For each regular homeomorphism $\Phi: T \to T$ let $\mathcal{F}_\Phi$ be the class of pairs $\{f, f^*\}$ of ($S$-normalized) conformal maps

$$f: D \to \Omega, \quad f^*: D^* \to \Omega^*,$$

$\Omega \cap \Omega^* = \emptyset$, $f^*(e^{i\theta}) = (f \circ \Phi)(e^{i\theta})$ (a.e. on $T$). The following is an immediate deduction from Section 2:

Lemma 6. $\mathcal{F}_\Phi$ is compact in the topology of uniform convergence on compact subsets of $D \cup D^*$.

Theorem 3. For every regular homeomorphism $\Phi: T \to T$ there is a conformal welding $\{f, f^*\}$ so that

(i) $\text{Area} \left( C - \{f(D) \cup f^*(D^*)\} \right) = 0$,  
(ii) Each impression of $\partial f(D)$ (and $\partial f^*(D^*)$) is a horizontal line segment.
On $\mathcal{F}_\Phi$ consider the problem of maximizing $\text{Re} \, a_1$ for $f^*(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \in \mathcal{F}_\Phi$.

Now for each pair

$$f^* = z + \sum_{k=1}^{\infty} a_k z^{-k}, \quad f = \sum_{k=0}^{\infty} b_k z^k$$

of $\mathcal{F}_\Phi$ we have the area formulae

$$\text{Area} \left( C - \{ f(D) \cup f^*(D^*) \} \right) = \pi - \pi \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2), \quad a_0 = 0.$$  

Thus

$$1 \geq \sum_{k=0}^{\infty} k(|a_k|^2 + |b_k|^2).$$

Regarding $\mathcal{F}_\Phi$ as a bounded subset of the obvious Hilbert space we see that \{f^*, f\} $\in \mathcal{F}_\Phi$ is an extreme point if and only if

$$\text{Area} \left( C - \{ f(D) \cup f^*(D^*) \} \right) = 0.$$  

Now by Lemma 6 there exists an extremal $f \in \varepsilon \times \tau(\mathcal{F}_\Phi)$ maximizing $\text{Re} \, a_1$. Thus we may assume the existence of an extremal satisfying (i).

Now let $\chi$ be any impression of $\partial f(D)$ which is not a horizontal line segment. By the variational theory of Schiffer, see [15], there is a function

$$g(z) = z + \sum_{k=1}^{\infty} d_k z^{-k}$$

conformal on $C - \chi$, so that $\text{Re} \, d_1 > 0$. Consider the pair \{g o $f^*$, g o f\} $\in \mathcal{F}_\Phi$. Then if $g o f^* = z + \sum_{k=1}^{\infty} e_k z^{-k}$ as $e_1 = d_1 + a_1$, we get $\text{Re} \, e_1 > \text{Re} \, a_1$ which is a contradiction.

**Remarks.** Using quasiconformal variations one can prove every extremal satisfies (i).

**References**


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