ON THE MOVEMENT OF THE POINCARÉ METRIC
WITH THE PSEUDOCONVEX DEFORMATION
OF OPEN RIEMANN SURFACES

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Abstract. The movement of the Poincaré metrics of open Riemann surfaces belonging to
an analytic family defined on a 2-dimensional complex manifold Ω is logarithmically plurisubhar-
monic in Ω if Ω is Stein. As a corollary, we get a theorem due to Nishino.

0. Introduction

It has been known since Riemann that differentiable surfaces having a common
constant Gauss curvature $k$ are locally isometric to each other. Hence, thanks to
Gauss–Bonnet’s theorem, we know that a Riemann surface $R$ of non-exceptional
type has the unique complete hermitian metric $ds_R^2$ with constant Gauss curvature
$k = -4$, which we call the Poincaré metric of $R$. Let $\tilde{R}$ denote the universal
covering surface of $R$ with the canonical projection $\pi: \tilde{R} \rightarrow R$. The induced
metric $\pi^*ds_R^2$ is the erstwhile Poincaré metric of $\tilde{R}$, which is biholomorphically
equivalent to the unit disc $D$.

Let $\Omega$ be a two-dimensional Stein manifold and let $f$ be a holomorphic function
defined on $\Omega$ such that $df \neq 0$ at each point of $\Omega$. We treat the foliation
defined by prime surfaces (irreducible components of level surfaces) of $f$ in this
paper. Let $S_c$ be a prime surface of $f$ with value $c$ and suppose that $S_c$ is not of
exceptional type. We denote the Poincaré metric of $S_c$ by $ds_c^2$. In the case where
$S_c$ is of exceptional type, set $ds_c^2 \equiv 0$ on $S_c$. We also call $ds_c^2$ the Poincaré metric
of $S_c$ in the latter case. We prove that the movement of $ds_c^2$ is logarithmically
plurisubharmonic in $\Omega$ in the following sense: Each point of $\Omega$ has a neighborhood
$U$ and a holomorphic function $g$ in $U$ such that $z = g(p)$, $w = f(p)$ ($p \in U$)
defines a biholomorphic mapping of $U$ onto a domain of $\mathbb{C}^2$. Suppose that the
Poincaré metrics $ds_w^2$ of prime surfaces $S_w$ satisfying $S_w \cap U \neq \emptyset$ have the expression
$ds_w = A(z,w)|d(z|S_w)|$ on $S_w \cap U$ with respect to the local holomorphic
coordinate system $(z,w)$. Then $\log A(z,w)$ must be a plurisubharmonic function
in $U$. This assertion is independent of the choice of the function $g$.

This fact was first noted by H. Yamaguchi [4, Corollary 3] in 1981 for the special
case that each level surface of $f$ is biholomorphically equivalent to the

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unit disc and that the boundaries of $\Omega$ and $S_c$ are smooth, where he has used Hadamard’s variational method. We prove this result generally and directly using a purely function-theoretic idea.

1. Robin constant and Poincaré metric of the unit disc

Let $ds^2$ denote the Poincaré metric of the unit disc $D$. Let $z$ be a local holomorphic coordinate system around a point $p$ of $D$ such that $z(p) = 0$. Assume that $ds^2$ has the expression

$$ds = A(z)|dz|$$

with respect to the local coordinate system $z$. Since $k = -(\Delta z \log A)/A^2 = -4$, $\log A$ is a subharmonic function on the variable $z$.

Let $\zeta$ be a standard holomorphic coordinate system of $D$ such that $\zeta(p) = 0$. Then $A(z) = |d\zeta/dz|/(1 - |z|^2)$, and so $A(0) = |d\zeta/dz|_{z=0}$. Let $g_p$ denote the Green function of $D$ with pole at $p$. The Robin constant $\lambda_z$ for $(D, p)$ with respect to the local coordinate system $z$ is the real number $\lim_{z \to 0} g_p(z) + \log |z|$.

Since $g_p = -\log |\zeta|$, we get $\lambda_z = \log(|dz/d\zeta|_{\zeta=0})$. Therefore $\lambda_z = -\log A(0)$. The following result can now be easily proved.

**Lemma 1.1.** Let $D_j (j = 1, 2, \ldots)$ be a sequence of simply connected subdomains of the unit disc $D$ such that $D_j \subset D_{j+1}$ and $D = \cup D_j$. Then the sequence of the Poincaré metrics $ds_j^2$ of $D_j$ converges monotonously to the Poincaré metric $ds^2$ of $D$.

2. The movement of $ds_{c, \alpha}^2$

Let $\Omega$ be a two-dimensional Stein manifold. Suppose that there exists a holomorphic function $f$ on $\Omega$ such that $df \neq 0$ at each point of $\Omega$. Fix a smooth strictly plurisubharmonic function $\varrho$ in $\Omega$ such that $\Omega^\alpha = \{p \in \Omega \mid \varrho(p) < \alpha\}$ is relatively compact in $\Omega$ for each real number $\alpha$.

For a point $p_0$ of $\Omega$, fix a holomorphic function $g$ in a relatively compact neighborhood $U$ of $p_0$ such that $z = g(p)$, $w = f(p)$ ($p \in U$) defines a biholomorphic mapping $G$ of $U$ onto a bidisc $B = \{(z, w) \in \mathbb{C} \mid |z| < 1, |w - f(p_0)| < \varepsilon\}$ for some positive constant $\varepsilon$. Fix a real number $\alpha$ such that $U \subset \subset \Omega^\alpha$. Set $O = G^{-1}(\{(z, w) \in B \mid z = 0\})$. Let $c$ be a complex number satisfying $|c - f(p_0)| < \varepsilon$. Let $S_c^\alpha$ denote the prime surface of $f \mid \Omega^\alpha$ with value $c$ which passes $O$, where $f\mid\Omega^\alpha$ is the restriction of $f$ to $\Omega^\alpha$, and $ds_{c, \alpha}^2$ the Poincaré metric of $S_c^\alpha$. In this section, we prove that the movement of $ds_{c, \alpha}^2$ is logarithmically plurisubharmonic in $U$.

Set $O_c = O \cap S_c^\alpha$. Because of the subharmonicity of the restriction $g \mid S_c$ of $g$ to $S_c$, we get the following lemma due to T. Nishino [2].
Lemma 2.2. Let $S_c$ denote the prime surface of $f$ with value $c$ which contains $S_c^\alpha$. Let $\gamma$ be a closed continuous curve on $S_c^\alpha$ beginning and ending at $O_c$. If $\gamma$ is not null-homotopic on $S_c^\alpha$ with base point $O_c$, then $\gamma$ is not null-homotopic on $S_c$ with base point $O_c$.

Set $a = f(p_0)$. Let $\tilde{S}$ be a domain in the prime surface $S_a$ such that $S_a^\alpha \subset \subset \tilde{S} \subset \subset S_a$. We also get the following

Lemma 2.3. There exists a tubular neighborhood $V$ of $\tilde{S}$ in $\Omega$ and a holomorphic mapping $\varphi$ of $V$ onto $\tilde{S}$ such that the mapping $\Phi: p \mapsto (\varphi(p), f(p))$ $(p \in V)$ maps $V$ onto the direct product $\tilde{S} \times \Gamma$ biholomorphically where $\Gamma = \{c \in \mathbb{C} | \|c - a\| < \delta\}$ for some positive number $\delta$ and such that $S_c^\alpha \subset \subset (S_c \cap V) \subset \subset S_c$ for each $c \in \Gamma$.

Proof. We prove this lemma using Nishino’s trick. Each point of $\Omega$ has a holomorphically convex neighborhood $W$ with a holomorphic vector field $X_W$ such that $(X_W)_p f = 1$ for each point $p$ in $W$. Since $\Omega$ is Stein, we can construct a global holomorphic vector field $X$ on $\Omega$ which satisfies $X_p f = 1$ for each point $p$ in $\Omega$. The system of local solutions of the partial differential equation $X_p g = 0$ defines a transversal holomorphic foliation on $\Omega$ with the holomorphic foliation defined by the prime surfaces of $f$. It proves the lemma.

Let $\tilde{V}$ denote the universal covering of the tubular neighborhood $V$ of $\tilde{S}$ in Lemma 2.3 whose canonical projection we denote by $\varpi: \tilde{V} \rightarrow V$. The analytic surface $\varpi^{-1}(S_c \cap V)$ is the universal covering surface of $S_c \cap V$ and the manifold $\tilde{V}$ is biholomorphically equivalent to the direct product $\mathbb{D} \times \Gamma$. So we identify $\tilde{V}$ with $\mathbb{D} \times \Gamma$ hereafter. Fix a connected component $U^*$ of $\varpi^{-1}(U \cap V)$. Set $\mathcal{D}^\alpha = \bigcup \{ \tilde{S}_c^\alpha \}$. Then $\mathcal{D}^\alpha$ is a subdomain of $V \cap \Omega^\alpha$ (cf. Nishino [1]). Let $\tilde{S}_c^\alpha$ denote a connected component of $\varpi^{-1}(S_c^\alpha)$ which passes $U^*$. Because of Lemma 2.2, each $\tilde{S}_c^\alpha$ is a simply connected subdomain of $\mathbb{D} \times \{c\}$. Hence $\tilde{S}_c^\alpha$ is the universal covering surface of $S_c^\alpha$ with the projection $\varpi | \tilde{S}_c^\alpha$. Set $\tilde{\mathcal{D}} = \bigcup \{ \tilde{S}_c^\alpha \}$, which is a subdomain of $\tilde{V}$. The manifold $\tilde{\mathcal{D}}$ is an unramified covering of $\mathcal{D}^\alpha$ and the section of $\tilde{\mathcal{D}}$ by the complex line $w = c$ is $\tilde{S}_c^\alpha$.

Let $\xi$ be a standard holomorphic coordinate system of $\mathbb{D}$. In the following, we treat the manifold $\tilde{V} = \mathbb{D} \times \Gamma$ as a domain of the direct product $P \times \Gamma$ where $P$ is the Riemann sphere equipped with the inhomogeneous coordinate system $\xi$. The subdomain $\tilde{\mathcal{D}}$ of $\mathbb{D} \times \Gamma$ is pseudoconvex in $P \times \Gamma$ since the frontier points of $\tilde{\mathcal{D}}$ in $\mathbb{D} \times \Gamma$ are strongly pseudoconvex. Let $ds_c^2$ denote the Poincaré metric of $\tilde{S}_c^\alpha$ which has the expression $d\tilde{s}_w = A(\xi, w) |d(\xi) | S_c^\alpha |_w|$ with respect to the coordinate system $(\xi, w)$ of $\tilde{\mathcal{D}}$. It suffices for us to prove that $\log A(\xi, w)$ is plurisubharmonic in $U^*$.

As is seen in the beginning of Section 1, $\log A(\xi, c)$ is a subharmonic function in $U^* \cap \tilde{S}_c^\alpha$ for each constant $c \in \Gamma$. So, for a subdomain $\Gamma'$ of $\Gamma$, we prove that $\log A(\psi(w), w)$ is a subharmonic function on the variable $w$ for an arbitrary
holomorphic function $\psi$ in $\Gamma'$ satisfying $(\psi(w), w) \in U^*$ for each $w \in \Gamma'$. Let $\lambda_{\xi_j}^{\omega}$ denote the Robin constant for $(\tilde{S}_w^\alpha, (\psi(w), w))$ with respect to the local coordinate system $\xi_j^\alpha(\tilde{S}_w^\alpha)$ where $\xi'$ is the meromorphic function $\xi - \psi(w)$ defined on $P \times \Gamma'$. Since $A(\xi, w)|d(\xi|\tilde{S}_w^\alpha)| = A(\xi + \psi(w), w)|d(\xi|\tilde{S}_w^\alpha)|$, it follows from Section 1 that $\lambda_{\xi_j}^{\omega} = -\log A(\psi(w), w)$. Set $\sigma = \{(\xi, w) \in P \times \Gamma' \mid \xi = \psi(w)\}$. Consider the mapping $\Psi$ of $(P \times \Gamma') - \sigma$ onto $\Gamma' \times C$ defined by $x = w(p)$, $y = 1/\xi(p)$ ($p \in (P \times \Gamma') - \sigma$). The complement $K$ of the image $\Psi(\tilde{\Gamma}' - \sigma)$ in $\Gamma' \times C$ is a pseudoconcave subset of $\Gamma' \times C$. Let $K_t$ denote the section $K \cap L_t$ of $K$ by the complex line $L_t = \{(x, y) \in \Gamma' \times C \mid x = t\}$. As H. Yamaguchi proved in 1971 by a function-theoretic deduction, the transfinite diameter $d_{\infty,t}$ of $K_t$ is a logarithmically subharmonic function on the variable $t$ (cf. Yamaguchi [5]). Thanks to G. Szegö [3], we know that $\lambda_{\xi_j}^{t} = -\log d_{\infty,t}$. Hence we have proved that $\log A(\psi(w), w)$ is a subharmonic function on the variable $w$.

3. Conclusions

Since $d_{s_c,\beta} \leq d_{s_c,\alpha}$ for real numbers $\alpha$ and $\beta$ satisfying $\alpha < \beta$, it is sufficient for the proof of the assertion in Introduction to prove that $d_{s_c,\alpha} \rightarrow d_{s_c} (\alpha \rightarrow \infty)$. Let $\hat{S}_c$ denote the universal covering surface of $S_c$ with the canonical projection $\pi: \hat{S}_c \rightarrow S_c$. Fix a point $\hat{p}$ of $\pi^{-1}(p_0)$. Let $D_c^\alpha$ denote the connected component of $\pi^{-1}(S_c^\alpha)$ which contains $\hat{p}$. Because of Lemma 2.2, $D_c^\alpha$ is a simply connected domain of $\hat{S}_c$ and $D_c^\alpha \subset D_c^\beta$ for real numbers $\alpha$ and $\beta$ satisfying $\alpha < \beta$. Suppose that $S_c$ is not of exceptional type. Since $\hat{S}_c = \bigcup_\alpha D_c^\alpha$, we get by Lemma 1.1 that $d_{s_c,\alpha} \rightarrow d_{s_c} (\alpha \rightarrow \infty)$. When $S_c$ is of exceptional type, we can prove easily that $d_{s_c,\alpha} \rightarrow 0 (\alpha \rightarrow \infty)$. Using a tubular neighborhood of $S_c^\alpha$, we can prove by this fact that $A(z, w)$ in Introduction is upper semi-continuous. Hence $A(z, w)$ must be logarithmically plurisubharmonic by the result of the previous section. Therefore we get the following

**Theorem.** Let $f$ be a holomorphic function on a two-dimensional Stein manifold $\Omega$ such that $df \neq 0$ at each point of $\Omega$. Then the movement of the Poincaré metrics of prime surfaces of $f$ is logarithmically plurisubharmonic in $\Omega$.

**Corollary** (T. Nishino [2]). Let $f$ be a holomorphic function on a two-dimensional Stein manifold. Set $e = \{c \in C \mid \text{at least one prime surface of } f \text{ with value } c \text{ is of exceptional type}\}$. If the logarithmic capacity of $e$ is not zero, then every prime surface of $f$ is smooth and of exceptional type.

In the case where $df \neq 0$ at each point, the proof of the above corollary is straightforward. For the general case, we must prove the fundamental lemma of T. Nishino [2] in a modified form to fit our situation. But the above theorem makes the proof of the modified fundamental lemma fairly easy.
References


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