EXACT COEFFICIENT ESTIMATES FOR UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSION

Samuel L. Krushkal
Bar-Ilan University, Research Institute for Mathematical Sciences
Department of Mathematics, 52900 Ramat-Gan, Israel; krushkal@bimacs.cs.biu.ac.il

Abstract. We give here a complete solution of the coefficient problem for normalized univalent functions on the unit disk, with $k$-quasiconformal extension for a small $k$, and derive an explicit bound for $k$.

1. Introduction

While the coefficient problem is completely solved in the class of all normalized univalent functions on the disk $\{|z|<1\}$, the question remains open for functions with quasiconformal extension.

The strongest result here is established for the functions with $k$-quasiconformal extension where $k$ is small; see [Kr2].

Let $S$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ univalent in the unit disk $\Delta = \{|z|<1\}$. The class $S(k)$ consists of $f \in S$ admitting $k$-quasiconformal extension onto the whole Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with additional normalization $f(\infty) = \infty$. Let

$$f_1(z) = \frac{z}{(1-ktz)^2}, \quad |z| < 1, \ |t| = 1,$$

$$f_{n-1} \{ f_1(z^{n-1}) \}^{1/(n-1)} = z + \frac{2kt}{n-1} z^n + \cdots, \quad n = 3, 4, \ldots.$$

Consider on $S$ a functional $F$ of the form

$$F(f) = a_n + H(a_{m_1}, a_{m_2}, \ldots, a_{m_s}),$$

where $a_j = a_j(f) \ ; \ n, m_j \geq 2$ and $H$ is a holomorphic function of $s$ variables in an appropriate domain of $\mathbb{C}^s$. We assume that this domain contains the origin $\mathbf{0}$ and that $H, \partial H$ vanish at $\mathbf{0}$.

The mentioned result of [Kr2] is:

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Theorem 1. For any functional of the above form there exists a $k(F) > 0$ such that, for $k \leq k(F)$,

$$\max_{S(k)} |F(f)| = |F(f_{n-1})|$$

for some $|t| = 1$.

As a corollary, one immediately gets for $f \in S(k)$ the sharp estimate

$$|a_n| \leq \frac{2k}{n-1}$$

for $k \leq k_n$, with equality only for the function $f_{n-1}$. This solves the well-known problem of Kühnau and Niske; see [KuN]. The estimate (2) is interesting only for $n \geq 3$, because for $n = 2$ there is the well-known bound $|a_2| \leq 2k$ for all $k \in [0, 1]$ with equality for the function $f_1$.

The purpose of this paper is to improve on Theorem 1, supplementing it with an explicit estimate for the quantity $k(F)$.

2. Statement of results

The main result of the paper is:

Theorem 2. Let $\sup_{S} |F(f)| = M_n$. Then the equality (1) holds for all

$$k \leq \frac{1}{2 + (n-1)(M_n + 1)} =: k_0(F).$$

The bound (3) is not sharp and can be improved.

Corollary. The estimate (2) is valid for all

$$k \leq \frac{1}{n^2 + 1}.$$

Proof. Take $F(f) = a_n$. Since $M_n = n$, by de Branges's theorem [dB], one immediately deduces from (3) that in this case

$$k_0(F) = \frac{1}{n^2 + 1}.$$

For simplicity, we consider here the functionals $F$ with holomorphic $H$ depending on a finite number of coefficients $a_m$. The latter condition is not essential; one can take $H$ depending on infinitely many $a_m$ (provided the series expansion of $H$ converges in some complex Banach space). The result shows that the main contribution here is given by the linear term $a_m$. The estimate (3) determines for which $k$ this is true.
3. Proof of Theorem 2

We shall show that for \( k \) satisfying (3) all arguments employed in the proof of Theorem 1 in [Kr2] remain valid after some modification. Actually, we only need to modify the proof of Lemma 1.

On \( \Delta^* = \{ z \in \mathbb{C} : |z| > 1 \} \) we have the Beltrami coefficients \( \mu_f = \partial \bar{z}f / \partial z f \) of the extensions \( f^\mu \) of functions \( f \in S(k) \); these coefficients range over the ball

\[
B(\Delta^*) = \{ \mu \in L_{\infty}(\mathbb{C}) : \mu \mid \Delta = 0, \| \mu \|_\infty < 1 \}.
\]

Let \( B(\Delta^*)_k = \{ \mu \in B(\Delta^*) : \| \mu \| \leq k \} \).

Note that the Beltrami coefficient for \( f_{n-1} \) can be taken to be \( kt\mu_n \), where \(|t| = 1 \) and

\[
\mu_n(z) = \frac{|z|^{n+1}}{z^{n+1}}.
\]

We shall also use the following notations. For a functional \( L : S \to \mathbb{C} \) define

\[
\hat{L}(\mu) = L(f^\mu), \quad \mu \in B(\Delta^*)
\]

If \( L \) is complex Gateaux differentiable, \( \hat{L} \) is a holomorphic functional on \( B(\Delta^*) \).

All our functionals have this property.

For \( \mu \in L_{\infty}(\Delta^*), \varphi \in L_1(\Delta^*) \) we define

\[
\langle \mu, \varphi \rangle = -\frac{1}{\pi} \iint_{\Delta^*} \mu \varphi \, dx \, dy \quad (z = x + iy).
\]

For small \( k \), the functions \( f^\mu \in S(k) \) can be represented by

\[
f^\mu(\zeta) = \zeta - \frac{\zeta^2}{\pi} \iint_{\Delta^*} \frac{\mu(z) \, dx \, dy}{z^2(z - \zeta)} + O(\| \mu \|^2),
\]

where the estimate of the remainder term is uniform on compact subsets of \( \mathbb{C} \) (see e.g. [Kr1, Ch. 2]); this easily implies

\[
\hat{F}(\mu) = \langle \mu, \frac{1}{z^{n+1}} \rangle + O_n(\| \mu \|^2)
\]

and hence

\[
\| \hat{F}'(0) \| = \sup \left\{ \left| \langle \mu, \frac{1}{z^{n+1}} \rangle \right| : \| \mu \| \leq 1 \right\} = \frac{1}{\pi} \iint_{\Delta^*} \frac{dx \, dy}{|z|^{n+1}} = \frac{2}{n - 1}.
\]
Now, applying the Schwarz lemma to the function

\[ h_\mu(t) = \hat{F}(t\mu) - \hat{F}'(0)t\mu: \Delta \to \mathbb{C}, \]

where \( \mu \in B(\Delta^*) \) is fixed, we get

\[ |\hat{F}(\mu) - \hat{F}'(0)\mu| \leq (M_n + \|\hat{F}'(0)\|)\|\mu\|^2 = \left( M_n + \frac{2}{n-1} \right)\|\mu\|^2. \]

Consider the auxiliary functional

\[ \hat{F}_p(\mu) = \hat{F}(\mu) + (p - 1)\xi\left\langle \mu, \frac{1}{z_p+1} \right\rangle, \]

where \( p \neq n \) is fixed and \( |\xi| < \frac{1}{2} \). Then

\[ \sup_{B(\Delta^*)} |\hat{F}_p(\mu)| < M_n + 1 \]

and, similarly to (7),

\[ \left| \hat{F}_p(\mu) - \hat{F}'(0)\mu - (p - 1)\xi\left\langle \mu, \frac{1}{z_p+1} \right\rangle \right| \leq \left( M_n + 1 + \frac{2}{n-1} \right)\|\mu\|^2. \]

We shall require that

\[ \left( M_n + 1 + \frac{2}{n-1} \right)\|\mu\|^2 < \frac{1}{n-1}\|\mu\| \]

or, equivalently,

\[ \|\mu\| \leq \frac{1}{2 + (n-1)(M_n + 1)} = k_0(F). \]

Consider now any function \( f_0 \) in \( S(k) \) maximizing \( |F| \) over \( S(k) \) (the existence of such functions follows from compactness). Let \( \mu_0 \) be an extremal dilatation of \( f_0 \), i.e.

\[ \|\mu_0\|_\infty = \inf\{\|\mu\|_\infty \leq k : f^\mu | \Delta = f_0 | \Delta\}. \]

Note that \( \|\mu_0\|_\infty = k \) by the maximum modulus principle. Suppose that \( \mu_0 \neq kt\mu_n \), where \( |t| = 1 \), and \( \mu_n \) is defined by (5). We show that this leads to contradiction for \( k \) satisfying (3). First of all, we may establish the following important property of extremal maps:
Lemma 1. If \( k \) satisfy (3), then for all \( 2 \leq p \neq n \),

\[
\langle \mu_0, \frac{1}{z^{p+1}} \rangle = 0.
\]

Proof. Note that, from (6),

\[
\langle \mu_0, \frac{1}{z^{p+1}} \rangle = \lim_{\tau \to \infty} \frac{a_p(f^\tau \mu_0)}{\tau}.
\]

Consider the classes \( S(\tau k_0) \) where \( k_0 = k_0(F) \) is defined in (3) and \( 0 < \tau < 1 \).

It follows from (6) that, as \( \tau \to 0 \),

(12) \( \max \{ |\hat{F}(\mu)| : \|\mu\| \leq \tau k_0 \} = \frac{\tau k_0}{\pi} \int \int_{\Delta^*} \frac{dx \, dy}{|z|^{n+1}} + O_n(\tau^2) = |\hat{F}(\tau \mu_0)| + O_n(\tau^2). \)

A similar calculation for functional (8) implies

(13) \( \max_{B(\Delta^*)} |\hat{F}_p(\mu)| = \frac{\tau k_0}{\pi} \int \int_{\Delta^*} \frac{1}{|z|^{n+1}} + \frac{(p-1)\xi}{z^{p+1}} \frac{dx \, dy}{|z|^{n+1}} + O_n(\tau^2), \)

where the remainder term estimate follows from (10) and depends (as in (12)) only on \( M_n \) and \( k_0 \).

Using the known properties of the norm

\[
h_p(\xi) = \int \int_{\Delta^*} |z^{-n-1} + (p-1)\xi z^{-p-1}| \, dx \, dy
\]

following from the Royden [Ro] and Earle–Kra [EK] lemmas, we deduce from (12), (13) that for small \( \xi \) there should be

(14) \( \max_{B(\Delta^*)} |\hat{F}_p(\mu)| = \max_{B(\Delta^*)} |\hat{F}(\mu)| + \tau a_p(\xi) + O_p(\tau^2 \xi) + O_n(\tau^2). \)

On the other hand, we have as \( \xi \to 0, \tau \to 0 \), from (8)

\[
|\hat{F}_p(\tau \mu_0)| = |\hat{F}(\tau \mu_0)| + \text{Re} \frac{\hat{F}(\tau \mu_0)}{|\hat{F}(\tau \mu_0)|} (p-1)\xi \langle \mu_0, \frac{1}{z^{p+1}} \rangle + O(\tau^2 \xi^2)
\]

\[
= |\hat{F}(\tau \mu_0)| + \tau (p-1)|\xi| \left| \langle \mu_0, \frac{1}{z^{p+1}} \rangle \right| + O(\tau^2 \xi^2)
\]

with suitable choices of \( \xi \to 0 \). Comparing this with (14), (10), (11), we conclude that \( \langle \mu_0, z^{-p-1} \rangle = 0 \). The proof of Lemma is completed.
This lemma is one of the central points in the proof of the Theorems 1 and 2. The crucial point in the proof of Lemma 1 is that we now have to check here that simultaneously an infinite (countable) number of orthogonality conditions remain valid for all \( k \) satisfying (3).

The next part of the proof is similar to [Kr2]. We briefly check that the arguments remain valid for all \( k \).

Consider the Grunsky coefficients of the function \( \sqrt{f(z^2)} \) which are defined from the series expansion

\[
\log \left( \frac{(f(z^2))^{1/2} - (f(\zeta^2))^{1/2}}{z - \zeta} \right) = -\sum_{m,n=1}^{\infty} \omega_{mn} z^m \zeta^n,
\]

taking the branch of logarithm which vanishes at 1. The diagonal coefficients \( \omega_{n-1,n-1}(f) \) are related to the Taylor coefficients of \( f \) by

\[
\omega_{n-1,n-1} = \frac{1}{2} a_n + P(a_2, \ldots, a_{n-1})
\]

where \( P \) is a polynomial without constant or linear terms (see [Hu]). Moreover, for \( f \in S(k) \) there is the well-known bound

\[
|\omega_{n-1,n-1}| \leq \frac{k}{n-1}
\]

with equality only for the functions \( f_{n-1} \).

Therefore, the map \( \Lambda_{n-1}: B(\Delta^*) \to B(\Delta^*) \) defined by

\[
\Lambda_{n-1}(\mu) = \{(n-1)\omega_{n-1,n-1}(\mu)\} \mu_n
\]

is holomorphic and fixes the disk \( \{t \mu_n : |t| < 1\} \). The differential of \( \Lambda_{n-1} \) at \( \mu = 0 \) can be easily computed from (6), (15). It is an operator \( P_n: L_\infty(\Delta^*) \to L_\infty(\Delta^*) \) given by

\[
P_n(\mu) = \beta_n \langle \varphi_n, \mu \rangle \mu_n, \quad \varphi_n = \frac{1}{z^{n+1}}.
\]

Let us define \( P_n(\mu) = \alpha(k) \mu_n \). Since, by assumption, \( f_0 \) is not equivalent to \( f_{n-1} \), we have

\[
\left\{ \Lambda_{n-1} \left( \frac{t}{k} \mu_0 \right) : |t| < 1 \right\} \not\subseteq \left\{ |t| < 1 \right\}.
\]

Thus, by the Schwarz lemma,

\[
|\alpha(k)| < k.
\]

Now consider the function

\[
\nu_0 = \mu_0 - \alpha(k) \mu_n
\]
and show that $\nu_0$ eliminates integrable holomorphic functions on $\Delta^*$.

From Lemma 1 and the mutual orthogonality of the powers $z^m$, $m \in \mathbb{Z}$,

$$\langle \nu_0, \frac{1}{z^{p+1}} \rangle = 0$$

for $p = 2, 3, \ldots, p \neq n$. To establish that

$$\langle \nu_0, \frac{1}{z^{n+1}} \rangle = 0,$$

consider the conjugate operator

$$P_n^*(\varphi) = \beta_n \langle \mu_n, \varphi \rangle \varphi_n, \quad \varphi_n = \frac{1}{z^{n+1}},$$

which maps $L_1(\Delta^*)$ onto $L_1(\Delta^*)$ and fixes the subspace $\{\lambda \varphi_n : \lambda \in \mathbb{C}\}$. The definition of $\nu_0$ implies $P_n(\nu_0) = 0$. Thus, for some $\lambda$,

$$\langle \nu_0, \varphi_n \rangle = \lambda \langle \nu_0, P_n^* \varphi_n \rangle = \lambda \langle P_n \nu_0, \varphi_n \rangle = 0.$$

Now consider in $L_1(\Delta^*)$ the subspace $A_1(\Delta^*)$ of functions $\varphi$ which are holomorphic on $\Delta^*$ and satisfy the condition $\varphi(z) = O(|z|^{-3})$ as $|z| \to \infty$. Let

$$A_1(\Delta^*)^\perp = \{\mu \in L_\infty(\Delta^*) : \langle \mu, \varphi \rangle = 0 \text{ for all } \varphi \in A_1(\Delta^*)\}.$$

Since the functions $\varphi_n = 1/z^{n+1}$, $n = 2, 3, \ldots$, form a complete set in $A_1(\Delta^*)$, we have proved that $\nu_0 \in A_1(\Delta^*)^\perp$.

Now we use the well-known properties of extremal quasiconformal maps (see e.g. [Ga], [Kr1], [RS]). First of all, since $\mu_0$ is extremal for $f_0$,

$$\|\mu_0\|_\infty = \inf \{|\langle \mu_0, \varphi \rangle| : \varphi \in A_1(\Delta^*), \|\varphi\| = 1\};$$

moreover, such an equality is necessary and sufficient for $\mu \in B(\Delta^*)$ to be extremal for $f^\mu$. Hence, for any $\nu \in A_1(\Delta^*)^\perp$,

$$\|\mu_0\|_\infty = \inf \{|\langle \mu_0 + \nu \rangle| : \varphi \in A_1(\Delta^*), \|\varphi\| = 1\} \leq \|\mu_0 + \nu\|_\infty.$$

Thus we have

**Lemma 2.** If $f_0$ is extremal,

$$\|\mu_0\|_\infty = k \leq \|\mu_0 - \nu_0\|_\infty. \tag{17}$$

We may now complete the proof of Theorem 2. By (17)

$$k \leq \|\mu_0 - \nu_0\|_\infty = \|\alpha(k) \mu_n\|_\infty = |\alpha(k)|,$$

which contradicts (16). Hence $f_0$ is equivalent to $f_{n-1}$ and we can take $\mu_0 = kt \mu_n$ for some $|t| = 1$. 

4. Complementary remarks and open questions

1) The estimates (1)–(3) also hold in the class \( S_k(1) \) of functions \( f \in S \) with \( k \)-quasiconformal extensions \( \tilde{f} \) normalized by \( \tilde{f}(1) = 1 \).

The proof is similar, only (6) should be replaced with the corresponding representation formula for \( f \in S_k(1) \) [Kr1, Ch. 5]:

\[
f^\mu(\zeta) = \zeta - \frac{\zeta^2(\zeta - 1)}{\pi} \iint_{\Delta^*} \frac{\mu(z) \, dx \, dy}{z^2(z - 1)(z - \zeta)} + O(\|\mu\|), \quad \text{as } \|\mu\| \to 0.
\]

2) Similar results are valid for the class \( \Sigma(k) \) of functions \( g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \), \( z \in \Delta^* \), with \( k \)-quasiconformal extensions to \( \hat{C} \) which fix the origin.

The next two problems still remain open:

1) Does there exist an estimate of coefficients \( a_n \) \((n \geq 3)\) for \( f \in S(k) \) which holds for \( k \leq k_0 \) with a single \( k_0 > 0 \)?

2) Can we find exact estimates of coefficients \( a_n \) for univalent functions on the disk with quasiconformal extension in the general case when the dilatation \( k < 1 \) is arbitrary?

For \( f \in S(k) \), one gets from (7) the estimate

\[
|a_n| \leq \frac{2k}{n-1} + \left(n + \frac{2}{n-1}\right)k^2
\]

for any \( k \), \( 0 \leq k < 1 \), (cf. [KrKu, Part 1, Ch. 2]). Note also that Grinshpan [Gr] established the exact growth order, with respect to \( n \), of the coefficients \( a_n \) of \( f \in S \) with \( k \)-quasiconformal extension, without any additional normalization: \( |a_n| \leq cn^k \).

References

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