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LIPSCHITZ CLASSES OF SOLUTIONS TO CERTAIN ELLIPTIC EQUATIONS

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Abstract. We consider weak solutions u in the Sobolev space $W^1_{\alpha,\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbf{R}^n$, to elliptic equations of the form $\operatorname{div} A(x,\nabla u) = B(x,\nabla u)$. The Lipschitz continuity of u over Ω is characterized by the growth of the local L^α-average of ∇u. Previous results cover the case in which the structure exponent $\alpha \geq n$. We give here a proof valid for all $1 < \alpha < \infty$.

1. Introduction and main results

Theorem 1.1 follows from results in [HL]. Throughout Ω will be a connected open subset of $\mathbf{R}^n$. When $f\colon \Omega \to \mathbf{R}^m$, $0 < k \leq 1$, we write

$$\|f\|^k = \sup_{\substack{x_1,x_2\in\Omega \\ x_1\neq x_2}} \big|f(x_1) - f(x_2)\big|/|x_1 - x_2|^k.$$

Theorem 1.1. *Let u be harmonic in the unit disk $D \subset \mathbf{R}^2$ and $0 < k \leq 1$. If there exists a constant C_1 such that*

$$\big|\nabla u(z)\big| \leq C_1\big(1 - |z|\big)^{k-1} \tag{1.2}$$

for all $z \in D$, then there exists a constant C_2, depending only on α and C_1, such that

$$\|u\|^k \leq C_2. \tag{1.3}$$

Conversely, (1.3) *implies that* (1.2) *holds for all $z \in D$ with C_1 depending only on α and C_2.*

An analogue of Theorem 1.1 is given in [N2] for solutions of certain elliptic equations in divergence form. The main result of this paper, Theorem 1.11, extends this analogue to a larger class of such equations and other domains in $\mathbf{R}^n$.

More precisely, we consider weak solutions u to equations of the form

$$\operatorname{div} A(x,\nabla u) = B(x,\nabla u) \tag{1.4}$$

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in domains $\Omega \subset \mathbf{R}^n$. Here we assume that there exists constants $1 < \alpha < \infty$, $0 < a, b$, such that the measurable functions A: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, B: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ satisfy

$$
\begin{aligned}
&\big|A(x,\xi)\big| \le a|\xi|^{\alpha-1}, \\
(1.5) \qquad &\big|B(x,\xi)\big| \le b|\xi|^{\alpha-1}, \qquad \text{and} \\
&\xi \cdot A(x,\xi) \ge |\xi|^{\alpha}
\end{aligned}
$$

for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$. By a weak solution to (1.4) we mean a function u locally in the Sobolev class $W^1_\alpha(\Omega)$ so that

$$
\int_\Omega (\nabla\varphi \cdot A + B\varphi)\, dx = 0 \tag{1.6}
$$

for all $\varphi \in C_0^\infty(\Omega)$. We remark that any such solution is continuous when, if necessary, it is redefined on a set of measure zero [S]. As such, we assume throughout that u is continuous in Ω.

Theorem 1.11 is given in [N2] for the case $n \le \alpha$. We show here that in fact Theorem 1.11 holds for all $1 < \alpha < \infty$. The proof is the same as previously given for the case $n < \alpha$ once we have established Lemma 2.7.

We write $B(x,R) = \{y \in \mathbf{R}^n \mid |x-y| < R\}$ and $|E|$ for the Lebesgue measure of $E \subset \mathbf{R}^n$. If U: $E \to \mathbf{R}^m$ is measurable, then we write, for $0 < p < \infty$,

$$
\|U\|_{p,E} = \left(\int_E |U(x)|^p\, dx\right)^{1/p}.
$$

If u is a weak solution to (1.4), then we write

$$
D_u(x) = |B|^{-1/\alpha}\, \|\nabla u\|_{\alpha,B}
$$

where $B = B\big(x, d(x,\partial\Omega)/2\big)$, $d(x,\partial\Omega) =$ distance between x and the boundary of Ω, $\partial\Omega$.

We need to discuss functions "Lipschitz at the boundary" and write, for f: $\Omega \to \mathbf{R}^m$, $0 < k \le 1$,

$$
\|f\|^k_\partial = \sup_{\substack{x_1,x_2\in\Omega \\ x_1\ne x_2}} \big|f(x_1) - f(x_2)\big| / \big(|x_1 - x_2|^k + d(x_1,\partial\Omega)\big)^k.
$$

We also need the following local definitions.

$$
\|f\|^k_{\text{loc}} = \sup\Big\{\big|f(x_1) - f(x_2)\big|/|x_1 - x_2|^k \mid x_1, x_2 \in \Omega, x_1 \ne x_2, |x_1 - x_2| < d(x_1,\partial\Omega)/2\Big\},
$$

$$\|f\|^k_{\mathrm{loc},\partial} = \sup\Big\{ |f(x_1) - f(x_2)| / \big(|x_1 - x_2| + d(x_1, \partial\Omega)\big)^k \;\Big|\; x_1, x_2 \in \Omega, x_1 \neq x_2, |x_1 - x_2| < d(x_1, \partial\Omega)/2 \Big\}.$$

Clearly,

$$\|f\|^k_{\mathrm{loc},\partial} \leq \min\left(\|f\|^k_\partial, \|f\|^k_{\mathrm{loc}} \right) \leq \max\left(\|f\|^k_\partial, \|f\|^k_{\mathrm{loc}} \right) \leq \|f\|^k.$$

The following definition is given in [L] and with $k = k'$, in [GM].

Definition 1.7. For $0 < k' \leq k \leq 1$, Ω is a $\mathrm{Lip}_{k,k'}$-extension domain if there is a constant N such that every pair of points $x_1, x_2 \in \Omega$ can be joined by a continuous curve $\gamma \subset \Omega$ for which

$$\int_\gamma d(\gamma(s), \partial\Omega)^{k-1} ds \leq N |x_1 - x_2|^{k'}.$$

The class of $\mathrm{Lip}_{k,k'}$-extension domains is wide, including uniform domains and quasiballs, see [GM] and [L]. When $k' < k$ a $\mathrm{Lip}_{k,k'}$-extension domain is necessarily bounded [L] while a $\mathrm{Lip}_{k,k'}$-extension domain may be unbounded. See Section 4 for an example.

We use the following facts about these domains.

Lemma 1.8. *Suppose that Ω is a $\mathrm{Lip}_{k,k'}$-extension domain with constant N. There exists a constant M, independent of $f\colon \Omega \to \mathbf{R}^m$, such that*

$$\|f\|^{k'} \leq M \|f\|^k_{\mathrm{loc}} \tag{1.9}$$

and

$$\|f\|^{k'}_\partial \leq M \|f\|^k_{\mathrm{loc},\partial}. \tag{1.10}$$

Moreover, $M \leq 5\big(N + (2 \operatorname{diam} \Omega)^{k-k'}\big)$ and when $k = k'$, $M \leq 5N$.

For the proof of (1.9) see [GM] and [L]. The proof of (1.10) is similar to the proof given in [N1] for the case $k = k'$. We remark that (1.9) actually characterizes $\mathrm{Lip}_{k,k'}$-extension domains. See [GM] and [L]. We now state the main results.

Theorem 1.11. *Suppose that u is a weak solution to (1.4) in Ω and $0 < k \leq 1$. If there is a constant C_1 such that*

$$D_u(x) \leq C_1\, d(x, \partial\Omega)^{k-1}, \tag{1.12}$$

for all $x \in \Omega$, then there is a constant C_2, depending only on n, α, a, b, k and C_1, such that

$$\|u\|^k_{\mathrm{loc},\partial} \leq C_2. \tag{1.13}$$

Conversely, if (1.13) holds, then (1.12) holds for all $x \in \Omega$ with C_1 depending only on n, α, a, b, k and C_2.

We also have the following global result.

Theorem 1.14. *Suppose that u is a weak solution to (1.4) in a $\mathrm{Lip}_{k,k'}$-extension domain Ω, with constant N, $0 < k' \le k \le 1$. If there is a constant C_1 such that*

$$D_u(x) \le C_1\, d(x, \partial\Omega)^{k-1} \tag{1.15}$$

for all $x \in \Omega$, then there is a constant C_2, depending only on n, α, a, b, k', k, N and C_1, such that

$$\|u\|_{\partial}^{k'} \le C_2. \tag{1.16}$$

In which case, u extends continuously to the closure of Ω, $\overline{\Omega}$. Moreover there are constants C_3, depending only on n, α, a, b, k', k and C_2, and β, depending only on n, α, a and b, such that (1.16) (and hence (1.15)) is equivalent to

$$\|u\|^{k'} \le C_3$$

if $k' \le \beta$. Otherwise, (1.16) only implies that

$$\|u\|^{\beta} \le C_3 (\operatorname{diam} \Omega)^{k'-\beta}.$$

When $k' = k$ and $\alpha \ge n$, he above results appear in [N2]. The BMO case when $k = 0$ is also in [N2].

2. A preliminary lemma

We establish Lemma 2.7 for the proof of Theorem 1.11.

Lemma 2.1. *If u is a solution to (1.4) in Ω, $0 < s < \infty$, then there exists a constant C, depending only on n, s, α, a, and b, such that*

$$|u(x)| \le C |B|^{-1/s} \|u\|_{s,B} \tag{2.2}$$

for all balls $B = B(x, R)$ with $R < d(x, \partial\Omega)$.

By now, Lemma 2.1 is well-known. It can be obtained by combining results in [S] and [IN].

Lemma 2.3. *Let u be a solution to (1.4) in Ω and let $0 < s < \infty$ and $1 < \sigma < \infty$. There exists an exponent α', depending only on n, α, a, and b, with $\alpha < \alpha'$ and a constant C, depending only on n, s, α, σ, a, and b such that*

$$\|\nabla u\|_{\alpha',B} \le C |B|^{(s-\alpha')/s\alpha'} \|\nabla u\|_{s,\sigma B} \tag{2.4}$$

for all balls B with $\sigma B < \Omega$. Here σB is the ball with the same center as B and with radius equal to σ times that of B.

Lemma 2.3 can be obtained from results in [ME], and [IN].

Lemma 2.5. *Let $A \subset \Omega$ with $|A| > 0$ and let $u \in L^p(\Omega)$ with $1 \le p < \infty$. Then for each $c \in \mathbf{R}$,*

$$\|u - u_A\|_{p,\Omega} \le 2\Big(\frac{|\Omega|}{|A|}\Big)^{1/p} \|u - c\|_{p,\Omega}\,. \tag{2.6}$$

Here u_A is the average value, $|A|^{-1}\int_A u(x)\,dx$.

Lemma 2.5 is obtained with elementary calculations. See [H].

We use the above lemmas to obtain the following result.

Lemma 2.7. *Let u be a solution to (1.4) in Ω, $0 < s < \infty$ and $1 < \sigma < \infty$. There exists a constant C, depending only on n, α, s, σ, a, and b, such that*

$$\sup_B u - \inf_B u \le C|B|^{(s-n)/sn} \|\nabla u\|_{s,\sigma B} \tag{2.8}$$

for all balls B with $\sigma B \subset \Omega$.

Proof. Let x = center of B, $r(B)$ = radius of B and $y \in B$. Then letting $B_2 = B\big(y,(\sigma-1)r(B)\big)$ we have, using (2.2) with $s = \alpha$, (2.6) with $p = \alpha$, $A = B_2$, $\Omega = \sigma B$, the Poincaré inequality and (2.4),

$$\begin{aligned} \big|u(x)-u(y)\big| &\le \big|u(x) - u_{B_2}\big| + \big|u(y) - u_{B_2}\big| \\ &\le C_1|B|^{-1/\alpha}\|u - u_{B_2}\|_{\alpha,\sigma B} + C_2|B_2|^{-1/\alpha}\|u - u_{B_2}\|_{\alpha,B_2} \\ &\le C_3\big(\sigma/(\sigma-1)\big)^{n/\alpha}|B|^{-1/\alpha}\|u - u_{\sigma B}\|_{\alpha,\sigma B} + C_2|B_2|^{-1/\alpha}\|u - u_{B_2}\|_{\alpha,B_2} \\ &\le C_4|B|^{(\alpha-n)/\alpha n}\|\nabla u\|_{\alpha,\sigma B} + C_4|B_2|^{(\alpha-n)/\alpha n}\|\nabla u\|_{\alpha,B_2} \\ &\le C_5|B|^{(s-n)/sn}\|\nabla u\|_{s,\sigma B}\,. \end{aligned}$$

Hence, if $y, z \in B$, then

$$\big|u(y) - u(z)\big| \le \big|u(y) - u(x)\big| + \big|u(z) - u(x)\big| \le 2C_5|B|^{(s-n)/sn}\|\nabla u\|_{s,\sigma B}\,.$$

Remark 2.9. In the case that $n < \alpha$ and $u \in W^1_\alpha(\Omega)$, Lemma 2.7 is valid whether or not u is a solution to (1.4) even when $\sigma = 1$. See [BI].

3. Proof of Theorem 1.11 and Theorem 1.14

We need the following facts.

Lemma 3.1. *Suppose that* $f\colon \Omega \to \mathbf{R}^m$, $0 < k \le 1$ *and* $0 < \eta < 1$. *There exists a constant* C_1, *independent of* f, *such that*

$$\left|f(x_1) - f(x_2)\right| \le C_1 |x_1 - x_2|^k$$

for all $x_1, x_2 \in \Omega$ *with* $|x_1 - x_2| \le \eta\, d(x_1, \partial\Omega)$ *if and only if there is a constant* C_2, *independent of* f, *such that*

$$\|f\|^k_{\mathrm{loc}} < C_2.$$

A similar statement holds for $\|f\|^k_{\mathrm{loc},\partial}$.

A proof of Lemma 3.1 appears in [L].

Lemma 3.2. *Suppose* $f\colon \Omega \to \mathbf{R}^m$, $0 < k \le 1$ *and* $0 < \eta < 1$. *There exists a constant* C_1, *independent of* f, *such that*

$$\left|f(x_1) - f(x_2)\right| \le C_1 |x_1 - x_2|^k \tag{3.3}$$

for all $x_1, x_2 \in \Omega$ *with* $|x_1 - x_2| = \eta\, d(x_1, \partial\Omega)$ *if and only if there is a constant* C_2, *independent of* f, *such that*

$$\|f\|^k_{\mathrm{loc},\partial} < C_2.$$

Proof. Assume that $\|f\|^k_{\mathrm{loc},\partial} < C_2$. It follows from Lemma 3.1 that there is a constant C such that

$$\left|f(x_1) - f(x_2)\right| \le C\big(|x_1 - x_2| + d(x_1, \partial\Omega)\big)^k$$

for all $x_1, x_2 \in \Omega$ with $|x_1 - x_2| \le \eta\, d(x_1, \partial\Omega)$. Hence if $|x_1 - x_2| = \eta\, d(x_1, \partial\Omega)$, then

$$\left|f(x_1) - f(x_2)\right| \le C(1 + 1/\eta)^k |x_1 - x_2|^k.$$

Conversely, suppose that

$$\left|f(x) - f(y)\right| \le C_1 |x - y|^k$$

for all $x, y \in \Omega$ with $|x - y| = \eta\, d(x, \partial\Omega)$. Fix $x_1, x_2 \in \Omega$ with $|x_1 - x_2| < \eta\, d(x_1, \partial\Omega)$. Let $R_1 = \eta\, d(x_1, \partial\Omega)$ and $R_2 = \eta\, d(x_2, \partial\Omega)$. Then $R_2 \le R_1 +$

$\eta|x_1 - x_2| \le R_1 + |x_1 - x_2|$ and $R_1 - |x_1 - x_2| \le R_1 - \eta|x_1 - x_2| \le R_2$. Hence $\partial B(x_1, R_1) \cap \partial B(x_2, R_2) \ne \emptyset$. Let $x_3 \in \partial B(x_1, R_1) \cap \partial B(x_2, R_2)$. We obtain

$$\begin{aligned} \big|f(x_1) - f(x_2)\big| &\le \big|f(x_1) - f(x_3)\big| + \big|f(x_2) - f(x_3)\big| \\ &\le C_1\big(|x_1 - x_3|^k + |x_2 - x_3|^k\big) \\ &\le C_1\Big(\eta^k d(x_1, \partial\Omega)^k + \eta^k\big(|x_1 - x_2| + d(x_1, \partial\Omega)\big)^k\Big) \\ &\le C\big(|x_1 - x_2| + d(x_1, \partial\Omega)\big)^k. \end{aligned}$$

It then again follows from Lemma 3.1 that $\|f\|^k_{\mathrm{loc},\partial} < \infty$.

We show that (1.12) implies (1.13). In view of Lemma 3.2, we only need to verify (3.3) for some $\eta < 1$. Fix x_1 and x_2 with $|x_1 - x_2| = \frac{1}{4}\, d(x_1, \partial\Omega)$ and let $B = B\big(x_1, 2|x_1 - x_2|\big)$. Using (2.8) and (1.12) we obtain

$$\big|u(x_1) - u(x_2)\big| \le C_1|B|^{(\alpha-n)/\alpha n}\,\|\nabla u\|_{\alpha,B} = C_1|B|^{1/n} D_u(x_1) \le C_2|x_1 - x_2|^k.$$

Conversely, we assume (1.13) and use the fact that if U is a solution to (1.4) in Ω, then there is a constant C_3, depending only on n, α, a, and b, so that

$$\|\nabla U\|_{\alpha,B'} \le C_3|B'|^{-1/n}\,\|U\|_{\alpha,2B'} \tag{3.4}$$

for all balls B' with $2B' \subset \Omega$. See [S]. Using (3.4) with $U = u - u(x_1)$ we obtain, with B as above,

$$\begin{aligned} D_u(x_1) = |B|^{-1/\alpha}\,\|\nabla u\|_{\alpha,2B} &\le C_4|B|^{-(\alpha+n)/\alpha n}\,\|u - u(x_1)\|_{\alpha,4B} \\ &\le C_5\, d(x_1, \partial\Omega)^{k-1}. \end{aligned}$$

This completes the proof of Theorem 1.11.

Next it follows from Theorem 1.11 with Lemma 1.8 that (1.15) implies (1.16) in a $\mathrm{Lip}_{k,k'}$-extension domain. The last part of Theorem 1.14 follows by applying Lemma 3.5. Its proof appears in [N2].

Lemma 3.5. *Let u be a solution to* (1.4) *in Ω, continuous in $\overline{\Omega}$ and $0 < k \le 1$. There exists a constant β, depending on n, α, a, and b, such that if $k \le \beta$ and if there exists a constant C_1 such that*

$$\big|u(x_1) - u(x_2)\big| \le C_1|x_1 - x_2|^k \tag{3.6}$$

for all $x_1 \in \Omega$ and $x_2 \in \partial\Omega$, then

$$\|u\|^k_\Omega \le C_2$$

where C_2 depends only on n, α, a, b, and C_1. If $\beta < k$, (3.6) *only implies that*

$$\|u\|^\beta \le C_2(\operatorname{diam}\Omega)^{k-\beta}.$$

4. Examples

Example 4.1. We give an example to show that at least in the case that $\alpha = n$ the term $d(x_0, \partial\Omega)$ cannot in general be omitted from (1.13). The radial stretching $f(x) = (f_1, f_2, \dots, f_n) = x|x|^{\gamma-1}$ where $\gamma = K^{1/(1-n)}$ is K-quasiconformal in $\mathbf{B}^n = \{x \in \mathbf{R}^n \mid |x| < 1\}$. As such each component f_i of f satisfies an equation of the form (1.4) with $B = 0$ and

$$A(x,h) = \begin{cases} J_f Df^{-1}\left|(Df^{-1})^t h\right|^{n-2}(Df^{-1})^t h, & \text{if } J_f \neq 0 \\ |h|^{n-2}h, & \text{if } J_f = 0 \text{ or does not exist.} \end{cases}$$

Here Df is the derivative of f and J_f is the Jacobian determinant. Also, $\alpha = n$ in (1.5). See [GLM], [M]. It is easy to see that there is a constant C, depending only on n and K, such that

$$D_{f_i}(x) \le C$$

for all $x \in \mathbf{B}^n$ and all $1 \le i \le n$. However $\|f_i\|^1 = \infty$ since

$$f_i(0,0,\underset{i\text{th}}{r},0,\dots,0) - f_i(0,\dots,0) = r^\gamma.$$

In this case $\beta = K^{1/(1-n)}$ in Lemma 3.5 is sharp. See [N1].

We next give an example in a $\mathrm{Lip}_{k,k'}$-extension domain where $k' \le k$.

Example 4.2. Let $0 < k' \le k < 1$, $u = r^{k'} \cos k'\theta$ and $\Omega = \{(x,y) \mid |y| < x^\gamma$ with $\gamma = (1-k')/(1-k)$, $0 < x < 1\}$. Then Ω is a $\mathrm{Lip}_{k,k'}$-extension domain, see [L], and u is harmonic in Ω. Notice that

$$\|u\|^k_{\mathrm{loc}} < \infty,$$

and

$$D_u(x) \le C\, d(x, \partial\Omega)^{k-1},$$

while we only have

$$\|u\|^{k'} < \infty.$$

A similar situation holds for the function $v = r^{k'}$ in Ω. When $k' = (\alpha-2)/(\alpha-1)$, v is a solution to the α-harmonic equation, $\alpha > 2$,

$$\operatorname{div}\left(|\nabla v|^{\alpha-2}\nabla v\right) = 0.$$

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