HOLOMORPHIC CURVATURE OF INFINITE DIMENSIONAL SYMMETRIC COMPLEX BANACH MANIFOLDS OF COMPACT TYPE

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Abstract. We show that the tangent norm on an infinite dimensional symmetric complex Banach manifold of compact type has constant positive holomorphic curvature (in the sense of [2], 8.7).

Introduction

Symmetric Banach manifolds generalise to infinite dimensions the Hermitian symmetric spaces. The finite dimensional Hermitian symmetric spaces were classified by Cartan, in [1], using the theory of Lie groups and Lie algebras. The theory of $J^*$-triple systems appears to be the more appropriate tool in infinite dimensions and a classification of the symmetric Banach manifolds in terms of $J^*$-triple systems is given by Kaup in [5].

The $J^*$-triple systems, or $J^*$-triples, which are Banach spaces endowed with a certain triple product structure, generalise the concepts of $C^*$-algebra, $J^*$-algebra, and $JB^*$-algebra. The $J^*$-triples which are positive in a certain sense, corresponding to a certain class of linear operators on the space having positive spectrum, are called $JB^*$-triples. It is known that, for the above classification, the $JB^*$-triples characterise the bounded symmetric domains, or, in finite dimensions, the Hermitian symmetric spaces of non-compact type.

Changing the sign of the triple product on a $J^*$-triple, $U$ say, produces another $J^*$-triple which is referred to as the dual triple of the $J^*$-triple $U$. In particular, the dual triple of a $JB^*$-triple is negative in a certain sense. Moreover, the symmetric Banach manifold associated, via Kaup’s construction, to the dual triple of a $JB^*$-triple is called a symmetric manifold of compact type, since in finite dimensions these manifolds are exactly the Hermitian symmetric spaces of compact type.

In this way, to every $JB^*$-triple we can associate a bounded symmetric domain and a symmetric manifold of compact type (classified by the dual triple).

1991 Mathematics Subject Classification: Primary 58B20; Secondary 17C36, 53C60.
This work is part of a doctoral dissertation submitted to the National University of Ireland.
This gives an exact analogy of the duality in finite dimensions between the Hermitian symmetric spaces of compact and non-compact type. While the bounded symmetric domains are relatively well understood and studied, little is known of the symmetric manifolds of compact type in infinite dimensions.

Examples include certain Grassmann manifolds (cf. [6]) and certain function spaces (cf. [7]).

As infinite dimensional analogues of the compact Hermitian symmetric spaces, these symmetric manifolds of compact type may be expected to display some properties reminiscent of compactness. In this paper we show this to be the case, by proving that the holomorphic curvature of an arbitrary symmetric manifold of compact type is constant and positive (with a value of $+4$).

The concept of holomorphic curvature that we use, is one that may be defined, for an arbitrary infinitesimal Finsler metric on a complex Banach manifold, in terms of a generalised Hessian. It is important to note, that this does not coincide with the concept of holomorphic curvature as understood in the Riemannian sense. For a survey in this direction, see [2].

**Notation**

Throughout, we let $\mathbf{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and for $r, 0 < r < 1$, $r\mathbf{D} = \{ rz : z \in \mathbf{D} \}$. We use $\text{Dom}(f)$ to denote the domain of a function $f$. For complex Banach spaces $E$ and $F$, let $L(E, F)$ denote the Banach space of all continuous linear maps: $E \to F$ and let $L(E) = L(E, E)$. We denote by $L^k(E)$ the space of all continuous homogeneous polynomials: $E \to E$ of degree $k$. We say $T \in L(E)$ is hermitian if $e^{i\lambda T}$ is an isometry for all $\lambda \in \mathbb{R}$.

Let $U$ be a complex Banach space with continuous conjugate-linear mapping $*: U \to L^2(U)$ and write $a^*$ for $*(a)$. For all $a, b, z$ in $U$ define

$$\{a, b^*, z\} := \frac{1}{2} (b^*(a + z) - b^*(a) - b^*(z))$$

and define $a \Box b^*$ in $L(U)$ by $a \Box b^*(z) := \{a, b^*, z\}$, for all $z$ in $U$.

Then $(U, *)$ is called a $J^*$-triple system, or $J^*$-triple, if and only if

(i) $\{\alpha, \beta^*, \{x, y^*, z\}\} = \{\{\alpha, \beta^*, x\}, y^*, z\} - \{x, \{\beta, \alpha^*, y\}^*, z\} + \{x, y^*, \{\alpha, \beta^*, z\}\}$

for all $\alpha, \beta, x, y$ and $z$ in $U$.

(ii) $\alpha \Box \alpha^* \in L(U)$ is hermitian for all $\alpha$ in $U$.

If, in addition, the following hold

(iii) $\alpha \Box \alpha^* \geq 0$ for all $\alpha \in U$

(iv) $\|\alpha \Box \alpha^*\| = \|\alpha\|^2$ for all $\alpha$ in $U$

we call $U$ a $JB^*$-triple.

We denote by $U_z$ the closed $J^*$-subtriple of $U$ generated by $z$ in $U$. 
A morphism of $J^∗$-triple systems $\lambda: (U, *) \to (V, *)$ is a continuous linear map $\lambda: U \to V$ such that
\[
\lambda\{u, v^*, w\} = \{\lambda(u), \lambda(v)^*, \lambda(w)\}
\]
for all $u, v, w$ in $U$.

A manifold $M$ modelled locally on open subsets of complex Banach spaces with biholomorphic coordinate transformations is called a complex Banach manifold. Let $TM$ denote the tangent bundle of $M$.

A mapping $\alpha: TM \to \mathbb{R}$ is called a norm on $TM$ if the restriction of $\alpha$ to every tangent space $T_x$, $x \in M$, is a norm on $T_x$ with the following property: there is a neighbourhood $U$ of $x$ in $M$ which can be realised as a domain in a complex Banach space $E$ such that
\[
c \|a\| \leq \alpha(u, a) \leq C \|a\|
\]
for all $(u, a) \in TU \cong U \times E$ and suitable constants $0 < c \leq C$. We then refer to $(M, \alpha)$, or simply $M$, as a normed manifold. If $(\tilde{M}, \tilde{\alpha})$ is another complex normed manifold, we say that a holomorphic mapping $\varphi: M \to \tilde{M}$ is an isometry if for all $(z, v) \in TM$,
\[
\tilde{\alpha}(\varphi(z), \varphi'(z)v) = \alpha(z, v).
\]

Let Aut$(M)$ denote the group of all biholomorphic isometries of $M$.

A connected complex normed manifold $M$ is called symmetric if for every $a \in M$ there exists an involution $s_a \in \text{Aut}(M)$ having $a$ as an isolated fixed point.

A morphism of the symmetric manifolds $M$ and $\tilde{M}$ is a holomorphic mapping $h: M \to \tilde{M}$ such that $h \circ s_x = s_{h(x)} \circ h$ for all $x \in M$. The characterisation of symmetric manifolds in terms of $J^∗$-triple systems is given in [5].

Let $u$ be a real-valued upper semicontinuous function with Dom$(u) \subseteq \mathbb{C}$. For $p \in \text{Dom}(u)$, we define the generalised Hessian of $u$ at $p$, as
\[
\Delta u(p) = 4 \lim_{r \to 0+} \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{i\theta}) \, d\theta - u(p) \right\} \in \mathbb{R} \cup \{\infty\}.
\]

Let $(p, v) \in TM$. We say $f \in \mathcal{H}(p, v)$ if for some $r$, $0 < r < 1$, $f$ is a holomorphic mapping: $rD \to \tilde{M}$ with $f(0) = p$, $f'(0) = v$. For $f \in \mathcal{H}(p, v)$ we define $\lambda_f: rD \to \mathbb{R}$ by $\lambda_f(z) = \alpha^2(f(z), f'(z))$.

For $(p, v) \in TM$ with $v \neq 0$, we may then define the holomorphic curvature of $\alpha$ at $p$ in the direction of $v$, cf. [1], denoted $\kappa_\alpha(p, v)$, as
\[
\kappa_\alpha(p, v) = \sup \left\{ \frac{\Delta \log \lambda_f(0)}{-2\alpha^2(p, v)} : f \in \mathcal{H}(p, v) \right\}.
\]
Clearly, for $\lambda \neq 0 \in \mathbb{C}$, $\kappa_\alpha(p, v) = \kappa_\alpha(p, \lambda v)$, for all $(p, v) \in TM$ with $v \neq 0$.

If $\alpha_1$ and $\alpha_2$ are tangent norms on the complex manifolds $D_1$ and $D_2$ respectively and $\varphi: D_1 \to D_2$ is a biholomorphic mapping which is an isometry for $\alpha_1$ and $\alpha_2$ then

$$\kappa_{\alpha_1}(p, v) = \kappa_{\alpha_2}(\varphi(p), \varphi'(p)v) \quad \text{for } (p, v) \in TD_1.$$

**Results**

Let $U$ be an arbitrary $JB^*$-triple and let $M$ be the associated symmetric manifold of compact type (i.e. $M$ is the symmetric manifold associated via Kaup’s construction to the dual triple of $U$). Let $m_0$ denote a base point of $M$ and let $\alpha: TM \to \mathbb{R}$ be the tangent norm of $M$. Since $\text{Aut}(M)$ acts transitively on $M$, it suffices to consider $\kappa_\alpha(m_0, v)$ where $v \in T_{m_0}M = U$, $\|v\| = 1$. Let $Q(z)$ denote the conjugate linear mapping on $U$ given by

$$Q(z)a = \{z, a^*, z\},$$

for all $a$ in $U$.

The following linear operators on $U$, called Bergman operators, play an important role:

$$B(z, w^*) = \mathbf{I}_U - 2z \square w^* + Q(z)Q(w) \quad \text{for } z, w \in U.$$

For all $z$ in $U$,

$$D_z := B(z, -z^*)^{1/2} \quad \text{and} \quad D_z^{-1}$$

are well defined.

By [4, 4.5], there exists a canonical chart on $M$, identifying a neighbourhood $W$ of $0 \in U$, with a neighbourhood of $m_0 \in M$, with respect to which the tangent norm on $M$ may be expressed as follows:

$$\alpha(z, v) = \|D_z^{-1}(v)\| \quad \text{for } (z, v) \in TW \cong W \times U.$$

We may chose $0 < \varepsilon < 1$ such that the ball $B(0, 2\varepsilon) := \{x \in U : \|x\| < 2\varepsilon\} \subseteq W$.

Fix $v \in U$, $\|v\| = 1$, arbitrary. The mapping $\varphi: \varepsilon \mathbf{D} \to M$, given by $\varphi(t) = tv$, is in $\mathcal{H}(0, v)$. Showing now that

$$\frac{\Delta \log \lambda_\varphi(0)}{-2\|v\|^2} \geq 4$$

implies $\kappa_\alpha(0, v) \geq 4$.

By [5, Corollary 4.8], the $JB^*$-subtriple of $U$ generated by $v$, $U_v$, is isometrically $J^*$-isomorphic to $C_0(S)$, for some locally compact space $S$. 
For any $z, w \in C_0(S)$, then

$$D_z^{-1}(w) = \left( \frac{1}{1 + |z|^2} \right) w$$

and hence

$$\left\| D_{re^{i\theta}}^{-1}(v) \right\| = \sup_{s \in S} \left( \frac{|v(s)|}{1 + r^2 |v(s)|^2} \right) = \frac{\|v\|}{1 + r^2 \|v\|^2}.$$  

Then

$$\Delta \log \lambda_\varphi(0) = -8 \liminf_{r \to 0} \frac{1}{r^2} \left( \log \left( 1 + r^2 \|v\|^2 \right) \right).$$

By elementary calculus, the limit involved here is exactly

$$g'(0) = \|v\|^2 \quad \text{for} \quad g(t) = \log \left( 1 + t \|v\|^2 \right).$$

Therefore

$$\Delta \log \lambda_\varphi(0) = -8 \|v\|^2.$$ 

In particular

$$\kappa_\alpha(0, v) \geq \frac{\Delta \log \lambda_\varphi(0)}{-2\|v\|^2} = 4.$$ 

In the other direction, we examine

$$\frac{\Delta \log \lambda_\psi(0)}{-2\|v\|^2}$$

for arbitrary $\psi \in \mathcal{H}(0, v)$.

**Lemma 1.1.** Let $U$ be a $JB^*$-triple. Then

$$\|B(z, -z^*)\| = (1 + \|z\|^2)^2$$

for all $z \in U$.

**Proof.** By [3, Corollary 3], we have that for all $x, y, z$ in a $JB^*$-triple $U$, $\|\{x, y^*, z\}\| \leq \|x\|\|y\|\|z\|$. Therefore $\|B(z, -z^*)\| \leq (1 + \|z\|^2)^2$, for all $z \in U$. On the other hand, since $U_z$ is isometrically $J^*$-isomorphic to $C_0(S)$, for some locally compact space $S$,

$$\|B(z, -z^*)(z)\| = (1 + \|z\|^2)^2 \|z\|$$

and the result follows. □
Fix $\psi$ arbitrary in $\mathcal{H}(0, v)$. Then
\[
\alpha(\psi(re^{i\theta}), \psi'(re^{i\theta})) = \left\| D_{\psi(re^{i\theta})}^{-1}\psi'(re^{i\theta}) \right\| \geq \left\| \psi'(re^{i\theta}) \right\| \left\| D_{\psi(re^{i\theta})} \right\|
\]
From this
\[
\Delta \log \lambda(0) \geq 4 \liminf_{r \to 0} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left( 2 \log \left\| \psi'(re^{i\theta}) \right\| - 2 \log \left\| D_{\psi(re^{i\theta})} \right\| \right) d\theta - 2 \log \| v \| \right\}
\]
As the mapping $z \mapsto 2 \log \| \psi'(z) \|$ is subharmonic
\[
\frac{1}{2\pi} \int_0^{2\pi} 2 \log \| \psi'(re^{i\theta}) \| d\theta \geq 2 \log \| v \|
\]
and
\[
\Delta \log \lambda(0) \geq -8 \liminf_{r \to 0} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \left( \left\| D_{\psi(re^{i\theta})} \right\| \right) d\theta \right\}
\]
By the Lebesgue dominated convergence theorem
\[
\liminf_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\left\| D_{\psi(re^{i\theta})} \right\|}{r^2} \right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left( \liminf_{r \to 0} \frac{\log \left( \left\| D_{\psi(re^{i\theta})} \right\| \right)}{r^2} \right) d\theta.
\]
Let $\sum_{n=0}^{\infty} a_n (t - 1)^n$ denote the Taylor expansion of the function $f(t) = \sqrt{t}$ for $|t - 1| < 1$.
Since $D_z = f(B(z, -z^*))$ in the sense of the functional calculus, then
\[
D_z = \sum_{n=0}^{\infty} a_n (B(z, -z^*) - I_U)^n
\]
whenever
\[
\sigma(B(z, -z^*)) \subset \{ t \in \mathbb{R} : |t - 1| < 1 \}.
\]
In particular, for $\|z\| < 1/2$, we have
\[
\|D_z\| \leq \sum_{n=0}^{\infty} |a_n| ((1 + \|z\|^2)^2 - 1)^n.
\]
As
\[
a_n = \frac{(-1)^{n+1}1.3.5\cdots(2n-3)}{2^n n!}
\]
for $n \geq 2$, $a_0 = 1$, and $a_1 = 1/2$, the power series

$$\sum_{n=0}^{\infty} |a_n|(t - 1)^n$$

represents the function $g(t) = 2 - \sqrt{2-t}$ for $|t-1| < 1$, and hence

$$\|Dz\| \leq g((1 + \|z\|^2)^2)$$

for $\|z\| < 1/2$.

The mapping $k(t) = \log(2 - \sqrt{2-t})$ is represented about $t = 1$ by the expansion

$$k(t) = \frac{(t-1)}{2} + \frac{(t-1)^3 k'''(\xi(t))}{3!}$$

where $\xi(t)$ lies between $t$ and 1.

From above,

$$\log(\|Dz\|) \leq k((1 + \|z\|^2)^2), \quad \text{for } \|z\| < 1/2.$$ 

Moreover, since $\psi(0) = 0$ and $\psi'(0) = v$, it follows that

$$\liminf_{r \to 0} \frac{\|\psi(re^{i\theta})\|}{r} = \|v\|$$

and therefore

$$\liminf_{r \to 0} \frac{\log(\|D\psi(re^{i\theta})\|)}{r^2} \leq \|v\|^2.$$ 

On the other hand, Lemma 1.1 implies that $\|Dz\| \geq 1 + \|z\|^2$, for all $z \in U$, so

$$\liminf_{r \to 0} \frac{\log(\|D\psi(re^{i\theta})\|)}{r^2} \geq \liminf_{r \to 0} \frac{\log \left(1 + \|\psi(re^{i\theta})\|^2\right)}{r^2} = \|v\|^2.$$ 

Therefore

$$\liminf_{r \to 0} \frac{\log(\|D\psi(re^{i\theta})\|)}{r^2} = \|v\|^2.$$ 

It follows then that

$$\frac{\Delta \log \lambda \psi(0)}{-2\|v\|^2} \leq 4.$$ 

Since $\psi$ was arbitrary

$$\kappa_\alpha(0, v) \leq 4$$

and we have proved the desired result.
Theorem 1.2. Let $U$ be a JB$^*$-triple and let $M$ be the associated symmetric complex manifold of compact type with tangent norm $\alpha$ on $TM$.

The holomorphic curvature of $\alpha$ at any point of $M$ is identically equal to 4.

Acknowledgements. I offer sincerest thanks to my Ph.D. supervisor Professor Sean Dineen for his advice, support and encouragement over the last number of years. I am also grateful to Professor Rodriguez Palacios and the referee for helpful comments.

References


Received 27 January 1992