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# ON CONVERGENCE RATES IN ONE-SIDED LAW OF LARGE NUMBERS

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Convergence rates in two-sided law of large numbers for sums $S_n = X_1 + \cdots + X_n$ of independent identically distributed random variables $\{X_k, k \geq 1\}$ have been studied in detail by Baum and Katz, Brillinger, Erdős, Hsu and Robbins, Spitzer, Heyde and Rohatgi, Gafurov, Shirokova and others (see references in [9]). Necessary and sufficient conditions were obtained for a wide class of normalizing sequences and rates of decrease of probabilities.

Convergence rates in one-sided law of large numbers were investigated by Petrov [10], Petrov and Shirokova [11], Chow and Lai [4], Gafurov and Slastnikov [6] and Amosova [1–3]. But necessary and sufficient conditions are unknown with two exceptions. Petrov and Shirokova [10] derived necessary and sufficient conditions for the exponential rate of decrease of $\mathbf{P}(S_n \geq n\varepsilon)$. Another exception is due to Erickson [5], it related to the series $\sum_{n=1}^{\infty} \mathbf{P}(S_n > n\varepsilon)/n$.

All the papers mentioned above deal with centering by zeros (or by $n\mathbf{E}X_1$, in case the mean is finite). We shall study probabilities $\mathbf{P}(S_n - a_n > b_n)$ with an arbitrary centering sequence of constants $\{a_n\}$. The only assumption made on the sequence $\{a_n\}$ is that for every $\varepsilon > 0$

$$\liminf \mathbf{P}(S_n - a_n > -\varepsilon b_n) > 0, \qquad \liminf \mathbf{P}(S_n - a_n < \varepsilon b_n) > 0$$

(unless otherwise stated all limits are taken as $n \to \infty$). In fact, it is possible to choose $a_n = \operatorname{median}(S_n)$. If $S_n/b_n \xrightarrow{P} 0$, then $a_n = o(b_n)$. Thus, in this case we can choose $a_n = 0$. Furthermore, if $\mathbf{E}X_1$ is finite and $(S_n - n\mathbf{E}X_1)/b_n \xrightarrow{P} 0$, then we can choose $a_n = n\mathbf{E}X_1$. Either of these two situations is under investigation in all the papers mentioned above (with the two exceptions indicated). This allows us to obtain most of the results mentioned above as corollaries of our results. Part of them (for the case $b_n = n$) are presented in [8] and [7].

We begin from the simplest case $b_n = n$. Let $L$ stand for the class of positive functions $f$, defined on the set of positive real numbers, satisfying the conditions: $f(x) \to 0$ as $x \to \infty$ and if $\{X_n\}$ is a sequence of independent random variables

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Part of the symposium lecture; we do not touch here one-sided strong limit theorems.

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having a common symmetric distribution satisfying $n\mathbf{P}(X_1 > n) = o\big(f(n)\big)$, then $\mathbf{P}(S_n - a_n > n) = o\big(f(n)\big)$.

It is well known that $f(x) = x^t \in L$ for every $t \le 0$ and $f(x) = x^t M(x) \in L$ for the same $t$ and arbitrary non-decreasing, positive, slowly-varying (at infinity) function $M(x)$ (see [9, Chapter 9, Theorem 28 and supplements]). So, the definition of the class $L$ includes some information about two-sided convergence rates.

**Theorem 1.** *Let $f \in L$. Then*

$$\mathbf{P}(S_n - a_n > \varepsilon n) = o\big(f(n)\big) \quad \text{for every } \varepsilon > 0, \tag{1}$$

*if and only if*

$$n\mathbf{P}(X_1 < -n) \to 0,$$

$$n\mathbf{P}(X_1 > n) = o\big(f(n)\big), \tag{2}$$

$$\mathbf{E}X_1 I(X_1 < 0) > -\infty \qquad \text{or} \qquad \int_{-n}^{-n/\log(1/f(n))} x\, dF(x) = o(1);$$

*here $F(x) = \mathbf{P}(X_1 < x)$.*

Amosova [2] has already earlier derived necessary and sufficient conditions for (1) under the additional condition $\mathbf{E}|X_1| < \infty$.

**Corollary 1.** *Let $f \in L$. If the conditions* (2) *and*

$$n\mathbf{P}\left(X_1 < -n/\log\left(1/f(n)\right)\right) = o(1)$$

*hold, then* (1) *holds.*

Now we will study the exponential rate of decrease of the sequence of probabilities $\mathbf{P}(S_n - a_n > n)$.

**Theorem 2.** *The following conditions are equivalent:*

(i) *For every $\varepsilon > 0$ there exist $\varrho \in (0,1)$ and $C > 0$ such that*

$$\mathbf{P}(S_n - a_n > \varepsilon n) \le C\varrho^n \tag{3}$$

*for every sufficiently large $n$;*

(ii) *The condition* (3) *holds for some $\varrho \in (0,1)$, $\varepsilon > 0$ and $C > 0$;*

(iii) $\mathbf{E}|X_1| < \infty$, $\mathbf{E}e^{tX_1} < \infty$ *for some $t > 0$.*

Earlier, Petrov and Shirokova [11] obtained an analogous criterion for $\mathbf{P}(S_n > \varepsilon n) \leq C\varrho^n$. They showed that this inequality is true for some $\varrho \in (0,1)$, $\varepsilon > 0$, $C > 0$ and all $n$ sufficiently large, if and only if $\mathbf{E}e^{tX_1} < \infty$ for some $t > 0$. It is interesting to notice that the result of Petrov and Shirokova is purely one-sided while Theorem 2 involves information about the left tail of the distribution of $X_1$.

Next we will study the case $b_n = n^{1/s}$, $0 < s < 2$, $s \neq 1$. The situation is very simple if $s < 1$. A traditional although slightly improved technique may be used in proving the next theorem (which may be known).

**Theorem 3.** *Let $0 < s < 1$, $t \leq 0$; $M(x)$, $x > 0$, be a non-increasing, positive, slowly-varying (at infinity) function and $r > s$.*
*The relation*

$$\mathbf{P}(S_n - a_n > \varepsilon n^{1/s}) = o\big(n^t M(n)\big) \qquad \text{for each } \varepsilon > 0 \tag{4}$$

*holds if and only if*

$$\begin{gathered} n\mathbf{P}(X_1 < -n^{1/s}) \to 0, \\ n\mathbf{P}(X_1 > n^{1/s}) = o\big(n^t M(n)\big). \end{gathered} \tag{5}$$

*The relation*

$$\sum_{n=1}^{\infty} n^{(r/s)-2}\mathbf{P}(S_n - a_n > \varepsilon n^{1/s}) < \infty \qquad \text{for each } \varepsilon > 0$$

*holds if and only if the conditions* (5) *and* $\int_0^\infty |x|^s dF(x) < \infty$ *are satisfied.*

In our proof of the theorem we essentially use the fact that "the negative part" of $S_n$ (i.e., $X_1 I(X_1 < 0) + \cdots + X_n I(X_n < 0)$) is negligible in the full sum. However, this is not true when $s \geq 1$.

**Theorem 4.** *Let $1 < s < 2$, $f \in L$. The relation*

$$\mathbf{P}(S_n - a_n > \varepsilon n^{1/s}) = o\big(f(n)\big) \qquad \text{for each } \varepsilon > 0$$

*holds, if and only if the conditions* (5),

$$n\mathbf{P}(X_1 > n^{1/s}) = o\big(f(n)\big) \tag{6}$$

*and*

$$\int_{-\infty}^{0} x^2 dF(x) < \infty$$

*or*

$$D\big(n^{1/s}/\log\big(1/f(n)\big)\big) = o\big(n^{(1/s)-1}\big)$$

*are satisfied. Here* $D(x) = \int_{-x}^0 y^2 dF(y) + \int_{-\infty}^{-x} y\, dF(y)$.

Note that condition (5) implies $D(x) < \infty$. Thus, $D(x)$ is finite in Theorem 4.

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