ON A THEOREM OF NEHARI AND QUASIDISCS

Martin Chuaqui
University of Pennsylvania, Department of Mathematics
David Rittenhouse Laboratory, 209 South 33rd Street, Philadelphia, PA 19104-6395, U.S.A.

Abstract. Let \( f \) be a locally injective analytic map of the unit disc \( D \) and let \( \{f, z\} \) be its Schwarzian derivative. Suppose \( |\{f, z\}| \leq 2p(|z|) \). We use the classical connection between Schwarzian derivative and second order linear equations to show that, for a particular class of functions \( p \), the image \( f(D) \) is a quasidisc. The analysis centers on the differential equation \( y'' + py = 0 \) and a finiteness condition of a positive solution \( y \). The proofs are based on Sturm comparison theorems. When \( p \) in the class is analytic and \( x = 1 \) is a regular singular point of the linear equation, it is possible to obtain precise information about Hölder continuity of \( f \) from considerations on the Frobenius solutions at that point. The main result in this paper resolves the complementary case in a general theorem of univalence of Nehari.

1. Introduction

Let \( f \) be analytic and locally univalent, and let \( \{f, z\} = (f''/f')' - \frac{1}{2}(f''/f')^2 \) be its Schwarzian derivative. Two main features of the Schwarzian derivative are that all solutions to \( \{f, z\} = 0 \) are given by fractional linear transformations \( T(z) = (az + b)/(cz + d) \) and that \( \{T \circ f, z\} = \{f, z\} \). The second property is a consequence of the first and an important addition formula for the Schwarzian derivative of a composition. There is a classical connection between the Schwarzian derivative and second order linear equations: any solution of \( \{f, z\} = 2p(z) \) is given by \((au + bv)/(cu + dv), ad - bc \neq 0\), where \( u, v \) are two linearly independent solutions of the equation

\[
y'' + py = 0.
\]

A well known fact that follows from this is that \( f \) is univalent on a given domain if and only if any nontrivial solution of (1.1) vanishes at most once in the domain (see, e.g., [D]). This characterization of univalence was systematically used by Nehari, who derived several sufficient conditions for global injectivity. Briefly, estimates on the size of \( |\{f, z\}| \) together with comparison theorems for the solutions of differential equations imply the absence of multiple zeroes of nontrivial solutions of (1.1). In the unit disc \( D \), some of the conditions of this type that imply univalence are

\[
|\{f, z\}| \leq \frac{\pi^2}{2},
\]

1991 Mathematics Subject Classification: Primary 30C62; Secondary 34C11.
The constants $\pi^2/2$, $2$ and $4$ are sharp in each case. All of these are particular instances of the following general result due to Nehari (Theorem 1 in [N 2]):

Let $p(x) \geq 0$ be an even function on $(-1,1)$ such that $(1 - x^2)^2p(x)$ is nonincreasing for $x > 0$.

Suppose that the solution $y$ of

$$y'' + py = 0, \quad y(0) = 1, \ y'(0) = 0$$

does not vanish on $(-1,1)$. If $|\{f, z\}| \leq 2p(|z|)$ then $f$ is univalent in the unit disc.

With the choices for $p$ as in (1.2), (1.3) and (1.4) the respective solutions of (1.5) are given by $\cos(\frac{1}{2}\pi x)$, $\sqrt{1 - x^2}$ and $1 - x^2$. The odd solution of the linear equation is

$$y(x) \int_0^x y^{-2}(s) \, ds$$

and since the functions involved here are analytic on $(-1,1)$ they extend to the unit disc. Consequently

$$F(z) = \int_0^z y^{-2}(\zeta) \, d\zeta$$

gives in each case the extremal map with the normalizations $F(0) = 0$, $F'(0) = 1$ and $F''(0) = 0$.

By using standard comparison theorems for solutions of differential equations one can go a step further and derive upper and lower bounds for $|f|$ and $|f'|$ when $f$ is normalized as $F$ and $|\{f, z\}| \leq 2p(|z|)$ ([C-O]).

In Nehari’s theorem $p$ is assumed to be continuous, and for such $p$ we will continue to denote by $F$ the associated function defined on $(-1,1)$. Nehari also showed under what circumstances the condition $|\{f, z\}| \leq 2p(|z|)$ is sharp. It states that if $F(x) \to \infty$ as $x \to 1$ then for any positive function $r(x)$ on $(-1,1)$ the condition

$$|\{f, z\}| \leq 2p(|z|) + r(|z|)$$

is in general not sufficient for univalence (Theorem 2 in [N 2]).

In this paper we shall be concerned with the question of what happens when $F(1) < \infty$. Our main result is
Theorem 1. Let \( p(x) \geq 0 \) be an even function on \((-1,1)\) with \((1-x^2)^2 p(x)\) nonincreasing for \(x > 0\). Suppose that the even solution \( y \) of (1.1) is positive and is such that
\[
\int_0^1 y^{-2}(x) \, dx < \infty.
\]
If \(|\{f,z\}| \leq 2p(|z|)\) then \( f(D) \) is a quasidisc.

A quasidisc is the image of \( D \) under some map which is quasiconformal in the entire plane. Let \( \Omega \) be simply-connected with its Poincaré metric \( \lambda(z) |dz| \). A combined result of Ahlfors and later Gehring gives a characterization of quasidiscs which is necessary and sufficient: there exists a positive constant \( \eta \) such that the inequality
\[
|\{\phi, z\}| \leq \eta \lambda^2(z)
\]
implies that \( \phi \) is univalent in \( \Omega \) (see, e.g., [L]).

The function \( p \) in Theorem 1 is assumed to be continuous. As previously shown, there are important cases when \( p \) is actually analytic and \( x = 1 \) is a regular singular point of (1.1). This allows to simplify the analysis by considering the possible Frobenius solutions at \( x = 1 \). The assumption that \( F(1) \) is finite implies that either \( y \sim (1 - x)^m \) as \( x \to 1 \), for some \( 0 < m < \frac{1}{2} \), or else that \( y(1) > 0 \). In the latter case, the normalized function \( f \) will be Lipschitz continuous on \( D \) while in the former case, it is possible to prove Hölder continuity.

I would like to thank C. Epstein for helpful discussions concerning the proof of Lemma 1. The referee’s valuable comments allowed a simplification of the original proof and gave greater clarity to other parts of the exposition.

2. Proofs

The proof of Theorem 1 will be divided in a series of lemmas. In what follows, let \( p \) and \( y \) satisfy the hypothesis of the theorem. Let \( \alpha \in [0,1) \). Most of the analysis ahead depends on the solution \( u \) of
\[
(2.1) \quad u'' + \frac{\alpha}{(1-x^2)^2} u = 0, \quad u(0) = 1, \quad u'(0) = 0.
\]
This function is given explicitly by
\[
u(x) = \frac{1}{2} \sqrt{1-x^2} \left\{ \left( \frac{1+x}{1-x} \right)^{\beta} + \left( \frac{1-x}{1+x} \right)^{\beta} \right\}
\]
where \( \beta = \frac{1}{2} \sqrt{1-\alpha} \) [K, p. 492]. In particular,
\[
u(x) \sim (1-x)^{\frac{1}{2}-\beta}, \quad x \to 1,
\]
and therefore \( \int_0^1 u^{-2}(x) \, dx < \infty \). Let \( \mu = \lim_{x \to 1} (1 - x^2)^2 p(x) \). Clearly \( \mu \geq 0 \) and we claim that \( \mu < 1 \). If not then \( p(x) \geq (1 - x^2)^{-2} \). Let \( P(x) = (1 - x^2)^{-2} \) so that the function \( q(x) = p(x) - P(x) \) is non-negative. Then \( z(x) = \sqrt{1 - x^2} \) satisfies

\[
(2.2) \quad z'' + Pz = 0, \quad z(0) = 1, \quad z'(0) = 0.
\]

Multiplying (2.2) by \( y \), (1.5) by \( z \), and subtracting, we get

\[
z''y - zy'' = qyz.
\]

We integrate this equation, using the initial condition on \( y \) and \( z \), to obtain

\[
\left( \frac{z}{y} \right)'(x) = \frac{\int_0^x (uqy)(s) \, ds}{y(x)^2}.
\]

Hence \( (z/y)' \) has the same sign as \( x \) and therefore \( y \leq z \) on \((-1, 1)\) since \( z(0) = y(0) = 1 \). It follows that either \( y \) vanishes on \((-1, 1)\) or else \( F(1) = \infty \). This contradiction proves our claim. Choose \( \alpha \) such that \( \mu < \alpha < 1 \) and let now

\[
q(x) = p(x) - \frac{\alpha}{(1 - x^2)^2}.
\]

**Lemma 1.** Let \( l = \lim \inf_{x \to 1} (1 - x)(\frac{y'}{y}) \). Then \( -1/2 < l \leq 0 \).

**Proof.** Following an argument almost identical to the one given above, we can write

\[
\frac{y'}{y} = \frac{u'}{u} - \frac{\int_0^x (uqy)(s) \, ds}{u(x)y(x)}.
\]

Since \( y'' = -py \leq 0 \) we have \( y' \leq 0 \) on \((0, 1)\) because of the initial condition. Hence \( l \leq 0 \).

By considering the graph of the function \( F \) it follows from elementary geometry that

\[
\lim_{x \to 1} \frac{1 - x}{y^2} = 0.
\]

Hence \((1 - x)(uy)^{-1} \to 0\) as \( x \to 1 \).
On the other hand, the limit of \((1 - x)(u' / u)\) as \(x \to 1\) can be computed directly and it equals \(-\left(\frac{1}{2} - \beta\right)\). This, together with equation (2.3) and the fact that \(q(x) < 0\) for \(x\) sufficiently close to 1 imply the lemma.

**Lemma 2.** There exists a constant \(M\) such that

\[
F(1) - F(x) \leq M \left(\frac{1 - x}{y^2}\right).
\]

**Proof.** The derivative of the left hand side of (2.4) is \(-y^2\) while the derivative of \((1 - x)y^{-2}\) is

\[-y^2 \left(1 + 2(1 - x)\frac{y'}{y}\right).\]

Lemma 1 implies that \(1 + 2(1 - x)(y'/y) \geq \sigma > 0\) provided \(x\) is sufficiently close to 1. Hence for all such \(x\)

\[F(1) - F(x) \leq \frac{1}{\sigma} \left(\frac{1 - x}{y^2}\right)
\]

and the lemma follows.

Now we state the key result in this chain.

**Lemma 3.** There exists a constant \(\eta > 0\) such that the solution \(\varphi\) of

\[
\varphi'' + \left(p(x) + \frac{\eta}{(1 - x^2)^2}\right)\varphi = 0, \quad \varphi(0) = 1, \quad \varphi'(0) = 0
\]

does not vanish on \((-1, 1)\).

**Proof.** Let \(c = F(1)\). On the image interval \((-c, c)\) we consider the “Poincaré density”

\[
\lambda(w) = \frac{1}{F'(x)(1 - x^2)} = \frac{y^2}{1 - x^2}
\]

where \(w = F(x)\). We will show that for \(\eta > 0\) sufficiently small the solution \(h\) of

\[
h'' + \eta \lambda^2(w)h = 0, \quad h(0) = 1, \quad h'(0) = 0
\]

is positive on \((-c, c)\). By Lemma 2,

\[
\lambda(w) = \frac{y^2}{1 - x^2} \leq \frac{y^2}{1 - x} \leq \frac{M}{c - w} \leq \frac{2Mc}{c^2 - w^2}.
\]

Thus it suffices to show that the solution of (2.6) with \(4M^2c^2(c^2 - w^2)^{-2}\) instead of \(\lambda^2(w)\) does not vanish. This will be the case as long as \(4M^2\eta \leq 1\). To see this, we rescale. The function \(\tilde{h}(x) = h(cx)\) solves \(\tilde{h}'' + 4M^2\eta(1 - x^2)^{-2}\tilde{h} = 0\) with even initial conditions. Then \(\tilde{h} > 0\) on \((-1, 1)\) if and only if \(4M^2\eta \leq 1\) [K, p. 492]. With \(h\) the positive solution of (2.6) we define \(\varphi\) by

\[\varphi(x) = y(x)h(F(x)).\]

A straightforward computation shows that \(\varphi\) is the solution of (2.5). This finishes the proof of Lemma 3.
This lemma together with Nehari’s first theorem shows that

\begin{equation}
|\{g, z\}| \leq 2 \left( p(|z|) + \frac{\eta}{1 - |z|^2} \right)
\end{equation}

is a sufficient condition for univalence. The proof of Theorem 1 is now quite simple. Assume \( |\{f, z\}| \leq 2p(|z|) \) and let \( \lambda(\zeta) \, |d\zeta| \) be the Poincaré metric on \( \Omega = f(D) \).

We will show that \( |\{\phi, \zeta\}| \leq 2\eta \lambda^2(\zeta) \) implies the univalence of the map \( \phi \). Let \( g(z) = \phi(f(z)) \). Then

\[
\{g, z\} = \{\phi, f(z)\} f'(z)^2 + \{f, z\}
\]

and therefore

\[
(1 - |z|^2)^2 |\{g, z\}| \leq \lambda^{-2}(\zeta) |\{\phi, \zeta\}| + 2(1 - |z|^2)^2 p(|z|)
\]

where \( \zeta = f(z) \). It follows that \( g \) satisfies (2.7), hence \( g \) and consequently \( \phi \) are univalent. This shows that \( \Omega \) is a quasidisc.

3. The analytic case

In this section we shall assume that, in addition, \( p \) is analytic and that \( x = 1 \) is a regular singular point of the equation (1.1). The assumptions on the even solution \( y \) are as before. Recall that \( \mu = \lim_{x \to 1} (1 - x^2)^2 p(x) \). From the analysis of the possible Frobenius solutions at \( x = 1 \) we will prove Hölder or Lipschitz continuity for maps \( f \) that satisfy \( |\{f, z\}| \leq 2p(|z|) \). Because of the invariance of the Schwarzian derivative under Möbius changes one can not expect such a result unless \( f \) is properly normalized. The right normalization turns out to be \( f''(0) = 0 \). Let \( u \) solve

\[
u'' + \frac{1}{2} \{f, z\} u = 0, \quad u(0) = 1, \quad u'(0) = 0
\]

and let

\[
v(z) = u(z) \int_0^z u^{-2}(\zeta) \, d\zeta
\]

be the solution with odd initial conditions. If \( f''(0) = 0 \) then

\[
f(z) = f(0) + f'(0) \int_0^z u^{-2}(\zeta) \, d\zeta.
\]

From Lemma 2 in [C-O] it follows that if \( |\{f, z\}| \leq 2p(|z|) \) then

\[
|u(z)| \geq y(|z|)
\]
and therefore
\[ |f'(z)| \leq |f'(0)|y^{-2}(|z|). \]

We distinguish the cases \( \mu > 0 \) and \( \mu = 0 \). Suppose \( \mu \) is positive. Then \( p(x) \geq \mu(1 - x^2)^{-2} \) and hence the function \( y \) must vanish at \( x = 1 \). The possible orders \( m \) of vanishing are given by the roots of the inditial equation
\[ m^2 - m + \frac{\mu}{4} = 0, \]
i.e.,
\[ m_1 = \frac{1 + \sqrt{1 - \mu}}{2}, \quad m_2 = \frac{1 - \sqrt{1 - \mu}}{2}. \]

Note that \( 0 < m_2 < \frac{1}{2} < m_1 < 1 \). Since \( m_1 - m_2 \) is not an integer both orders of vanishing can occur [H].

**Theorem 2.** Let \( f \) satisfy \( |\{f, z\}| \leq 2p(|z|) \), \( f''(0) = 0 \) and suppose \( \mu > 0 \). If \( F(1) \) is finite then \( f \) is Hölder continuous on \( D \) with Hölder exponent \( \sqrt{1 - \mu} \).

**Proof.** The assumption that \( F(1) < \infty \) implies that \( y \sim (1 - x)^{m_2} \) as \( x \to 1 \). Therefore
\[ |f'(z)| = O((1 - |z|)^{-2m_2}). \]
A standard technique of integrating along hyperbolic segments (see, e.g., [G-P]) gives
\[ |f(z_1) - f(z_2)| = O(|z_1 - z_2|^{1-2m_2}), \]
and the theorem follows.

Suppose now \( \mu = 0 \). In this case, the roots of the inditial equation are 1 and 0. Hence two linearly independent solutions are \( y_1 = (1 - x)h_1 \) and \( y_2 = h_2 + cy_1 \log(1 - x) \), where \( h_1, h_2 \) are analytic and nonvanishing at \( x = 1 \) [H, Theorem 5.3.1].

**Theorem 3.** Let \( f \) satisfy \( |\{f, z\}| \leq 2p(|z|) \), \( f''(0) = 0 \) and suppose \( \mu = 0 \). If \( F(1) \) is finite then \( f \) is Lipschitz continuous on \( D \).

**Proof.** The finiteness condition and the discussion preceding the theorem imply that in fact \( y \) cannot vanish at \( x = 1 \). Hence \( |f'| \) is uniformly bounded.

The following situation describes accurately the case \( \mu = 0 \). For \( 0 \leq t < 1 \) let \( p_t(x) = tp(x) \), where \( p(x) = 2(1 - x^2)^{-1} \). Since the inequality \( |\{f, z\}| \leq 2p_t(|z|) \) is sufficient for univalence then \( |\{f, z\}| \leq 2p_t(|z|) \) implies that \( f(D) \) is a quasidisc (Theorem 6 in [G-P]). As mentioned in the introduction, the even solution of (1.1)
is in this case $y = 1 - x^2$. We claim that the even solution $y_t$ of (1.1) with $p$ replaced by $p_t$ must be positive at the endpoints. To show this, let

$$F_t(z) = \int_0^z y_t^{-2}(\zeta) d\zeta.$$ 

This function is odd and has Schwarzian derivative equal to $2p_t(z)$. Therefore $F_t(D)$ is a quasidisc and hence $F_t(1) < \infty$, otherwise the point at infinity would be a point of self-intersection of $\partial F_t(D)$. This in turn would contradict the fact that $\partial F_t(D)$ is a Jordan curve. Since $\mu_t = t\mu = 0$ it follows that $y_t$ is a linear combination of the functions $y_1, y_2$ as in the paragraph preceding the statement of Theorem 3. Thus $F_t(1) < \infty$ forces $y_t(1) > 0$.

We consider finally examples for any $\mu \in (0, 1)$. For $s \in (1, 2)$ let

$$p(x) = s \frac{1 - (s - 1)x^2}{(1 - x^2)^2}.$$ 

Then $\mu = s(2 - s)$ and the even solution of (1.1) is

$$y = (1 - x^2)^{s/2}.$$ 

(The exponent $s/2$ corresponds to $m_1$ in (3.1) and $m_2 = (2 - s)/2$.) This shows that the function $F$ has $F(1) = \infty$. On the other hand, by the argument given above, changing $p$ to $tp$ has the effect of making $F_t(1)$ finite. Consider now equation (3.1) with $\mu$ replaced by $t\mu$. Since $y_t \sim (1 - x)^{m_1}$, $x \to 1$, would make $F_t(1)$ infinite, we conclude that the order of vanishing of the solution $y_t$ must be the other root, $m_2$.

References


Received 24 February 1992