NULL SETS FOR DOUBLING AND DYADIC DOUBLING MEASURES

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Abstract. In this note, we study sets on the real line which are null with respect to all doubling measures on \( \mathbb{R} \), or with respect to all dyadic doubling measures on \( \mathbb{R} \). We give some sufficient conditions for the former, a test for the latter, and some examples.

Our work is motivated by a characterization of dyadic doubling measures by Fefferman, Kenig and Pipher [5], and by a result of Martio [8] on porous sets and sets of total \( \mathscr{A} \)-harmonic measure zero for certain class of nonlinear \( \mathscr{A} \)-operators.

A measure \( \mu \) on \( \mathbb{R} \) is said to have the doubling property with constant \( \lambda \) if, whenever \( I \) and \( J \) are two neighboring intervals of same length then \( \mu(I) \leq \lambda \mu(J) \); denote by \( \mathcal{D}(\lambda) \) the collection of all doubling measures with constant \( \lambda \), and \( \mathcal{D} = \bigcup_{\lambda \geq 1} \mathcal{D}(\lambda) \). A measure \( \mu \) on \( \mathbb{R} \) has the dyadic doubling property with constant \( \lambda \) if \( \mu(I) \leq \lambda \mu(J) \) whenever \( I \) and \( J \) are two dyadic neighboring intervals of same length and \( I \cup J \) is also a dyadic interval; denote by \( \mathcal{D}_d(\lambda) \) and \( \mathcal{D}_d \) the corresponding collections of dyadic doubling measures.

Given \( \{\alpha_n\}, 0 < \alpha_n < 1 \), a set \( E \subseteq \mathbb{R} \) is called \( \{\alpha_n\} \)-porous if there exists a sequence of coverings \( \mathcal{E}_n = \{E_{n,j}\} \) of \( E \), by intervals with mutually disjoint interiors, so that each \( E_{n,j} \setminus E \) contains an interval \( J_{n,j} \) of length \( \geq \alpha_n |E_{n,j}| \), \( \cup_{k} E_{n+1,k} \) is contained in \( \cup_{\mathcal{E}_n} (E_{n,j} \setminus J_{n,j}) \) and \( \sup |E_{n,j}| \to 0 \) as \( n \to \infty \). The Cantor ternary set is \( \{\frac{1}{3}\} \)-porous. The porous sets studied by Martio [8] are \( \{\alpha\} \)-porous for some \( \alpha > 0 \).

Theorem 1. If \( 0 < \alpha_n < 1 \), \( \sum_1^\infty \alpha_n^K = \infty \) for all \( K \geq 1 \), and \( E \) is \( \{\alpha_n\} \)-porous, then \( E \) is null for all doubling measures on \( \mathbb{R} \).

Corollary. There exist sets of Hausdorff dimension one which are null for all doubling measures.

The condition given in Theorem 1 cannot be improved:

Theorem 2. If \( 0 < \alpha_n < \frac{1}{4} \) is a decreasing sequence satisfying \( \sum |\alpha_n^K| < \infty \) for some \( K \geq 1 \), then there exists a perfect set which is \( \{\alpha_n\} \)-porous, but carries a positive measure for some \( \mu \in \mathcal{D} \).

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Let $E$ be a closed set in $[0, 1]$ and $\lambda > 1$. In Theorem 3, we give a deterministic procedure of testing whether $E$ is $D_d(\lambda)$-null. In this process, an optimal measure $\mu_{E, \lambda}$ among $D_d(\lambda)$ is selected for $E$. The precise statement is given in Section 2.

Denote by $\mathcal{N}$ the collection of null sets for doubling measures $\{E : \mu(E) = 0 \text{ for all } \mu \in D_d\}$, and $\mathcal{N}_d$ its dyadic counterpart $\{E : \mu(E) = 0 \text{ for all } \mu \in D_d\}$. Clearly $\mathcal{N}_d \subseteq \mathcal{N}$ and $\mathcal{N}$ is translation invariant. The assertion that $\mathcal{N}_d \neq \mathcal{N}$ is not surprising, however it requires a lot of work.

**Theorem 4.** There exists a perfect set $S \subseteq [0, 1]$ which is in $\mathcal{N} \setminus \mathcal{N}_d$. And corresponding to this $S$, there exists a set $T$ of dimension one, so that $t + S \in \mathcal{N}_d$ for each $t \in T$.

It would be interesting to know whether a pair of sets $S, T$ can be chosen to satisfy length $(T) > 0$ in addition to the properties in Theorem 4.

**Theorem 5.** Let $t$ be any number whose binary expansion has infinitely many zeros and infinitely many ones. Then there exists a perfect set $S_t$ so that $S_t \in \mathcal{N} \setminus \mathcal{N}_d$ but $t + S_t \in \mathcal{N}_d$.

Finally, in Section 4, we shall comment on relations between sets in $\mathcal{N}$ and null sets of the harmonic measures with respect to the $p$-Laplacians in the upper half plane.

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### 1. Proofs of Theorems 1 and 2

We first state a useful lemma.

**Lemma 1.** Let $\mu$ be a dyadic doubling measure on $[0, 1]$ and $I$ be any subinterval. Then there exists $K > 1$ depending on the dyadic doubling constant $\lambda$ only, so that

$$4|I|^{1/K} \mu([0, 1]) \geq \mu(I) \geq \frac{1}{4}|I|^{K} \mu([0, 1]).$$

**Proof.** Let $I_1$ and $I_2$ be two adjacent dyadic closed intervals in $[0, 1]$ such that $I \subseteq I_1 \cup I_2$ and $2|I| \geq |I_1| + |I_2|$. Then

$$\mu(I) \leq \mu(I_1) + \mu(I_2)$$

$$\leq \left(\frac{\lambda}{1 + \lambda}\right)^{-\log_2|I_1|} + \left(\frac{\lambda}{1 + \lambda}\right)^{-\log_2|I_2|} \mu([0, 1])$$

$$\leq 2(2|I|)^{\log_2((1+\lambda)/\lambda)} \mu([0, 1]) \leq 4|I|^{\log_2((1+\lambda)/\lambda)} \mu([0, 1]).$$

If $|I| \geq 1 - 16^{-(\log_2((1+\lambda)/\lambda))^{-1}} \equiv A$, 

$$4|I|^{1/K} \mu([0, 1]) \geq \mu(I) \geq \frac{1}{4}|I|^{K} \mu([0, 1]).$$


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\[ \mu([0, 1] \setminus I) \leq 8(1 - |I|) \log_2((1 + \lambda)/\lambda) \mu([0, 1]) \leq \frac{1}{2} \mu([0, 1]). \]

Hence \( \mu(I) \geq \frac{1}{2} \mu([0, 1]). \) If \(|I| < A\), let \( J \) be the largest dyadic interval contained in \( I \). Therefore \(|J| \geq |I|/4\) and

\[ \mu(I) \geq \mu(J) \geq \left( \frac{1}{1 + \lambda} \right)^{-\log_2 |J|} \mu([0, 1]) \geq (|I|/4)^{\log_2(1 + \lambda)} \mu([0, 1]) \geq \mu([0, 1]) |J|^{\log_2(1 + \lambda)}(1 - 2(\log_2 A)^{-1}). \]

**Proof of Theorem 1.** Assume \( E \subseteq [0, 1] \), and let \( E_n = \{E_{n,j}\} \) be the coverings of \( E \) and \( \{J_{n,j}\} \) be the subintervals of \( E_{n,j} \setminus E \) in defining \( \{\alpha_n\}\)-porosity. Let \( E \in \mathcal{D} \), it follows from Lemma 1 that \( \mu(J_{n,j}) \geq \frac{1}{4} \alpha_n^K \mu(E_{n,j}) \) for some \( K \geq 1 \) depending on \( \mu \) only. Thus

\[ \mu(E_{n,j} \setminus J_{n,j}) \leq (1 - \frac{1}{4} \alpha_n^K) \mu(E_{n,j}). \]

Summing over \( j \), we obtain

\[ \sum_k \mu(E_{n+1,k}) \leq (1 - \frac{1}{4} \alpha_n^K) \sum_j \mu(E_{n,j}). \]

Therefore

\[ \mu(E) \leq \prod_n (1 - \frac{1}{4} \alpha_n^K) \mu([0, 1]) = 0. \]

**Proof of Theorem 2.** Let \( N_n \) be a rapidly increasing sequence of odd integers with \( N_1 = 1 \), \( N_n \geq \alpha_{n-1}^{-1} \) for \( n \geq 2 \). After replacing \( \alpha_n \) by a number which is at most twice its size, we may assume that \( \alpha_n = m_n N_{n+1}^{-1} \) for some odd integer \( m_n \). The construction of \( E \) resembles that of the Cantor set. First we remove the open interval which constitutes the middle \( \alpha_1 \) position of \([0, 1]\), and subdivide the two remaining closed intervals into subintervals of equal length \( N_2^{-1} \), call this collection of subintervals \( \mathcal{S}_1 \). This subdivision is possible due to the modification on \( \alpha_n \)'s. On each interval in \( \mathcal{S}_1 \), remove the open interval which constitutes its middle \( \alpha_2 \) portion, and subdivide the remaining intervals into subintervals of equal length \((N_2 N_3)^{-1}\), call this new collection of subintervals \( \mathcal{S}_2 \). Continue the process indefinitely and let

\[ E = \bigcap_n \left( \bigcup_{I \in \mathcal{S}_n} I \right). \]

Clearly \( E \) is \( \{\alpha_n\}\)-porous.

It remains to choose \( N_n \) so that \( \mu(E) > 0 \) for some \( \mu \in \mathcal{D} \). Our idea comes from Ahlfors and Beurling [2; Theorem 3]. First, we construct a function \( h \) which plays the role of \( 1 + \lambda \cos \) in [2].
Lemma 2. Given $0 < \alpha < \frac{1}{2}$, $K > 2$, there exists a function $h$ continuous on $\mathbb{R}$, of period 1, monotonic in $[0, \frac{1}{2}]$ and in $[\frac{1}{2}, 1]$ respectively, which satisfies $\int_0^1 h(x) \, dx = 1$,

$$h(x) = \begin{cases} \alpha^{K-1} & \text{on } \left[\frac{1}{2}(1 - \alpha), \frac{1}{2}(1 + \alpha)\right], \\ 1 + \sqrt{\alpha} & \text{on } \left[0, \frac{1}{2} - \sqrt{\alpha}\right] \cup \left[\frac{1}{2} + \sqrt{\alpha}, 1\right], \end{cases}$$

and $h(x) \, dx$ is in $\mathcal{D}(B^K)$ for some absolute constant $B > 2$.

As an example, we may choose

$$h(x) = \alpha^{K-1} + \alpha^{-K} \left(x - \frac{1 + \alpha}{2}\right)^{2K-1}$$

on

$$\left[\frac{1 + \alpha}{2}, \frac{1 + \alpha}{2} + \frac{\alpha(2K-1)/(4(K-1))}{4}\right],$$

piecewise linear on

$$\left[\frac{1 + \alpha}{2} + \frac{\alpha(2K-1)/(4(K-1))}{4}, \frac{1}{2} + \sqrt{\alpha}\right]$$

with derivatives between $1/4\sqrt{\alpha}$ and $4/\sqrt{\alpha}$, and $h(x) = h(1 - x)$ for $x$ in $[\frac{1}{2} - \sqrt{\alpha}, \frac{1}{2}(1 - \alpha)]$, so that the continuity, monotonicity and $\int_0^1 h(x) \, dx = 1$ are satisfied. For this $h$, $h \, dx \in \mathcal{D}(B^K)$ for some absolute constant $B > 2$.

In the hypothesis $\sum \alpha_n^K < \infty$, we may assume $K > 2$. Corresponding to each pair $(\alpha_n, K)$, we fix a function $h_n$ which satisfies properties in Lemma 2 with $\alpha = \alpha_n$. Denote by

$$A_n = \bigcup_{k=-\infty}^{\infty} ([k, k + \frac{1}{2} - \frac{1}{2}\alpha_n]) \cup ([k + \frac{1}{2} + \frac{1}{2}\alpha_n, k + 1]),$$

$$F_k = \bigcup_{I \in \mathcal{I}_k} I,$$

$$M_n = \prod_{1}^{n} N_k,$$

and

$$f_n(x) = \prod_{1}^{n} h_k(M_kx).$$

We shall choose $N_n$ inductively so that $N_{n+1} \gg N_n$ and that $f_n(x)$ is “nearly constant” on each interval of length $M_{n+1}^{-1}$.

Recall that $N_1 = 1$ and assume that odd integers $N_2, N_3, \ldots, N_n$ have been chosen so that

$$\int_{F_k} f_k(x) \, dx \geq \prod_{j=1}^{k} (1 - 2\alpha_j^K) \quad (1 \leq k \leq n),$$

(1.1)
and whenever $|x - x'| \leq M_{k}^{-1}$, (2 \leq k \leq n),

\begin{equation}
\frac{k - 1}{k} < h_{k-1}(M_{k-1}x)/h_{k-1}(M_{k-1}x') < \frac{k + 1}{k},
\end{equation}

and

\begin{equation}
\frac{k - 1}{k} < f_{k-1}(x)/f_{k-1}(x') < \frac{k + 1}{k}.
\end{equation}

Note that $\int_0^1 h_k(x) \, dx = 1$, $\int_{[0,1] \cap A_k} h_k(x) \, dx = 1 - \alpha_k^K$ and that $f_k(x)$ is uniformly continuous on $\mathbb{R}$ for each $k \geq 1$. Let $F \subseteq [0, 1]$ be any measurable set. Then $\chi_F f_n$ is the pointwise a.e. and $L^1$ limit of an increasing sequence of simple functions, each of which has the form $\sum a_j \chi_{I_j}$, where $\{a_j\}$ are constants and $\{I_j\}$ are finitely many mutually disjoint open intervals with rational end points.

Therefore,

\begin{equation*}
\int_0^1 \chi_F f_n(x) \chi_{A_{n+1}}(Mx) h_{n+1}(Mx) \, dx \to (1 - \alpha_{n+1}^K) \int_F f_n(x) \, dx
\end{equation*}

as $M \to \infty$. Thus a large odd integer $N_{n+1}$ can be found so that $N_{n+1} > \alpha_n^{-1}$, and (1.1), (1.2) and (1.3) hold with $k = n + 1$. Here we have used the fact that $F_{n+1} = \{x \in F_n : M_{n+1}x \in A_{n+1}\}$.

Let $\mu$ be a weak limit point of $f_n(x) \, dx$. Then $\mu(E) > 0$ in view of (1.1) and $\sum \alpha_n^K < \infty$. To verify that $\mu$ is a doubling measure we consider two neighboring intervals $I$ and $I'$ satisfying

\begin{equation*}
M_n^{-1} \leq |I| = |I'| \leq M_n^{-1}.
\end{equation*}

In view of (1.3)

\begin{equation*}
\left(\frac{n - 1}{n}\right)^2 < \frac{f_{n-1}(x)}{f_{n-1}(x')} \leq \left(\frac{n + 1}{n}\right)^2
\end{equation*}

whenever $x \in I$ and $x' \in I'$. We note that $f_m(x)/f_n(x)$ has period $M^{-1}_{n+1}$ if $m \geq n + 1$, and that

\begin{equation*}
\frac{n}{n + 1} < \frac{h_n(M_n x)}{h_n(M_n x')} < \frac{n + 2}{n + 1}
\end{equation*}

whenever $|x - x'| \leq M_n^{-1}$. Writing

\begin{equation*}
f_m(x) = f_{n-1}(x) h_n(M_n x) f_m(x) / f_n(x), \quad \text{for } m \geq n + 1,
\end{equation*}

we deduce from the fact $h_n(M_n x) \, dx \in \mathcal{D}(B^K)$ that

\begin{equation*}
(CB^K)^{-1} \leq \int_I f_m(x) \, dx / \int_{I'} f_m(x) \, dx \leq CB^K
\end{equation*}

for every $m \geq n + 1$. Therefore $\mu \in \mathcal{D}$. 

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2. A test for $\mathcal{D}_d(\lambda)$-null sets

Given a closed set $E$ in $[0,1]$, we shall develop a procedure to test whether $E$ is in $\mathcal{D}_d(\lambda)$ for some $\lambda > 1$.

Let $I_{n,j}$, $1 \leq j \leq 2^n$, be the dyadic closed intervals in $[0,1]$ of length $2^{-n}$, $I_{n,j}^r$ be the closed interval which forms the right half of $I_{n,j}$ and let $I_{n,j}^l = I_{n,j} \setminus I_{n,j}^r$. Let

$$h_{n,j}(x) = \begin{cases} 1 & \text{on } I_{n,j}^r \\ -1 & \text{on } I_{n,j}^l \\ 0 & \text{on } \mathbb{R} \setminus I_{n,j}. \end{cases}$$

For a fixed integer $n \geq 2$, define

$$f_n^{(n)} \equiv \prod_{j=1}^{2^n} \left( 1 + \delta(n,j) \tau h_{n,j} \right),$$

and $d\nu_n^{(n)} \equiv f_n^{(n)} \, dx$, where $\tau = (\lambda - 1)/(\lambda + 1)$ and

$$\delta(n,j) = \begin{cases} 1, & E \cap I_{n,j}^r \neq \emptyset, \\ -1, & E \cap I_{n,j}^l \neq \emptyset. \end{cases}$$

Denote by $E_n = \bigcup \{I_{n+1,j} : I_{n+1,j} \cap E \neq \emptyset, 1 \leq j \leq 2^{n+1}\}$. Let

$$f_{n-1}^{(n)} \equiv \prod_{j=1}^{2^{n-1}} \left( 1 + \delta(n-1,j) \tau h_{n-1,j} \right),$$

and $d\nu_{n-1}^{(n)} \equiv f_{n-1}^{(n)} \, d\nu_n^{(n)}$, where

$$\delta(n-1,j) = \begin{cases} 1, & \nu_n^{(n)}(I_{n-1,j}^r \cap E_n) \geq \nu_n^{(n)}(I_{n-1,j}^l \cap E_n), \\ -1, & \text{otherwise}. \end{cases}$$

After defining $f_k^{(n)}$ and $d\nu_k^{(n)}$, we let

$$f_{k-1}^{(n)} \equiv \prod_{j=1}^{2^{k-1}} \left( 1 + \delta(k-1,j) \tau h_{k-1,j} \right),$$

and $d\nu_{k-1}^{(n)} \equiv f_{k-1}^{(n)} \, d\nu_k^{(n)}$, where

$$\delta(k-1,j) = \begin{cases} 1, & \nu_k^{(n)}(I_{k-1,j}^r \cap E_n) \geq \nu_k^{(n)}(I_{k-1,j}^l \cap E_n), \\ -1, & \text{otherwise}. \end{cases}$$
Continue this process until we arrive at $\nu_1^{(n)}$. Define $\mu_n \equiv \nu_1^{(n)}$.

Notice that

\[(2.1) \quad \nu_k^{(n)}(I_{k,j}) = 2^{-k} \quad (1 \leq k \leq n)\]

with the understanding that $I_{0,j} \equiv I_{0,1} \equiv [0,1]$; and that from $d\nu_{k}^{(n)}$ to $d\nu_{k-1}^{(n)}$, total measure in each $I_{k-1,i}$ is kept unchanged, but the total measures of $I_{k-1,i}$ and $I_{k-1,i}$ are redistributed in the most advantageous way.

Repeat for each $n \geq 2$, to obtain a sequence of measures $\{\mu_n\}$. Let $\mu_{E,\lambda}$ be a weak limit point of $\{\mu_n\}$, extended to $\mathbb{R}$ with period 1.

**Theorem 3.** Among all the measures in $\mathcal{D}(\lambda)$ which have mass one on $[0,1]$, $\mu_{E,\lambda}$ has the maximum measure on $E$. In particular, $E$ is $\mathcal{D}(\lambda)$-null if and only if $\mu_{E,\lambda}(E) = 0$.

**Proof.** It is clear that $\mu_{E,\lambda}([0,1]) = 1$ and $\mu_{E,\lambda} \in \mathcal{D}(\lambda)$.

Let $\omega$ be any measure in $\mathcal{D}(\lambda)$ with $\omega([0,1]) = 1$. We claim, in fact, that

\[(2.2) \quad \omega(E_n) \leq \mu_n(E_n).\]

Let $m$ be the largest integer in $[1,n]$, if it exists, such that there exists at least one interval $I_{m,j}$ on which

\[(2.3) \quad \frac{\omega(I_{m,j}^r)}{\omega(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)}.\]

(If such $m$ does not exist, then $\omega(E_n) = \mu_n(E_n)$.) We shall redistribute the measure $\omega$ on these $I_{m,j}$’s and keep $\omega$ unchanged elsewhere. Denote by $\mathcal{I}_m = \{I_{m,j} : (2.3) \text{ holds on } I_{m,j}\}$; and let $\omega_m = \omega$ on $[0,1] \setminus \bigcup_{\mathcal{I}_m} I_{m,j}$, and

\[(2.4) \quad d\omega_m = \begin{cases} 
\frac{\omega(I_{m,j}) \mu_n(I_{m,j}^r)}{\omega(I_{m,j}^r) \mu_n(I_{m,j})} d\omega & \text{on } I_{m,j}^r, \\
\frac{\omega(I_{m,j}) \mu_n(I_{m,j}^l)}{\omega(I_{m,j}^l) \mu_n(I_{m,j})} d\omega & \text{on } I_{m,j}^l,
\end{cases}\]

for each $I_{m,j} \in \mathcal{I}_m$. Clearly, if $I_{m,j} \in \mathcal{I}_m$, then

\[
\frac{\omega_m(I_{m,j}^r)}{\omega_m(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},
\]

and $\omega_m$ is dyadic on $I_{m,j}$ with constant $\leq \lambda$. Actually, $\omega_m$ is in $\mathcal{D}(\lambda)$ by the following lemma.
Lemma 3. Let $I$ be a dyadic subinterval of $[0, 1]$, and $\mu$ and $\nu$ be dyadic doubling measures on $[0, 1]$ and $I$ respectively, satisfying $\mu(I) = \nu(I)$. Then the new measure $\omega$ defined by $\omega \equiv \nu$ on $I$, $\equiv \mu$ on $[0, 1] \backslash I$ is dyadic doubling on $[0, 1]$ with constant bounded by the maximum of those of $\mu$ and $\nu$.

First, we shall verify that $\omega(E_n) \leq \omega_m(E_n)$. To show this, it is enough to prove
\[
\omega(E_n \cap I_{m,j}) \leq \omega_m(E_n \cap I_{m,j})
\]
for each $I_{m,j} \in \mathcal{I}_m$.

Fix $I_{m,j} \in \mathcal{I}_m$, clearly (2.5) holds when $m = n$. Thus we assume $m < n$ and note from the definition of $m$ that
\[
\frac{\omega(I_{m,j}^r) \mu_n(I_{m,j}^r)}{\omega(I_{m,j}^r) \mu_n(I_{m,j}^r)}, \quad \text{and} \quad \frac{\omega(I_{m,j}^l) \mu_n(I_{m,j}^l)}{\omega(I_{m,j}^l) \mu_n(I_{m,j}^l)}
\]
for $m + 1 \leq k \leq n$. Therefore
\[
\frac{\omega(I_{m,j} \cap E_n)}{\omega(I_{m,j}^r)} = \frac{\mu_n(I_{m,j} \cap E_n)}{\mu_n(I_{m,j}^r)}
\]
and
\[
\frac{\omega(I_{m,j} \cap E_n)}{\omega(I_{m,j}^l)} = \frac{\mu_n(I_{m,j} \cap E_n)}{\mu_n(I_{m,j}^l)}.
\]
Moreover, from the construction of $\mu_n$,
\[
\frac{\mu_n(I_{n+1,l})}{\mu_n(I_{k,i})} = \frac{\nu_k^{(n)}(I_{n+1,l})}{\nu_k^{(n)}(I_{k,i})},
\]
if $1 \leq k \leq n$ and $I_{n+1,l} \subseteq I_{k,i}$. Thus by (2.1) and the above identities,
\[
\frac{\omega(I_{m,j} \cap E_n)}{\omega(I_{m,j}^r)} = \nu_{m+1}^{(n)}(I_{m,j} \cap E_n)2^{m+1}
\]
and
\[
\frac{\omega(I_{m,j} \cap E_n)}{\omega(I_{m,j}^l)} = \nu_{m+1}^{(n)}(I_{m,j} \cap E_n)2^{m+1}.
\]
Writing $I$ in place of $I_{m,j}$ for the rest of this paragraph, we obtain
\[
\omega(E_n \cap I) = \frac{\omega(E_n \cap I^r) \omega(I^r)}{\omega(I^r) \omega(I)} + \frac{\omega(E_n \cap I^l) \omega(I^l)}{\omega(I^l) \omega(I)} \omega(I)
\]
\[
= 2^{m+1}\omega(I) \left[ \nu_{m+1}^{(n)}(E_n \cap I^r) \frac{\omega(I^r)}{\omega(I)} + \nu_{m+1}^{(n)}(E_n \cap I^l) \frac{\omega(I^l)}{\omega(I)} \right]
\]
\[
\leq 2^{m+1}\omega(I) \left[ \nu_{m+1}^{(n)}(E_n \cap I^r) A + \nu_{m+1}^{(n)}(E_n \cap I^l)(1 - A) \right]
\]
where \( A = \frac{1}{2}(1+\tau) \) if \( \nu^{(n)}_{m+1}(E_n \cap I^r) \geq \nu^{(n)}_{m+1}(E_n \cap I^l) \), and \( A = \frac{1}{2}(1-\tau) \) otherwise. From the definition of \( \nu^{(n)}_m \), (2.6), (2.7), and (2.8), it follows that

\[
\omega(E_n \cap I) \leq 2^{m+1} \omega(I) \left[ \nu^{(n)}_{m+1}(E_n \cap I^r) \frac{\nu^{(n)}_m(I^r)}{\nu^{(n)}_m(I)} + \nu^{(n)}_{m+1}(E_n \cap I^l) \frac{\nu^{(n)}_m(I^l)}{\nu^{(n)}_m(I)} \right] = \omega(I) \left[ \frac{\omega(E_n \cap I^r) \mu_n(I^r)}{\mu_n(I)} + \frac{\omega(E_n \cap I^l) \mu_n(I^l)}{\mu_n(I)} \right] = \omega_m(E_n \cap I).
\]

This proves (2.5) and hence \( \omega(E_n) \leq \omega_m(E_n) \).

We proceed to make modifications of \( \omega_m \) on each dyadic interval \( I_{m-1,i} \) of size \( 2^{-m+1} \) on which (2.3) holds with \( m, j, \omega \) replaced by \( m-1, i \) and \( \omega_m \) respectively, according to the rule (2.4) adapted for \( m-1, i \) and \( \omega_m \); call this new measure \( \omega_{m-1} \). Continue to modify \( \omega_{m-1} \) on dyadic intervals of size \( 2^{-m+2} \) if necessary to obtain \( \omega_{m-2}, \ldots \). Finally we arrive at a measure \( \omega_1 \), and obtain

\[
\omega(E_n) \leq \omega_m(E_n) \leq \omega_{m-1}(E_n) \leq \cdots \leq \omega_1(E_n)
\]

and

\[
\frac{\omega_1(I_{m,j}^r)}{\omega_1(I_{m,j}^l)} = \frac{\mu_n(I_{m,j}^r)}{\mu_n(I_{m,j}^l)},
\]

for all \( 1 \leq m \leq n, 1 \leq j \leq 2^m \). Therefore \( \omega_1(E_n) = \mu_n(E_n) \) and (2.2) is proved.

We note that \( E = \cap_m E_m \). Therefore for any \( \varepsilon > 0 \) and sufficiently large \( m \) and \( n \) with \( m > m(\varepsilon) \) and \( n > n(\varepsilon, m) \), we have

\[
\mu_{E,\lambda}(E) \geq \mu_{E,\lambda}(E_m) - \varepsilon \geq \mu_n(E_m) - 2\varepsilon \geq \mu_n(E) - 2\varepsilon \\
\geq \omega(E_n) - 2\varepsilon \geq \omega(E) - 2\varepsilon.
\]

This shows that \( \omega(E) \leq \mu_{E,\lambda}(E) \).

### 3. Proofs of Theorems 4 and 5

Given \( a, \varepsilon, \delta \in (0, 1), \varepsilon a < \delta < \varepsilon \), we choose a sequence of integers \( \{n_k\} \) satisfying \( n_k \geq 4 \) and

\[
n_{n+1} > n_k + [\varepsilon \log_2 k].
\]

For \( k \geq 2 + [2^{1/\delta}] \) and \( 0 \leq j \leq 2^{n_k} - 1 \), denote by

\[
L_{k,j} = \left[ \frac{j}{2^{n_k}}, \frac{j + 1}{2^{n_k}} \right],
\]

\[
I_{k,j} = \left[ \frac{j}{2^{n_k}}, \frac{j}{2^{n_k}} + \frac{1}{2^{n_k} k^{\delta}} \right],
\]
and

\[ J_{k,j} = \left[ \frac{j + 1}{2^{n_k}} - \frac{1}{2^{n_k} k^\varepsilon}, \frac{j + 1}{2^{n_k}} \right], \]

where \( k^\delta = 2^{[\delta \log_2 k]}, k^\varepsilon = 2^{[\varepsilon \log_2 k]} \) and \([ \cdot ]\) is the greatest integer function. Note that intervals \( I \)'s, \( I \)'s and \( J \)'s are dyadic,

(3.2)

\[ |J_{k,j}|/|I_{k,j'}| = O(k^{\delta-\varepsilon}) = o(1) \quad \text{as } k \to \infty, \]

and

(3.3)

\[ |J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1} - n_k - [\varepsilon \log_2 k]} > 2. \]

The construction of a set \( S \in \mathcal{N} \setminus \mathcal{N}_d \) is similar to the Cantor set; collections of nested intervals from \( \{J_{k,j}\} \) are used. The measure \( \mu \) in \( \mathcal{D}_d \) to be produced with \( \mu(S) > 0 \) will satisfy

\[ \frac{\mu(J_{k,j})}{\mu(I_{k,j})} = \left( \frac{|J_{k,j}|}{|L_{k,j}|} \right)^a \]

on infinitely many \( J_{k,j} \)'s.

Let \( \{K_i\} \) be an increasing sequence of integers with \( K_0 \equiv 2 + \lfloor 2^1/\delta \rfloor \) and some other properties to be specified later. Let \( S_0 = [0, 1] \), \( \mathcal{C}^I_{1+K_0} \) be the collection of all \( I_{1+K_0,j} \subseteq S_0 \) and \( \mathcal{C}^J_{1+K_0} \) be the collection of all \( J_{1+K_0,j} \subseteq S_0 \). After \( \mathcal{C}^I_k \) and \( \mathcal{C}^J_k \) have been defined for some \( k, 1 + K_0 \leq k \leq K_1 - 1 \), we let

\[ \mathcal{C}^I_{k+1} = \mathcal{C}^I_k \cup \{I_{k+1,j} \subseteq S_0 : I_{k+1,j} \text{ is not contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k \}, \]

\[ \mathcal{C}^J_{k+1} = \mathcal{C}^J_k \cup \{J_{k+1,j} \subseteq S_0 : J_{k+1,j} \text{ is not contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k \}; \]

and let

\[ S^I_1 = \text{union of all intervals in } \mathcal{C}^I_{K_1}, \]

\[ S_1 = \text{union of all intervals in } \mathcal{C}^J_{K_1}. \]

Next let \( \mathcal{C}^I_{1+K_1} \) be the collection of all \( I_{1+K_1,j} \subseteq S_1 \) and \( \mathcal{C}^I_{1+K_1} \) be the collection of all \( J_{1+K_1,j} \subseteq S_1 \). And define for each \( k, 1 + K_1 \leq k \leq K_2 - 1 \),

\[ \mathcal{C}^I_{K+1} = \mathcal{C}^I_k \cup \{I_{K+1,j} \subseteq S_1 : I_{K+1,j} \text{ is not contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k \}, \]

\[ \mathcal{C}^J_{K+1} = \mathcal{C}^J_k \cup \{J_{K+1,j} \subseteq S_1 : J_{K+1,j} \text{ is not contained in any interval in } \mathcal{C}^I_k \cup \mathcal{C}^J_k \}, \]

\[ S^I_2 = \text{union of all intervals in } \mathcal{C}^I_{K_2}, \]

\[ S_2 = \text{union of all intervals in } \mathcal{C}^J_{K_2}. \]

Clearly \( S^I_2 \subseteq S_1 \) and \( S_2 \subseteq S_1 \).

Continue this procedure to obtain \( \mathcal{C}^I_{K_3}, \mathcal{C}^J_{K_3}, S^I_3 \) and \( S_3, \ldots, \) and so on, and let

\[ S = \bigcap_{1}^{\infty} S_m. \]

To construct \( \mu \in \mathcal{D}_d \) with \( \mu(S) > 0 \), we shall use scale invariant versions of Lemma 3 and the following lemma repeatedly.
Lemma 4. Given $a, \alpha, \beta \in (0, 1)$ with $\alpha^a + \beta < 1/16$ and $c_1, c_2 \in (\frac{1}{2}, 2)$, there exists a measure $\mu \in \mathcal{D}_d(10^{1/a})$, which satisfies $\mu([0,1]) = 1$, $\mu([0, \alpha]) = c_1\alpha^a$, and $\mu([1 - \beta, 1]) = c_2\beta$.

As an example, we may choose

$$
\mu([0, t]) = \begin{cases} 
    c_1 t^a, & 0 \leq t \leq t_0 \equiv (\frac{1}{8})^{1/a}, \\
    \frac{1}{8}c_1 + (1 - \frac{1}{8}(c_1 + c_2))(t - t_0)/(\frac{7}{8} - t_0), & t_0 \leq t \leq \frac{7}{8}, \\
    c_2 t + 1 - c_2, & \frac{7}{8} \leq t \leq 1.
\end{cases}
$$

Then extend $\mu$ periodically to $\mathbb{R}$ with period 1.

All measures $\mu_k$ defined below are periodic with period 1. Choose $\mu_{1+K_0} \in \mathcal{D}_d(10^{1/a})$ so that

$$
\mu_{1+K_0}(L_{1+K_0,j}) = |L_{1+K_0,j}|, \\
\mu_{1+K_0}(I_{1+K_0,j}) = |L_{1+K_0,j}|(1 + K_0)^{-\delta}, \\
\mu_{1+K_0}(J_{1+K_0,j}) = |L_{1+K_0,j}|(1 + K_0)^{-\varepsilon a}
$$

for each $0 \leq j \leq 2^{1+K_0} - 1$. After $\mu_k$ is selected for some $k$, $1 + K_0 \leq k \leq K_1$, we choose $\mu_{k+1} \in \mathcal{D}_d(10^{1/a})$, so that $\mu_{k+1} = \mu_k$ on each interval in $\mathcal{C}_k^I \cup \mathcal{C}_k^J$, and $\mu_{k+1}$ is a redistribution of $\mu_k$ on each $L_{k+1,j}$ which is not contained in any interval in $\mathcal{C}_k^I \cup \mathcal{C}_k^J$.

(3.4) \hspace{1cm} \mu_{k+1}(L_{k+1,j}) = \mu_k(L_{k+1,j}),

(3.5) \hspace{1cm} \mu_{k+1}(I_{k+1,j}) = (1 + k)^{-\delta}\mu_{k+1}(L_{k+1,j}),

(3.6) \hspace{1cm} \mu_{k+1}(J_{k+1,j}) = (1 + k)^{-\varepsilon a}\mu_{k+1}(L_{k+1,j}).

The measure $\mu_{K_1}$ so chosen has the properties that

$$
\mu_{K_1}(S_1^I \cup S_1) = 1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a} - k^{-\delta})
$$

and

$$
\mu_{K_1}(S_1) \geq \mu_{K_1}(S_1^I \cup S_1) \inf_{1+K_0 \leq k \leq K_1} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}}
\geq \left(1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a})\right)(1 - K_0^{-\varepsilon a - \delta}),
$$

because $\varepsilon a < \delta$. 

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Next choose \( \mu_{1+K_1} \in \mathcal{D}_d(10^{1/a}) \) so that \( \mu_{1+K_1} = \mu_{K_1} \) on \( S_0 \setminus S_1 \), and on each \( L_{1+K_1,j} \subseteq S_1 \) it is a redistribution of \( \mu_{K_1} \) satisfying (3.4), (3.5) and (3.6) with \( k = K_1 \). After \( \mu_k \) is constructed for some \( k, 1 + K_1 \leq k < K_2 \), build \( \mu_{k+1} \) from \( \mu_k \) following the same steps as in the case \( 1 + K_0 \leq k \leq K_1 \). The dyadic doubling measure \( \mu_{K_2} \) so obtained belongs to \( \mathcal{D}_d(10^{1/a}) \), moreover

\[
\mu_{K_2}(S_1^i \cap S_2) = \left( 1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a} - k^{-\delta}) \right) \mu_{K_1}(S_1),
\]

and

\[
\mu_{K_2}(S_2) \geq \mu_{K_2}(S_1^i \cup S_2) \inf_{1+K_1 \leq k \leq K_2} \frac{k^{-\varepsilon a}}{k^{-\varepsilon a} + k^{-\delta}} \\
\geq \mu_{K_2}(S_1^i \cup S_2)(1 - K_1^{-\varepsilon a} - \delta) \\
\geq \left( 1 - \prod_{1+K_0}^{K_1} (1 - k^{-\varepsilon a}) \right) \left( 1 - \prod_{1+K_1}^{K_2} (1 - k^{-\varepsilon a}) \right) (1 - K_0^{-\varepsilon a} - \delta)(1 - K_1^{-\varepsilon a} - \delta).
\]

Whenever \( \mu_{K_m} \) is constructed, keep \( \mu_{1+K_m} = \mu_{K_m} \) on \( S_0 \setminus S_m \), redistribute the mass on each \( L_{1+K_m,j} = S_m \) according to (3.4), (3.5) and (3.6) with \( k = K_m \), and keep the dyadic doubling constant bounded by \( 10^{1/a} \). Continue this indefinitely. Thus, we obtain a sequence of measures \( \mu_{K_m} \in \mathcal{D}_d(10^{1/a}) \), with \( \mu_{K_m}([0,1]) = 1 \) and

\[
\mu_{K_m}(S_m) \geq \prod_{i=0}^{m-1} \left( (1 - K_i^{-\varepsilon a} - \delta)(1 - A_i) \right)
\]

where \( A_i = \prod_{1+K_1}^{K_{i+1}} (1 - k^{-\varepsilon a}) \). Let \( \mu \) be a weak limit point of \( \{ \mu_{K_m} \} \). Clearly \( \mu \in \mathcal{D}_d(10^{1/a}) \).

Since \( \varepsilon a < 1 \), it is possible to choose \( \{K_i\} \) so that

\[
(3.7) \quad \sum_{i=1}^{\infty} K_i^{-\varepsilon a - \delta} + \sum_{i=1}^{\infty} A_i < +\infty.
\]

With respect to this choice of \( \{K_i\} \), we have \( \mu(S) > 0 \), hence \( \mu \notin \mathcal{N}_d \).

It remains to show that \( S \in \mathcal{N} \). Let \( \nu \in \mathcal{D} \). Recall that \( J_{k,j} \) and \( I_{k,j+1} \) have the common boundary point \( (j + 1)/(2^n) \); by the doubling property

\[
\nu(J_{k,j} \cup I_{k,j+1}) \geq A^{(\varepsilon - \delta) \log_2 k - 5} \nu(J_{k,j})
\]

for some \( A > 1 \) depending only on the doubling constant of \( \nu \). For \( m \geq 2 \), intervals in \( \mathcal{C}_{K_m}^J \cup \{I_{k,j+1} : J_{k,j} \in \mathcal{C}_{K_m}^J \} \) (\( \neq \mathcal{C}_{K_m}^J \cup \mathcal{C}_{K_m}^J \)) may meet in their
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interiors; however, because of (3.3), every point in \([0, 1]\) is covered by at most three such intervals. Therefore

\[
3\nu([0, 1]) \geq \sum_{J_{k,j} \in \mathcal{E}_{K_m}^J} \nu(J_{k,j} \cup I_{k,j+1}) \geq A^{(e-\delta)\log_2 K_{m-1}-5\nu(S_m)} \nu(S_m)
\]

\[
\geq A^{(e-\delta)\log_2 K_{m-1}-5\nu(S)}.
\]

Hence \(\nu(S) = 0\). Therefore \(S \in \mathcal{N} \setminus \mathcal{N}_d\).

Let

\[
T = \left\{ t = \sum_{n=1}^{\infty} t_n 2^{-n}, \text{ where } t_n = 0 \text{ or } 1, \right. \]

\[
\text{but } t_{n_k+\lfloor \delta \log_2 k \rfloor + 1} = 1 \text{ and } t_{n_k+\lfloor \delta \log_2 k \rfloor + 2} = 0 \]

\[
\text{for each integer } k > K_0 \}.
\]

In view of (3.1), it has Hausdorff dimension 1.

Fix \(t \in T\) and \(\nu \in \mathcal{D}_d\) and let \(J_{k,j}\) be any interval in \(\mathcal{E}_{K_m}^J\). We note that

\[
\frac{p + \frac{1}{2}}{2^{n_k} k^\delta} < t + \frac{j + 1}{2^{n_k}} < \frac{p + \frac{3}{2}}{2^{n_k} k^\delta}
\]

for some integer \(p\), because

\[
q + \frac{1}{2} < t 2^{n_k} k^\delta < q + \frac{3}{4}
\]

for some integer \(q\).

Therefore \(t + J_{k,j}\) is contained in the middle half of some dyadic interval

\[
M_{k,j} = \left[ \frac{p}{2^{n_k} k^\delta}, \frac{p + 1}{2^{n_k} k^\delta} \right].
\]

Recall that the interval \(I_{k,j+1}\) shares an end point \((j + 1)/2^{n_k}\) with \(J_{k,j}\) and has length \(1/2^{n_k} k^\delta\). Therefore

\[
\left| (t + I_{k,j+1}) \cap M_{j,k} \right| > \frac{1}{4} \frac{1}{2^{n_k} k^\delta}.
\]

The dyadic doubling property of \(\nu\), (3.2) and Lemma 1,

\[
\nu(t + (J_{k,j} \cup I_{k,j+1}) \cap M_{j,k}) \geq c(k, \nu) \nu(t + J_{k,j})
\]

with \(c(k, \nu) \to \infty\) as \(k \to \infty\). Summing over all \(J_{k,j}\) in \(\mathcal{E}_{K_m}^J\) and reasoning as before, we obtain

\[
3\nu([0, 1]) \geq c(K_{m-1}, \nu) \nu(t + S_m) \geq c(K_{m-1}, \nu) \nu(t + S).
\]

Letting \(m \to \infty\), we have \(\nu(t + S) = 0\). This completes the proof of Theorem 4.

It would be interesting to characterize those \(t\)’s so that \(t + S\) is in \(\mathcal{N}_d\). However this seems difficult.
To prove Theorem 5, we note that in the binary expansion of \( t \), the event that a digit 1 is followed immediately by a digit 0 occurs infinitely often. Choose \( \varepsilon, \delta \) and \( a \) as in Theorem 4, and \( \{n_k\} \) depending on \( t \), so that (3.1),

\[
t_{n_k + [\delta \log_2 k] + 1} = 1 \quad \text{and} \quad t_{n_k + [\delta \log_2 k] + 2} = 0
\]

hold for each \( k > k_0 \). Let \( S_t \equiv S \) in Theorem 4 associated with this sequence \( \{n_k\} \). Then \( S_t \in \mathcal{N} \setminus \mathcal{A}_d \). The proof of \( t + S_t \in \mathcal{A}_d \) is similar to that in Theorem 4.

4. Null sets for \( p \)-harmonic measures

Consider the \( p \)-Laplace equation \((1 < p < \infty)\)

\[
\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0
\]

in the half plane \( \Omega \equiv \{x \in \mathbb{R}^2 : x_2 > 0\} \). For the definition and properties of \( p \)-harmonic measure (the harmonic measure for \( p \)-Laplacian) see [6; Chapter 10].

Let \( E \) be a compact set on \( \partial \Omega \) which has positive \( p \)-harmonic measure for some \( p \). Then there exists a nonconstant solution \( u \) \((0 \leq u \leq 1)\) of the \( p \)-Laplacian in \( \Omega \), with continuous boundary value 0 on \( \partial \Omega \setminus E \).

Following [1], we may apply a linearization technique in [7] or an approximation technique in [4], and Theorem 4.5 in [3], to write

\[
u(x) = \int_{\partial \Omega} K(x, y) f(y) \, d\omega(y),
\]

where \( K \) is a limit of kernel functions and \( \omega \) is a weak limit of harmonic measures at a fixed point, corresponding to a sequence of uniformly elliptic operators of nondivergence form in \( \Omega \), with ellipticity constants depending only on \( p \). Moreover \( \omega \) has the doubling property and \( u \) has nontangential limit on \( \partial \Omega \) \( \omega \)-a.e.

Because \( u \) has zero boundary value on \( \partial \Omega \setminus E \), \( f(y) \, d\omega(y) \) is supported in \( E \). This implies that \( \omega(E) > 0 \). Therefore, we have

**Theorem 6.** Compact sets in \( \mathcal{N} \) are null sets for any \( p \)-harmonic measure with respect to the half plane \( \{x \in \mathbb{R}^2 : x_2 > 0\} \).

**Remark.** Martio has defined a version of porosity and proved that a porous set on \( \{x_2 = 0\} \) has zero \( \mathcal{A} \)-harmonic measure with respect to all those nonlinear operators \( \mathcal{A} \) on \( \{x_2 > 0\} \) considered in [8]; \( p \)-Laplacians are examples of such operators. We do not know whether Theorem 6 can be extended to all such \( \mathcal{A} \)-operators. However a compact set \( E \) on \( \{x_2 = 0\} \) is \( \mathcal{A} \)-harmonic measure null for all such \( \mathcal{A} \) if it satisfies a stronger \( \{a_n\} \)-porous condition for some \( \{a_n\} \)

\[
\sum a^K_n = \infty \quad \text{for all} \quad K > 1,
\]

namely, in defining \( \{a_n\} \)-porosity, \( E \cap E_{n,j} \) is required to lie in the middle \( 1 - 2a_n \) portion of \( E_{n,j} \) for each \( n \) and \( j \). Proof follows by combining the original proof of Martio and that of Theorem 1.
References


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