OPTIMAL TRANSMISSION OF GAUSSIAN SIGNALS INVOLVING COST OF FEEDBACK

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Methods to analyse the problem of signal transmission through a noiseless feedback channel using the results of nonlinear filtering theory are given in [2]. In [1] we constructed an optimal transmission model for Gaussian signals taking into account the cost of transmission. The aim of this paper is to construct an optimal transmission scheme for a Gaussian signal involving the cost of feedback.

Suppose the transmitted message is a Gaussian random variable $\theta$ with $E\theta = m$ and $E(\theta - m)^2 = \gamma > 0$, and the transmission of $\theta$ is carried out according to the following scheme

$$d\xi_t = \left\{ \beta_t [A_0(t, \xi) + A_1(t, \xi)\theta] + (1 - \beta_t) [B_0(t) + B_1(t)\theta] \right\} dt + dW_t,$$

$$\xi_0 = 0, \quad t \in [0, T],$$

where $\beta = (\beta_t)_{t \leq T}$ is a $(\mathcal{F}_t)_{t \leq T}$-adapted stochastic process taking two values 0 and 1. Under $\beta_t = 1$ the transmission is with feedback and under $\beta_t = 0$ it is without feedback. Let

$$\delta(t) = \inf_{A_0, A_1, B_0, B_1, \beta, \theta} E[(\theta - \hat{\theta}_t)^2 + c \int_0^t \beta_s ds], \quad c > 0,$$

where $c$ is the cost of the feedback.

Let the following moment conditions

$$E\left\{ [A_0(t, \xi) + A_1(t, \xi)\theta]^2 | \mathcal{F}_t^\xi \right\} \leq P, \quad E[B_0(t) + B_1(t)\theta]^2 \leq P$$

be satisfied, $P > 0$.

The functionals $A_0$, $A_1$, $B_0$, $B_1$, $\beta$ are supposed to be such that equation (1) has a unique strong solution.

Denote

$$m_t = E(\theta | \mathcal{F}_t^\xi), \quad \gamma_t = E[(\theta - m_t)^2 | \mathcal{F}_t^\xi].$$

The equations for \( m_t \) and \( \gamma_t \) are of the following form

\[
dm_t = \gamma_t \left[ \beta_t A_1(t, \xi) + (1 - \beta_t) B_1(t) \right],
\]

(3)

\[
\cdot \left\{ d\xi_t - \left[ \beta_t A_0(t, \xi) + (1 - \beta_t) B_0(t) + \frac{1}{m_0} \right] dt \right\}, \quad m_0 = m,
\]

(4)

\[
d\gamma_t = -\gamma_t^2 \left[ \beta_t A_1(t, \xi) + (1 - \beta_t) B_1(t) \right]^2 dt, \quad \gamma_0 = \gamma.
\]

**Theorem.** Suppose the transmission of a Gaussian variable \( \theta \) is carried out according to scheme (1).

If \( c \geq \frac{1}{4} \gamma P \), then the optimal strategy is \( \beta^*_s = 0 \), \( s \in [0, t] \), i.e., transmission without feedback. The optimal coding is \( B_1^*(s) = \sqrt{\frac{P}{\gamma}} \), \( B_0^*(s) = -m \sqrt{\frac{P}{\gamma}} \), and optimal decoding \( m_t^* \), for given \( \beta^* \), \( B_0^* \), \( B_1^* \), is determined through (2) and (3). In that case

\[
\delta(t) = \frac{\gamma}{1 + Pt}.
\]

Exactly the same statement is true if \( c < \frac{1}{4} \gamma P \) and \( t \leq a \) or \( t \geq b \) with

\[
a = \frac{\gamma P - 2c - \sqrt{2} P^2 - 4c\gamma P}{2cP}, \quad b = \frac{\gamma P - 2c + \sqrt{2} P^2 - 4c\gamma P}{2cP}.
\]

If \( c < \frac{1}{4} \gamma P \) and \( t \in (a, b) \), then the optimal strategy has the form \( \beta^*_s = I(t - x_0 \leq s \leq t) \), where \( x_0 \) is the unique solution of the equation

\[
\gamma e^{Px} [1 + P(t - x)]^2 = \gamma P^2 (t - x)
\]

in the interval \((0, t)\).

The optimal coding rules are determined as follows:

\[
B_1^*(s) = \sqrt{\frac{P}{\gamma}}, \quad B_0^*(s) = -m \sqrt{\frac{P}{\gamma}},
\]

\[
A_1^*(s, \xi) = \sqrt{\frac{P}{\gamma}} \exp \left\{ \frac{P}{2} \int_0^s \beta^*_u \, du \right\} \left[ 1 + \gamma \int_0^s (1 - \beta^*_u) (B_1^*(u))^2 \exp \left\{ -P \int_0^u \beta^*_r \, dr \right\} \, du \right\],
\]

\[
A_0^*(t, \xi) = -A_1^*(t, \xi) m_t^*,
\]

where \( m_t^* \) and \( \gamma_t^* \) are determined through equations (3) and (4). In this case

\[
\delta(t) = \frac{\gamma e^{-Px_0}}{1 + P(t - x_0)} + cx_0.
\]

To prove this theorem we need the following lemma.
Lemma. Let $\beta = (\beta_t)_{t \geq 0}$ be a real function with values in $[0,1]$ and let $P$ be a positive constant. Then the following inequality is valid

$$\int_0^t e^{-P \int_0^s \beta_u du} ds \leq t - \int_0^t \beta_s ds + \frac{1}{P} \left(1 - e^{-P \int_0^t \beta_s ds} \right).$$

If $\beta$ has the form $\beta_s = I(u \leq s \leq t)$, $u \in [0,t]$, then the inequality reduces into an equality.

Proof. Let $\int_0^t \beta_s ds = v$ and $\beta^*_s = I(t - v \leq s \leq t)$. Then $\int_0^t \beta^*_s ds = v$. Let us first show that

$$\int_0^s \beta^*_u du \leq \int_0^s \beta_u du$$

for any $s \in [0,1]$. This inequality is evident for $s < t - v$. Assume that for some $t_1 \geq t - v$ an inverse inequality occurs. Then

$$v = \int_0^t \beta^*_u du = \int_0^{t_1} \beta^*_u du + \int_{t_1}^t \beta^*_u du > \int_0^{t_1} \beta_u du + \int_{t_1}^t \beta_u du = \int_0^t \beta_u du,$$

which is not true. Thus,

$$\int_0^t e^{-P \int_0^s \beta_u du} ds \leq \int_0^t e^{-P \int_0^s \beta^*_u du} ds.$$

One can easily see that

$$\int_0^t e^{-P \int_0^s \beta^*_u du} ds = t - v + \frac{1}{P} \left(1 - e^{-Pv} \right),$$

and since $v = \int_0^t \beta_s ds$, we obtain the required inequality.

The validity of the second assertion of the lemma follows by a direct verification.

Proof of the theorem. For fixed coding functionals $A_0$, $A_1$, $B_0$, $B_1$ and strategy $\beta$ the optimal decoding is $m_t$. Hence,

$$\delta(t) = \inf_{A_0,A_1,B_0,B_1,\beta} \mathbb{E} \left[ \gamma t + c \int_0^t \beta_s ds \right].$$

We construct first the optimal coding functionals for a fixed strategy $\beta$. Rewrite the moment condition in the following form

$$[A_0(t,\xi) + A_1(t,\xi)m_t]^2 + \gamma_t A^2_1(t,\xi) \leq P, \quad [B_0(t) + B_1(t)m]^2 + \gamma B^2_1(t) \leq P.$$
The equation for $\gamma_t$ can be written in the form

$$\gamma_t = \gamma \exp \left\{ - \int_0^t \beta_s \gamma_s A_1^2(s, \xi) \, ds - \int_0^t (1 - \beta_s) \gamma_s B_1^2(s) \, ds \right\}.$$ 

From (5) it follows that $\gamma_t A_1^2(t, \xi) \leq P$. Therefore,

$$\gamma_t \geq \gamma \exp \left\{ - P \int_0^t \beta_s \, ds - \int_0^t (1 - \beta_s) B_1^2(s) \, ds \right\}. \tag{6}$$

If we choose $A_1^*(t, \xi) = \sqrt{P/\gamma_t^*}$, where $\gamma_t^*$ is a solution of (4) for a given functional $A_1^*$ and a fixed $B_1$, then the inequality reduces into an equality. Hence $A_1^*$ is the optimal coding. The moment condition is satisfied for $A_0^*(t, \xi) = -A_1^*(t, \xi)m_t^*$, where $m_t^*$ is a solution of (3) for given $A_0^*, A_1^*, \gamma_t^*$. If we solve the equation (4) for $A_1^*(t, \xi) = \sqrt{P/\gamma_t^*}$, we obtain

$$\gamma_t^* = \frac{\gamma e^{-P \int_0^t \beta_s \, ds}}{1 + \gamma \int_0^t (1 - \beta_s) B_1^2(s) e^{-P \int_0^s \beta_u \, du} \, ds},$$

and therefore the optimal coding has the form

$$A_1^*(t, \xi) = \sqrt{\frac{P}{\gamma}} e^{P/2} \int_0^t \beta_s \, ds \left[ 1 + \gamma \int_0^t (1 - \beta_s) B_1^2(s) e^{-P \int_0^s \beta_u \, du} \, ds \right].$$

Thus

$$\delta(t) = \inf_{B_0, B_1, \beta} E \left[ \frac{\gamma e^{-P \int_0^t \beta_s \, ds}}{1 + \gamma \int_0^t (1 - \beta_s) B_1^2(s) e^{-P \int_0^s \beta_u \, du} \, ds} + c \int_0^t \beta_s \, ds \right].$$

If follows from (5) that $B_1^2(t) \leq P/\gamma$. If we choose $B_1^*(t) = \sqrt{P/\gamma}$ and $B_0^*(t) = -m \sqrt{P/\gamma}$, then the condition (5) is fulfilled and

$$\delta(t) = \inf_{\beta} E \left[ \frac{\gamma e^{-P \int_0^t \beta_s \, ds}}{1 + P \int_0^t (1 - \beta_s) e^{-P \int_0^s \beta_u \, du} \, ds} + c \int_0^t \beta_s \, ds \right].$$

Thus, for a fixed strategy $\beta$, our construction gives the optimal coding functionals and the optimal decoding.

Now we construct the optimal strategy. It can be easily shown that

$$\delta(t) = \inf_{\beta} E \left[ \frac{\gamma}{1 + P e^{P \int_0^t \beta_s \, ds} e^{-P \int_0^t \beta_u \, du} \, ds} + c \int_0^t \beta_s \, ds \right].$$
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We denote the expression in the brackets by \( \delta_\beta(t) \). Using the inequality from the previous lemma we obtain

\[
\delta_\beta(t) \geq \frac{\gamma e^{-P \int_0^t \beta_s \, ds}}{1 + P(t - \int_0^t \beta_s \, ds)} + c \int_0^t \beta_s \, ds.
\]

Consider the function

\[
f(x) = \frac{\gamma e^{-Px}}{1 + P(t - x)} + cx, \quad x \in [0, t].
\]

The properties of the derivatives of this function easily show that this function increases for all \( x \in [0, 1] \), if \( c \geq \frac{1}{4}\gamma P \) and therefore \( f(x) \geq f(0) = \gamma/(1 + Pt) \). If \( c < \gamma P/4 \) and \( t \not\in (a, b) \), where the numbers \( a \) and \( b \) have been defined in the enunciation of the theorem, then again \( f(x) \geq \gamma/(1 + Pt) \), \( x \in [0, t] \). If \( c < \frac{1}{4}\gamma P \) and \( t \in (a, b) \), then \( f'(x) < 0 \) for \( x \in [0, x_0] \) and \( f'(x) > 0 \) for \( x \in (x_0, t] \), i.e., \( f(x) \geq f(x_0), x \in [0, t] \), where \( x_0 \) is the unique solution of the equation

\[
ce^{Px} \left[ 1 + P(t - x) \right]^2 = \gamma P^2(t - x).
\]

Using the properties of the function \( f \) we can obtain the following statements:

If \( c \geq \frac{1}{4}\gamma P \), then for any strategy \( \beta \) we have

\[
\delta_\beta(t) \geq \frac{\gamma}{1 + Pt}, \quad \text{(note that } 0 \leq \int_0^t \beta_s \, ds \leq t),
\]

and for \( \beta^*_s = 0, \ s \in [0, t] \),

\[
\delta_{\beta^*}(t) = \frac{\gamma}{1 + Pt},
\]

i.e., \( \beta^* \) is the optimal strategy.

If \( c < \frac{1}{4}\gamma P \) and \( t \leq a \) or \( t \geq b \), then the same results hold.

If \( c < \frac{1}{4}\gamma P \) and \( t \in (a, b) \), then

\[
\delta_\beta(t) \geq \frac{\gamma e^{P \int_0^t \beta_s \, ds}}{1 + P(t - \int_0^t \beta_s \, ds)} + cx_0.
\]

Let \( \beta^*_s = I(t - x_0 \leq s \leq t) \). Then, according to the second statement in the previous lemma, we obtain

\[
\delta_{\beta^*}(t) = \frac{\gamma e^{-P \int_0^t \beta^*_s \, ds}}{1 + P(t - \int_0^t \beta^*_s \, ds)} + c \int_0^t \beta^*_s \, ds
\]

\[
= \frac{\gamma e^{-P \int_0^t \beta^*_s \, ds} + cx_0}{1 + P(t - x_0)} + cx_0,
\]

which completes the proof of the theorem.
References
