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QUASICONFORMAL IMAGES OF HÖLDER DOMAINS

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Abstract. We introduce and study the k-cap condition and use it to prove that the quasiconformal image of a Hölder domain is itself Hölder if and only if it supports a Trudinger inequality. We compare and contrast the k-cap condition with related slice-type conditions.

0. Introduction

Smith and Stegenga [SS2] showed that every Hölder domain is a Trudinger domain, i.e., if G is a Euclidean domain on which quasihyperbolic distance to some fixed $x_0 \in G$ grows like the logarithm of distance to the boundary, then G supports a Trudinger imbedding. Subject to some rather mild restrictions, the converse is also true; see [BK2, Theorem 4.1] and [BO, Theorem 5.3]. We note that some restriction is essential for the converse direction to rule out easy counterexamples based on removability or extendability.

In particular, it follows from the results in [BK2] that the quasiconformal image of a uniform domain satisfies a slice condition, and hence that it is a Trudinger domain if and only if it is a Hölder domain. Here we generalize this result by showing that a quasiconformal image of a Hölder domain is a Trudinger domain if and only if it is a Hölder domain; the resulting proof is also simpler than the proofs based on slice conditions.

A key step in the earlier papers is the use of a global conformal capacity estimate (the so-called *Loewner estimate*) to prove that all quasiconformal images of a uniform domain satisfy slice conditions. Uniform domains satisfy such an estimate, but the typical Hölder domain does not. Indeed we shall see that, although Hölder domains satisfy weak slice conditions, their quasiconformal images may fail to do so. Instead, we introduce and use the *k-cap condition*, which relates conformal capacity and quasihyperbolic distance. This condition is implied by all previously defined (weak) slice conditions, but implies none of them. Crucially, it is conformally invariant but still strong enough to weed out all Trudinger domains that are not Hölder.

After some preliminaries in Section 1, we define the k-cap condition and prove the Trudinger–Hölder result in Section 2. We then discuss the relationship between the k-cap and various slice-type conditions in Section 3.

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1. Preliminaries

First let us introduce some general notation. Throughout, we look at domains in $\mathbf{R}^n$, $n > 1$. Suppose the Lebesgue measure $|D|$ of $D \subset \mathbf{R}^n$ is positive and finite. Given a function $u\colon D \to \mathbf{R}$, we denote by u_D the Lebesgue average of u on D. We define the Orlicz norm for functions $f\colon D \to \mathbf{R}$ with respect to the Orlicz function ϕ and normalized Lebesgue measure by the equation

$$\|f\|_{\phi(L)(D)} = \inf\left\{t > 0 : \frac{1}{|D|}\int_D \phi(|f(x)|/t)\,dx \le 1\right\}.$$

As a special case, $\|\cdot\|_{L^p(D)}$ denotes the usual L^p norm on D with respect to normalized Lebesgue measure. Various concepts that we introduce involve one or more parameters which we include only when needed; for instance we define $(\varepsilon, C; x_0)$-Hölder domains, but refer to such domains generically as Hölder domains. For any two numbers a, b, $a \vee b$ and $a \wedge b$ denote their maximum and minimum, respectively. For any set S, χ_S is its characteristic function. If S is either an open or closed ball, tS denotes its concentric dilate by a factor t. We state quantitative dependence in the usual manner: $C = C(Q_1, Q_2, \ldots)$ means that C depends only on the quantities $Q_1, Q_2, \ldots$.

Assume that $G \subsetneq \mathbf{R}^n$ is a domain. We write $\delta_G(x)$ for the boundary distance $\operatorname{dist}(x, \partial G)$, $x \in G$, and call $r(G) = \sup_{x\in G}\delta_G(x)$ the *inradius of* G. When it is clear from the context what domain G we have in mind, we use B_x and $\overline{B}_x$, respectively, to denote the open and closed balls around x of radius $\delta_G(x)$. Let $\Gamma_G(x,y)$ be the class of rectifiable paths $\lambda\colon [0,t] \to G$ for which $\lambda(0) = x$ and $\lambda(t) = y$. Writing ds for arclength measure, we define the *quasihyperbolic length* of a rectifiable path γ in G, and the *quasihyperbolic distance* between $x, y \in G$ by the equations

$$\begin{aligned}
\operatorname{len}_{k;G}(\gamma) &= \int_\gamma \frac{ds(z)}{\delta_G(z)},\\
k_G(x,y) &= \inf_{\gamma\in\Gamma_G(x,y)} \operatorname{len}_{k;G}(\gamma), \qquad x, y \in G.
\end{aligned}$$

Given $x, y \in G$, there always exists a *quasihyperbolic geodesic*, i.e., a path $\gamma \in \Gamma_G(x,y)$ with $\operatorname{len}_{k;G}(\gamma) = k_G(x,y)$; see [GO]. We write $B(x,r)$ for the open Euclidean ball of radius r about x, and $B_k(x,r)$ for the quasihyperbolic ball of radius r about x (when the domain G is understood). We denote by $\operatorname{len}(S)$ and $\operatorname{len}_{k;G}(S)$ the one-dimensional Hausdorff measures of a set $S \subset G$ with respect to the Euclidean and quasihyperbolic metrics, respectively; the sets S that interest us are all countable unions of image sets of paths, so $\operatorname{len}(S)$ and $\operatorname{len}_{k;G}(S)$ are just sums of the corresponding path-lengths. Whenever λ is a path, λ^* denotes its image set. We denote by $l_G(x,y)$ the inner Euclidean distance from x to y in G, i.e. the infimum of $\operatorname{len}(\gamma^*)$ over all $\gamma \in \Gamma_G(x,y)$.

For $1 \le p < \infty$, $L^{1,p}(G)$ is the space of functions $f\colon G \to \mathbf{R}$ with distributional gradients in $L^p(G)$, and $W^{1,p}(G) = L^p(G) \cap L^{1,p}(G)$ is the corresponding Sobolev space. We write $\|u\|_{W^{1,p}(G)} = \|u\|_{L^p(G)} + \|\nabla u\|_{L^p(G)}$.

The *conformal capacity*, $\operatorname{cap}(E, F; G)$, of the disjoint compact subsets $E, F \subset G$ relative to G, is the infimum of $\int_G |\nabla u|^n$, as u ranges over all functions which are locally Lipschitz continuous in G, equal 1 on E, and 0 on F. We write $\operatorname{cap}(E, F) = \operatorname{cap}(E, F; \mathbf{R}^n)$. Trivially, $\operatorname{cap}(E, F; G) \le \operatorname{cap}(E', F'; G')$ whenever $E \subset E'$, $F \subset F'$, $G \subset G'$, and $\operatorname{cap}(E, F; G) = \operatorname{cap}(\partial E, \partial F; G)$.

It is sometimes useful to use conformal modulus instead of capacity. The *conformal modulus*, $\operatorname{cap}(E, F; G)$, of the disjoint compact subsets $E, F \subset G$ relative to G, is the infimum of $\int_G \varrho^n$, where ϱ ranges over all *admissable weights*, meaning non-negative Borel measurable functions such that the line integral $\int_\gamma \varrho\, ds$ is always at least 1, for every locally rectifiable path γ that begins in E, ends in F, and remains inside G. The principle that modulus equals capacity has a long history going back to Ziemer [Z1] but, with our definition of capacity, the fact that $\operatorname{cap}(E, F; G) = \operatorname{mod}(E, F; G)$ is due to Kallunki and Shanmugalingam [KS], where the reader can also find many references to other results of this type.

We shall need a few capacity estimates, which we now state. Defining the *relative distance*

$$\Delta(E, F) \equiv \frac{\operatorname{dist}(E, F)}{\operatorname{dia}(E) \wedge \operatorname{dia}(F)},$$

it is well known (and is proven after Proposition 3.5) that there exists a dimensional constant C_n such that

$$\Delta(E, F) \ge 2 \quad \Longrightarrow \quad \operatorname{cap}(E, F; G) \le C_n \big(\log \Delta(E, F)\big)^{-n+1}. \tag{1.1}$$

In the case $G = \mathbf{R}^n$, there exists another dimensional constant c_n such that

$$\Delta(E, F) \ge 2 \quad \Longrightarrow \quad \operatorname{cap}(E, F) \ge c_n \big(\log \Delta(E, F)\big)^{-n+1}. \tag{1.2}$$

This follows, for instance, as a special case of [HnK, Theorem 3.6]. Our final capacity estimate is a transfer estimate given by [HrK, Lemma 3.2]. If E is a closed ball, with $\sigma E \subset G \subset \mathbf{R}^n$ for some $\sigma > 1$, then for all compact subsets F of $G \setminus \sigma E$, and all constants $0 < c < 1$, there is a constant $C = C(c, \sigma, n)$ such that

$$\operatorname{cap}(cE, F; G) \le \operatorname{cap}(E, F; G) \le C \operatorname{cap}(cE, F; G). \tag{1.3}$$

Given $C \ge 1$, we say that a domain $G \subset \mathbf{R}^n$ is a C*-Trudinger domain* if $|G| < \infty$ and it supports the Trudinger imbedding

$$\|u - u_G\|_{\phi(L)(G)} \le C r(G) \|\nabla u\|_{L^n(G)} \qquad \text{for all } u \in W^{1,n}(G),$$

where $\phi(x) = \exp(x^{n/(n-1)}) - 1$. The use of normalized Lebesgue measure and the presence of the inradius on the right-hand side ensures that the Trudinger imbedding is dilation invariant. More generally, given a non-empty open set $A \subset G$, we say that G is a $(C;A)$-*Trudinger domain* if

$$\|u - u_A\|_{\phi(L)(G)} \le Cr(G)\|\nabla u\|_{L^n(G)} \qquad \text{for all } u \in W^{1,n}(G).$$

As is well known, if $A' \subset G$ is also non-empty and open, then every $(C;A)$-Trudinger domain is a $(C';A')$-Trudinger domain with $C' = C'(n, |A'|/|G|)$.

Let $C \ge 1$, $x, y \in G \subsetneq \mathbf{R}^n$, and let $\gamma \in \Gamma_G(x,y)$ be a path of length l which is parametrized by arclength. We say that γ is a C-*uniform path* for $x, y \in G$ if $l \le C|x-y|$ and $t \wedge (l-t) \le C\delta_G\big(\gamma(t)\big)$. We say that G is a C-*uniform domain* if there is a C-*uniform path* for every pair $x, y \in G$. If there is a C-uniform path for the points $x, y \in G$, then

$$k_G(x,y) \le 2C \log\left(1 + \frac{|x-y|}{\delta_G(x) \wedge \delta_G(y)}\right) + C', \tag{1.4}$$

where $C' = 2(C + C\log C + 1)$. This result is due to Gehring and Osgood [GO], where they also show that (1.4) holds with a uniform constant C for all $x, y \in G$ if and only if G is uniform.

One can form *one-sided* versions of uniformity and (1.4), by assuming the defining conditions uniformly for all $x \in G$, but only for a fixed $y = x_0 \in G$. This yields the classes of John and Hölder domains, respectively, which are no longer equivalent. We shall, however, use a somewhat different defining inequality for Hölder domains to reflect the asymmetry between the roles of x and x_0.

Given $\varepsilon \in (0,1]$, $C \ge 0$, and a pair of points x, x_0 in a domain $G \subsetneq \mathbf{R}^n$, we say that the path $\gamma \in \Gamma_G(x, x_0)$ is an (ε, C)-*Hölder path* for the pair x, x_0 if $\operatorname{len}_{k;G}(\gamma) \le C + \varepsilon^{-1}\log\big(\delta_G(x_0)/\delta_G(x)\big)$. We say that G is an $(\varepsilon, C; x_0)$-*Hölder domain* if there is an (ε, C)-Hölder path for all pairs x, x_0, $x \in G$. The concept of a Hölder domain and the parameter ε, but not the parameter C, are independent of $x_0 \in G$. We note that the concept of a Hölder domain, and the associated numerical parameters, are dilation invariant.

All uniform domains are John domains, and all John domains are Hölder domains, but these classes are distinct. Uniform domains include all bounded Lipschitz and certain fractal domains (e.g. the region inside the von Koch snowflake). The domains in the proof of Proposition 2.11 below are Hölder domains, but are not John. For more on Hölder domains, see [SS1] and [K]; for more on uniform domains, see [V2] and [V3].

We close this section by stating a useful lemma for Hölder domains, which is implied by Corollary 1 of [SS1].

Lemma 1.5. *If $G \subset \mathbf{R}^n$ is an $(\varepsilon, C; x_0)$-Hölder domain, then* $\operatorname{dia}(G) \le C'\delta_G(x_0)$ *for some $C' = C'(\varepsilon, C)$.*

2. Trudinger, Hölder, and the k-cap condition

In this section we introduce the k-cap condition and use it to show that quasiconformal images of Hölder domains are themselves Hölder domains if they are Trudinger domains. We also show that the class of quasiconformal images of Hölder domains is strictly larger than the class of quasiconformal images of uniform domains, and so this result improves on [BK2] where the same conclusion is reached for the latter class of domains.

Theorem 2.1. *Suppose f is a quasiconformal mapping from one domain $G \subset \mathbf{R}^n$ onto another one, G'. If G is a Hölder domain and G' is a Trudinger domain, then G' is also a Hölder domain.*

Before we proceed, let us discuss the parameter dependence in this theorem. Suppose G is an $(\varepsilon, C; y)$-Hölder domain, G' is a C_1-Trudinger domain, f is a K-quasiconformal mapping, and $y' = f(y)$. In that case, we shall see that G' is an $(\varepsilon', C'; y')$-Hölder domain, where ε', C' depend only on ε, C, n, C_1, K, and the ratio $|G'|/|B_{y'}|$. Dependence on the last parameter might seem unpleasant, so let us discuss it further. First, a careful reading of the proof indicates that it is needed only to determine C', not the more important parameter ε'. Secondly, even this dependence can be removed by a reworking of the assumptions. Since G' is a Trudinger domain, it is also a $\left(C_2; \frac{1}{2}B_{y'}\right)$-Trudinger domain, for some C_2 dependent only on C_1 and $|G'|/|B_{y'}|$. We can then choose ε', C' to depend only on ε, C, n, C_2, and K. Finally, by taking f to be a Möbius self-map of the unit disk which takes the origin to a point close to the unit circle, one sees that with the original assumptions, dependence on $|G'|/|B_{y'}|$ is essential.

The main tool in our proof of Theorem 2.1 is the notion of a k-cap condition. First note that if G is a bounded subdomain of $\mathbf{R}^n$, $x_0 \in G$, and $0 < c \le \frac{1}{2}$, then there is a constant $C > 0$ such that

$$(2.2)\quad k_G(x,y) \ge 2 \quad\Longrightarrow\quad k_G(x,y)^{n-1}\operatorname{cap}(c\overline{B}_x, c\overline{B}_y; G) \ge C \qquad \text{for all } x \in G.$$

This fact is implicit, for instance, in the proof of [HrK, Theorem 6.1]. The k-cap condition, which is our main tool in the proof of Theorem 2.1, simply reverses this inequality. Specifically, for a given point $y \in G$ and constants $C > 0$, $0 < c \le \frac{1}{2}$, we say that G satisfies the $(C, c; y)$*-k-cap condition* if

$$(\mathrm{KC})\qquad k_G(x,y) \ge 2 \implies k_G(x,y)^{n-1}\operatorname{cap}(c\overline{B}_x, c\overline{B}_y; G) \le C \quad \text{for all } x \in G.$$

If the parameter c is omitted, it is assumed that $c = \frac{1}{2}$. We call any $(C; y)$-k-cap condition a *one-sided k-cap condition* if we do not wish to specify the parameters. The adjective "one-sided" is added to distinguish this condition from a *two-sided C-k-cap condition*, which means that G satisfies a $(C; y)$-k-cap condition for each $y \in G$. We say that the *C-k-cap inequality* holds for $x, y \in G$, $k_G(x,y) \ge 2$, if the main inequality in (KC) holds for this pair; formally the data here is a triple (x, y, G), but usually G is implicit.

By a simple estimate, the quasihyperbolic ball of radius $r > 0$ around a point $z \in G$ contains $\bigl(1 - \exp(-r)\bigr)B_z$. It follows that if $k_G(x, y) \geq 2$, then ϱB_x and ϱB_y are disjoint, where $\varrho = 1 - 1/e$. Since $\varrho > \frac{1}{2}$, it follows from the transfer estimate (1.3) that every $(C, c; y)$-*k-cap condition* implies a $(C_1 C, c'; y)$-*k-cap condition*, for some $C_1 = C_1(c, c', n)$. Additionally, using (1.1), we see that there exists a dimensional constant C_n such that

$$k(x, y) \geq 2 \quad \Longrightarrow \quad \operatorname{cap}\bigl(\tfrac{1}{2}\overline{B}_x, \tfrac{1}{2}\overline{B}_y; G\bigr) \leq \operatorname{cap}\bigl(\mathbf{R}^n \setminus \varrho B_y, \tfrac{1}{2}\overline{B}_y; \mathbf{R}^n\bigr) \leq C_n.$$

Thus if we want to prove that a domain satisfies a k-cap condition, but we do not care about the precise values of the parameters, it suffices to prove the estimate in (KC) only for large quasihyperbolic distance.

The following proposition is the first step in our proof of Theorem 2.1.

Proposition 2.3. *Let $G \subset \mathbf{R}^n$ be a $\bigl(C_1; \frac{1}{2}B_y\bigr)$-Trudinger domain that satisfies the $(C_2; y)$-k-cap condition for some $y \in G$. Then G is an $(\varepsilon, C; y)$-Hölder domain for some ε, C dependent only on C_1, C_2, and n.*

Proof. By the dilation invariance of the assumptions and the conclusion, we may assume that $|G| = 1$, and so $r(G) < 1$. Let $x \in G$ be arbitrary but fixed. The Hölder estimate is trivially true if $k_G(x, y) < 2$, so we may assume that $k_G(x, y) \geq 2$. Let $u \colon G \to \mathbf{R}$ be any locally Lipschitz function such that $u|_{(1/2)B_x} \equiv 1$ and $u|_{(1/2)B_y} \equiv 0$. The Trudinger imbedding implies that

$$\|\chi_{(1/2)B_x}\|_{\phi(L)(G)} \leq \|u - u_{(1/2)B_y}\|_{\phi(L)(G)} \leq C_1 \|\nabla u\|_{L^n(G)}$$

and so $\phi(1/C_1\|\nabla u\|_{L^n(G)})\bigl|\frac{1}{2}B_x\bigr| \leq 1$. Unravelling this and taking an infimum over all such functions u, we get

$$\operatorname{cap}\bigl(\tfrac{1}{2}\overline{B}_x, \tfrac{1}{2}\overline{B}_y; G\bigr) \geq C_1^{-n}\bigl[\log\bigl(1 + \bigl|\tfrac{1}{2}B_x\bigr|^{-1}\bigr)\bigr]^{1-n}.$$

Combining this inequality with (KC), we deduce that

$$k_G(x, y) \lesssim \log\bigl(1/\bigl|\tfrac{1}{2}B_x\bigr|\bigr).$$

This last inequality readily implies that G is an $(\varepsilon, C; y)$-Hölder domain, but with the parameter C depending on $\delta_G(y)$ as well as the allowed parameters. To deduce the desired Hölder condition, we find a positive lower bound for $\delta_G(y)$ which depends only on C_1 and n. Let $E \equiv \frac{1}{4}\overline{B}_y$, define the test function $u(x) = \operatorname{dist}(x, E)$, $x \in G$, and let $N_u \equiv \|u\|_{\phi(L)(G)}$. Then

$$|G \setminus tE|\, \phi\biggl(\frac{(t-1)\delta_G(y)}{4N_u}\biggr) \leq \int_{G \setminus tE} \phi\biggl(\frac{u}{N_u}\biggr) \leq 1, \qquad t > 1.$$

Choosing t_0 so that $|G \setminus t_0 E| = \frac{1}{2}$, and defining $r_0 = \frac{1}{4} t_0 \delta_G(y)$, it follows that $t_0 > 2$ and $r_0 < 2N_u \phi^{-1}(2) \lesssim N_u$. Moreover, u is a Lipschitz function with $u \equiv 0$ on E and $\|\nabla u\|_{L^n(G)} \le \|\nabla u\|_{L^\infty(G)} = 1$, and so Trudinger's inequality implies that $N_u \lesssim 1$. Thus $r_0 \lesssim 1$.

We now define another test function $v\colon G \to [0,\infty)$, by the equation

$$v(x) = \begin{cases} 0, & x \in E, \\ (\log t_0)^{-1/n} \log(4|x-y|/\delta_G(y)), & x \in t_0 E \setminus E, \\ (\log t_0)^{1-1/n}, & x \in G \setminus t_0 E. \end{cases}$$

Then v is Lipschitz and $\|\nabla v\|^n_{L^n(G)} \lesssim 1$. By Trudinger's inequality, we have $N_v \equiv \|v\|_{\phi(L)(G)} \lesssim 1$. It follows as before that $|G \setminus t_0 E| \phi(\log(t_0)^{1-1/n}/N_v) \le 1$. Since $|G \setminus t_0 E| = \frac{1}{2}$, we deduce that t_0 is bounded. Since $|G \cap t_0 E| = \frac{1}{2}$, a lower bound for $\delta_G(y)$ follows immediately. □

By establishing a lower bound for $\delta_G(y)$ in the last proof, we implicitly proved the following Trudinger version of Lemma 1.5.

Proposition 2.4. *If $G \subset \mathbf{R}^n$ is an $(C;B)$-Trudinger domain, then* $\operatorname{dia}(G) \le C' r(B)$ *for some* $C' = C'(C,n)$.

We next claim that there is a dimensional constant C_n such that for all $0 < c \le \frac{1}{2}$,

$$\text{(2.5)} \quad k_G(x,y) \ge 2 \quad \Longrightarrow \quad \operatorname{cap}(c\overline{B}_x, c\overline{B}_y; G) \le C_n \left[\log\left(\frac{|x-y|}{\delta_G(x) \wedge \delta_G(y)} \right) \right]^{-n+1}.$$

To see this, let $E = \frac{1}{6} B_x$ and $F = \frac{1}{6} B_y$ and note that if $k_G(x,y) \ge 2$, then $\Delta(E,F) \ge 2$ and $\Delta(E,F)$ is comparable with $|x-y|/\bigl(\delta_G(x) \wedge \delta_G(y)\bigr)$. We therefore deduce (2.5) from (1.1) in the case $c = \frac{1}{6}$ (and hence also if $0 < c \le \frac{1}{6}$). Using (1.3), our claim follows in all cases.

Using (2.5) we see that the k-cap inequality holds for any pair x,y satisfying (1.4), and so in particular whenever there is a uniform path for x,y. The following lemma now follows easily.

Lemma 2.6. *Every C-uniform domain $G \subsetneq \mathbf{R}^n$ satisfies a two-sided C'-k-cap condition for some $C' = C'(C,n)$. Every $(\varepsilon, C; y)$-Hölder domain $G \subsetneq \mathbf{R}^n$ satisfies a $(C'; y)$-k-cap condition for some $C' = C'(\varepsilon, C, n)$.*

The proof of Theorem 2.1 is now almost clear. Proposition 2.3 reduces the task to showing that G' satisfies a k-cap condition. By Lemma 2.6, G satisfies the k-cap condition, so it only remains to show that the k-cap condition is a quasiconformal quasi-invariant.

We pause to record some properties of K-quasiconformal mappings f from G onto G', where $G, G' \subsetneq \mathbf{R}^n$, and the *dilatation* K is at least 1. Suppose also that $x, y \in G$, with $x' = f(x)$, $y' = f(y)$. First, f quasipreserves conformal capacity, i.e. it distorts it by at most a positive factor $C = C(K, n)$. In many modern accounts, this is a special case of the definition of quasiconformality, but the original proof from an analytic definition was found by Gehring [G]; for related results in more general contexts, see [T] and Theorems 4.9 and 8.5 of [HnK]. Also, K-quasiconformal mappings quasipreserve large quasihyperbolic distance; in fact, according to [GO, Theorem 3], there are constants $C = C(K, n)$ and $\alpha = K^{1/(1-n)}$ such that

$$k_{G'}(x', y') \le C\big(k_G(x, y) \vee k_G(x, y)^\alpha\big).$$

Lastly, if $B = B(x, r) \subset G$, with $\operatorname{dist}(B, \partial G) = Cr$, then $c\overline{B}_{x'} \subset f(\overline{B})$, for some $c = c(C, K, n) > 0$; this follows, for instance, by applying [V1, 18.1] to the (quasiconformal) inverse of f.

From the quasi-invariance properties listed above, we see that if G satisfies the $(C; y)$-k-cap condition and $f\colon G \to G'$ is K-quasiconformal, then

$$k_G\big(f(x), f(y)\big)^{n-1} \operatorname{cap}\big(f\big(\tfrac{1}{2}\overline{B}_x\big), f\big(\tfrac{1}{2}\overline{B}_y\big); G'\big) \le C' \qquad \text{for all } x \in G.$$

Since, for some $c' = c'(K, n)$, we have

$$c'\overline{B}_{f(z)} \subset f\big(\tfrac{1}{2}\overline{B}_z\big), \qquad z = x, y,$$

the $(C', c'; y')$-k-cap condition follows. As mentioned previously, this implies a $(C''; y')$-k-cap condition, quantitatively. Thus (KC) is quasiconformally quasi-invariant and the proof of Theorem 2.1 is complete.

Quasiextremal distance domains, or QED domains, were introduced by Gehring and Martio [GM]. Later Herron and Koskela [HrK] introduced the weaker variation that they called $\mathrm{QED}^1_{\mathrm{b}}$. Given a domain $G \subset \mathbf{R}^n$, and a closed ball $F \subset G$, we say that G is a $(C; F)$-*QED^1_{b} domain* if

$$C \operatorname{cap}(E, F; G) \ge \operatorname{cap}(E, F), \tag{2.7}$$

whenever $E \subset G \setminus F$ is a closed ball.

$\mathrm{QED}^1_{\mathrm{b}}$ domains and Trudinger domains are closely related. It is shown in [HrK, Theorem 6.1] that every Hölder domain is a $\mathrm{QED}^1_{\mathrm{b}}$ domain, and we already know that every Hölder domain is a Trudinger domain. It follows from [HrK, Proposition 3.6] that the $\mathrm{QED}^1_{\mathrm{b}}$ condition is equivalent to the *a priori* weaker condition where (2.7) is assumed only in the case where $E = c\overline{B}_x$, $0 < c < 1$ is fixed, and $E \subset G \setminus F$. In fact, the capacity estimate is easily verified when x, y are quasihyperbolically close, so it suffices to take $F = \frac{1}{2}\overline{B}_y$ for some fixed $y \in G$, $E = \frac{1}{2}\overline{B}_x$, where $x \in G$ is an arbitrary point for which $k_G(x, y) \ge 2$. With these choices, and capacity estimates (1.1) and (1.2), the $\mathrm{QED}^1_{\mathrm{b}}$ condition for a bounded domain G reduces to the statement that there exist positive constants C, ε such that for all $x \in G$,

$$k_G(x,y) \geq 2 \quad \Longrightarrow \quad \operatorname{cap}\big(\tfrac12 \overline{B}_x, \tfrac12 \overline{B}_y; G\big) \geq \left[C + \varepsilon^{-1} \log\left(\frac{\delta_G(y)}{\delta_G(x)}\right)\right]^{1-n}.$$

Putting this estimate together with (KC), it immediately follows that G is a $(C', \varepsilon; y)$-Hölder domain. In fact using the quasi-invariance of the k-cap condition, we get the following result, which was proved by other methods in Section 6 of [HrK].

Theorem 2.8. *If $G \subset \mathbf{R}^n$ is a bounded QED^1_{b} domain that satisfies a k-cap condition, then G is a Hölder domain. Consequently, the quasiconformal image of a Hölder domain is bounded and QED^1_{b} if and only if it is a Hölder domain.*

Our next aim is to give an example for each dimension $n \geq 2$ of (a quasiconformal image of) a Hölder domain that is not the quasiconformal image of a uniform domain. We first state a lemma, which is essentially Lemma 3.3 of [BK1]; we have added an indication of parameter dependence that is implicit in the proof.

Lemma 2.9. *If $G \subset \mathbf{R}^n$ is a C-uniform domain, and f is a K-quasiconformal mapping from G onto G', then there exists a constant $C_0 = C_0(C,K,n)$ such that G' satisfies the following separation property: if $x,y,w \in G'$ and if w lies on a quasihyperbolic geodesic from x to y, then*

$$\lambda^* \cap B(w, C_0\delta_{G'}(w)) \neq \emptyset \qquad \text{for all } \lambda \in \Gamma_{G'}(x,y)\,. \tag{2.10}$$

Proposition 2.11. *For each $n \geq 2$, there exists a Hölder domain $G \subset \mathbf{R}^n$ which is not the quasiconformal image of any uniform domain.*

Proof. We first give an example G_1 that works for each $n \geq 3$. We treat the final coordinate direction as the "vertical" direction, and let $\pi_* : \mathbf{R}^n \to \mathbf{R}^{n-1}$ and π_n: $\mathbf{R}^n \to \mathbf{R}$ be projection onto the first $n-1$ coordinates and final coordinates respectively.

Letting $a_j = 2^{-j}$, $l_j = 3^{-j}$, and $\varepsilon_j = 4^{-j}$ for each $j \in \mathbf{N}$, we define the domain $G_1 = Q_0 \cup \big(\bigcup_{j=1}^\infty (Q_j \cup N_j)\big)$, which consists of a central cube $Q_0 = (0,1)^n$ to which are attached the peripheral cubes

$$Q_j = (a_j, a_j + l_j)^{n-1} \times (-l_j - \varepsilon_j, -\varepsilon_j), \qquad j \in \mathbf{N},$$

via the necks

$$N_j = (a_j, a_j + l_j)^{n-2} \times (a_j, a_j + \varepsilon_j) \times [-\varepsilon_j, 0], \qquad j \in \mathbf{N}.$$

Let us show, in every dimension $n \geq 2$, that G_1 is a Hölder domain with respect to z_0, the center of Q_0. Writing z_j for the center of Q_j, any quasihyperbolic geodesic γ_j from z_j to z_0 has to pass through the bottleneck N_j but this does not invalidate the Hölder condition because the inradius of N_j is comparable to a fixed power of the diameter of Q_j, and the length of this bottleneck is comparable with its inradius. In fact, by using a path consisting of three straight line segments as a test path, it is easy to see that

$$\begin{aligned}\operatorname{len}_{k;G_1}(\gamma_j) &= \operatorname{len}_{k;G_1}(\gamma_j^* \cap Q_j) + \operatorname{len}_{k;G_1}(\gamma_j^* \cap N_j) + \operatorname{len}_{k;G_1}(\gamma_j^* \cap Q_0) \\ &\lesssim \int_{4^{-j}}^{3^{-j}} \frac{dt}{t} + 1 + \int_{4^{-j}}^{1} \frac{dt}{t} \\ &\approx j \approx \log\bigl(1/\delta_{G_1}(z_j)\bigr).\end{aligned} \tag{2.12}$$

Additionally, it is easily verified that Q_j is itself a $\bigl(1/\sqrt{n}\,, 0; z_j\bigr)$-Hölder domain for each j (with all Hölder paths being straight lines). Putting together this fact, (2.12), the inequality $k_G(u, z_0) \le k_G(u, z_j) + k_G(z_j, z_0)$, and the fact that $k_G|_{Q_j}$ is a smaller metric that k_{Q_j}, we get a Hölder estimate (with respect to z_0) for all points $u \in Q_j$ which is uniform in j. As for Q_0, the Hölder estimate there follows almost immediately from the fact that Q_0 itself is a Hölder domain.

Finally, suppose u is a point in a neck N_j. Let $R_j \subset \mathbf{R}^n$ be the $(n-2)$-dimensional rectangle given by

$$R_j = \bigl\{z = (z', z_{n-1}, z_n) : z' \in [a_j + \varepsilon_j, a_j + l_j - \varepsilon_j]^{n-2},\ z_{n-1} = a_j + \tfrac{1}{2}\varepsilon_j,\ z_n = 0\bigr\},$$

let u' be the point in R_j closest to u, let u'' be the point with $\pi_*(u'') = \pi_*(u')$ and $\pi_n(u'') = \pi_n(u)$. We leave it to the reader to verify that the path which consists of three line segments from u to u'' to u' to z_0 is a Hölder path, with constants uniform over all such u and j.

If $n \ge 3$, G_1 does not satisfy the separation property (2.10) uniformly for all choices of data x, y, w; to see this, take $x = z_0$, $y = z_j$, and let $w = w_j$ be a point on the connecting quasihyperbolic geodesic whose final coordinate is $-\frac{1}{2}\varepsilon_j$. The elongated shape of cross-sections of N_j requires us to take $C_0 \ge l_j/\varepsilon_j$ in order for (2.10) to be valid. Thus (2.10) fails for any fixed C_0 when we let j tend to infinity, and so G_1 cannot be the quasiconformal image of a uniform domain.

Note that the above example G_1 cannot work in the plane because it is simply-connected and so the (quasi-)conformal image of a uniform domain (namely, the unit disk). The domain G in Theorem 3.6 below would suffice (since the quasiconformal image of a uniform domain would have to satisfy a wslice condition), but let us instead give a simpler example, namely

$$G_2 = (0,1)^2 \cup \left(\bigcup_{j=1}^{\infty} Q_j \cup N_j^1 \cup N_j^2 \right),$$

where

$$\begin{aligned} Q_j &= (a_j, a_j + l_j) \times (-l_j - \varepsilon_j, -\varepsilon_j), \\ N_j^1 &= (a_j, a_j + \varepsilon_j) \times [-\varepsilon_j, 0], \\ N_j^2 &= (a_j + l_j - \varepsilon_j, a_j + l_j) \times [-\varepsilon_j, 0], \end{aligned}$$

and a_j, l_j, and ε_j are as in the earlier example. As before, we see that G_2 is Hölder. It does not satisfy (2.10) uniformly because N_j and N_j' are much further apart than their inradius. Thus G_2 is not the quasiconformal image of a uniform domain. □

3. Weak slice versus k-cap

The original slice condition was defined in [BK2], where it was used to connect Sobolev imbeddings with the geometry of a domain. Weak slice conditions [1] were then introduced in [BO] and [BS1], and used to prove various refinements of these results. The fact that every quasiconformal image of a uniform domain satisfies a slice (and hence weak slice) condition, is exploited in [BS2] to classify the quasiconformal images of uniform domains which are Cartesian products of domains in lower dimensions. In [BB], it is shown that all such slice conditions hold on domains where the quasihyperbolic metric is Gromov hyperbolic, and conversely that a variant of the two-sided slice condition is equivalent to Gromov hyperbolicity.

In this section, we show that the k-cap condition is implied by almost all the slice-type conditions in the literature (and by a few new ones), but that there are no such results in the converse direction (with the possible exception of a capacitary weak slice condition that we introduce below). We also show that, unlike the k-cap condition, the weak slice condition is not quasiconformally quasi-invariant.

Let $C \ge 1$ and $x, y \in G \subsetneq \mathbf{R}^n$. A *set of C-wslices* for x, y is a finite collection $\mathscr{F}$ of pairwise disjoint open subsets of G such that for each $S \in \mathscr{F}$ we have for all $\lambda \in \Gamma_G(x,y)$:

$$\text{(W-1)} \qquad \operatorname{len}(\lambda^* \cap S) \ge \operatorname{dia}(S)/C;$$

$$\text{(W-2)} \qquad (C^{-1}B_x \cup C^{-1}B_y) \cap S = \emptyset.$$

Next let

$$d_{\mathrm{W}}(x,y;G;C) = \sup\bigl\{\operatorname{card}(\mathscr{F}) \mid \mathscr{F} \text{ is a set of C-wslices for } x,y\bigr\}.$$

A priori, $d_{\mathrm{W}}(x,y;G;C)$ could be any non-negative integer or even infinity but in reality it is bounded. In fact, there exists a constant $C' = C'(C)$ such that

$$\text{(3.1)} \qquad d_{\mathrm{W}}(x,y;G;C) \le C'[1 + k_G(x,y)].$$

This follows from Lemma 2.3 of [BS1], or from (3.3) below.

We define wslice conditions essentially by reversing (3.1) for large $k_G(x,y)$. More precisely, we say that x, y satisfy the *C-wslice inequality* on G if

$$\text{(W-3)} \qquad k_G(x,y) \le C(d_{\mathrm{W}}(x,y;G;C) + 1).$$

---

[1] so-called because they are weaker than the slice condition in [BK2].

If (W-3) holds for all $x \in G$, and fixed $y \in G$, we say that G is a *one-sided* $(C;y)$*-wslice domain*, while if (W-3) holds for all $x, y \in G$, we say that G is a *two-sided* C*-wslice domain.* This weak slice condition was introduced in [BO, Section 5], and is essentially the $\alpha = 0$ case of the Euclidean wslice conditions of [BS1] and [BS2]. It is clear that the concept of a one-sided wslice domain is independent of base point y (but different choices of y might necessitate different choices of C).

On a general domain $G \subsetneq \mathbf{R}^n$, we have the estimate

$$C \geq 4 \quad \Longrightarrow \quad d_{\mathrm{W}}(x, y; G; C) \geq m_0 \equiv 0 \vee \left\lfloor \log_2 \left(\frac{|x-y|}{\delta_G(x) \wedge \delta_G(y)} \right) \right\rfloor. \tag{3.2}$$

By swapping x and y if necessary, it suffices to prove this estimate under the assumption that $\delta_G(x) \leq \delta_G(y)$. We then pick as a set of wslices the concentric annuli $A_i = B\big(x, 2^{i-2}\delta_G(x)\big) \setminus B\big(x, 2^{i-3}\delta_G(x)\big)$ for $1 \leq i \leq m_0$.

Inequality (3.2) gives us the first of an important string of inequalities that hold on all bounded domains G, for all points x, y, $k_G(x, y) \geq 2$:

$$\begin{aligned} \log\left(1 + \frac{|x-y|}{\delta_G(x) \wedge \delta_G(y)}\right) &\lesssim d_{\mathrm{W}}(x, y; G, C) \\ &\lesssim \operatorname{cap}^{-1/(n-1)}\big(C^{-1}\overline{B}_x, C^{-1}\overline{B}_y; G\big) \lesssim k_G(x, y). \end{aligned}$$

The second inequality here follows from (3.4) below, and the third inequality follows from (2.2). Note that reversing the last inequality uniformly for all x gives a one-sided k-cap condition, similarly reversing the last two inequalities gives a wslice condition, and similarly reversing all three inequalities gives a Hölder domain. If the reversed inequalities hold uniformly for all x and y, we get two-sided k-cap and wslice conditions, and uniform domains. In particular, a Hölder domain always satisfies a one-sided wslice condition, and if (1.4) holds for a pair of points $x, y \in G$ (as it does if there exists a uniform path from x to y), then a wslice inequality holds for $x, y \in G$.

We now derive a simple but useful slice estimate. Given a subset E of G, a finite subset $\mathscr{F}$ of 2^G, and a number $t > 0$, let $\mathscr{L}(\mathscr{F}, E, t)$ be the collection of those $S \in \mathscr{F}$ such that $\operatorname{len}(E \cap S) \geq t \operatorname{dia}(S)$, and let $N(\mathscr{F}, E, t)$ be the cardinality of $\mathscr{L}(\mathscr{F}, E, t)$. Suppose the given family $\mathscr{F}$ is a set of C-wslices for $x, y \in G$, and so $\delta_G(w) < \frac{1}{2}(C+1)\operatorname{dia}(S)$ for all $w \in S \in \mathscr{F}$ according to [BS1, Lemma 2.2]. Consequently,

$$\operatorname{len}_{k;G}(A) > \frac{2\operatorname{len}(A)}{(C+1)\operatorname{dia}(S)}, \qquad A \subset S \in \mathscr{F},$$

and so if $E \subset G$ and if $\mathscr{F}$ is a set of C-wslices for $x, y \in G$, then

$$\begin{aligned}\operatorname{len}_{k;G}(E) &\geq \sum_{S\in\mathscr{L}(\mathscr{F},E,t)} \operatorname{len}_{k;G}(E\cap S) \geq \sum_{S\in\mathscr{L}(\mathscr{F},E,t)} \frac{2\operatorname{len}(E\cap S)}{(C+1)\operatorname{dia}(S)} \\ &\geq \frac{2tN(\mathscr{F},E,t)}{C+1}\end{aligned}$$

which we rewrite as the desired estimate

$$N(\mathscr{F},E,t) \leq \frac{(C+1)\operatorname{len}_{k;G}(E)}{2t}. \tag{3.3}$$

Note that if we take $t = 1/C$ and $E = \gamma^*$, where γ is a quasihyperbolic geodesic from x to y, then (3.3) gives (3.1).

A wslice inequality always implies a k-cap inequality. The key to proving this is a construction that associates a capacity test function $u_{\mathscr{F}}$ with any set $\mathscr{F}$ of wslices, although it suits us to define this in the more general context of a family $\mathscr{F} = \{S_i\}_{i=1}^m$ of open subsets of G (and a pair of points $x, y \in G$). We define

$$u_i(z) = \left[\inf_{\lambda\in\Gamma_G(z,x)} \operatorname{len}(\lambda^*\cap S_i)\right], \qquad z \in G,\ 1 \leq i \leq m,$$

and, assuming $u_i(y) > 0$ for all $1 \leq i \leq m$, we also define the *function $u_{\mathscr{F}}$ associated with $\mathscr{F}$* by the equation

$$u(z) = m^{-1}\sum_{i=1}^m \frac{u_i(z)}{u_i(y)}, \qquad z \in G.$$

Note that if $x \in E$, $y \in F$, where E, F are compact subsets of G and the sets in $\mathscr{F} \cup \{E, F\}$ are pairwise disjoint, then any such function $u_{\mathscr{F}}$ is a capacity test function for the triple $(E, F; G)$ in the sense that it is Lipschitz, is constantly zero on E, and is constantly 1 on F.

In particular, if $\mathscr{F} = \{S_i\}_{i=1}^m$ satisfies (W-1), and the family $\mathscr{F} \cup \{E, F\}$ is pairwise disjoint, where E and F are compact subsets of G containing x and y respectively, then $\|\nabla u_i(\,\cdot\,)/u_i(y)\|_{L^\infty(G)} \leq C/\operatorname{dia}(S_i)$, and thus

$$\operatorname{cap}(E,F;G) \leq \sum_{i=1}^m \int_{\overline{S_i}} |\nabla u_{\mathscr{F}}|^n \leq \sum_{i=1}^m \frac{C^n|\overline{S_i}|}{m^n \operatorname{dia}(S_i)^n} \leq m^{1-n}C^n. \tag{3.4}$$

Taking $E = C^{-1}\overline{B}_x$, $F = C^{-1}\overline{B}_y$, we readily deduce the following result.

Proposition 3.5. *Every one-sided $(C; y)$-wslice domain in $\mathbf{R}^n$ satisfies a $(C'; y)$-k-cap condition, where $C' = C'(C)$.*

Inequality (3.4) has many other uses. Together with (3.2), it implies the special case $E = C^{-1}\overline{B}_x$, $F = C^{-1}\overline{B}$ of (1.1). More generally, the concentric annuli used to prove (3.2) also give the full-strength version of (1.1). To see this, suppose E and F are compact subsets of G with $\Delta(E,F) \ge 2$. By symmetry, we may suppose that $\mathrm{dia}(E) \le \mathrm{dia}(F)$. Then (1.1) follows by taking $\mathscr{F} = \{S_i\}_{i=1}^m$, where

$$S_i = \bigl\{z \in G : 2^{i-1}\,\mathrm{dia}(E) < |z-x| < 2^i\,\mathrm{dia}(E)\bigr\}, \qquad 1 \le i \le m,$$

and $m+1$ is the least integer i for which $B\bigl(x_0, 2^i\,\mathrm{dia}(E)\bigr)$ intersects F.

We next wish to give a domain $G \subset \mathbf{R}^2$ which shows that one-sided wslice conditions are not conformally invariant (and so the reverse of the implication in Proposition 3.5 is false), but we first pause for some preliminary definitions. First, let us define one particular type of wslice sets that are needed repeatedly. By the *annular slices around* x with *maximum radius* r, $r \ge \delta_G(x)$, we mean the collection of open sets

$$S_i = \bigl\{z \in G : 2^{i-1}\delta_G(x) < |z-x| < 2^i\delta_G(x)\bigr\}, \qquad 0 \le i \le m,$$

where m is the largest non-negative integer i satisfying $2^i\delta_G(x) \le r$. The *inner annular slices around* x with *maximum radius* r, $r \ge \delta_G(x)$, are the analogous collection of inner Euclidean annuli, i.e. we simply replace $|z-x|$ by $l_G(z,x)$ in the previous definition.

For this paragraph, let d denote either the inner Euclidean or Euclidean metric, according to whether or not each sentence is read with the word “inner” included. For a collection of (inner) annular slices to be a set of wslices for x, y, we need that the maximal radius r is at most $d(x,y) - \frac{1}{2}\delta_G(y)$, but typically we use only enough annular slices to cover a part of the path from x to y, so r may be much smaller than this bound. We use (inner) annular slices only when there is an (inner) uniform path from x to a point z with $d(x,z)$ approximately equal to r, which guarantees that $m+1$ is comparable with $1 + k_G(x,z)$.

We are now ready to give the example of a domain satisfying a one-sided wslice condition which is quasiconformally equivalent to a domain that does not satisfy a wslice condition. Let $a_j = 2^{-j}$, $l_j = 3^{-j}$, $\varepsilon_j = 3^{-2j}/2$, and let $g_j\colon [-l_j, 0] \to \mathbf{R}$ be the function which linearly interpolates between the following values:

$$g_j(0) = g_j(-l_j) = a_j + 2\varepsilon_j, \qquad g_j\bigl(-\tfrac{1}{2}l_j\bigr) = a_j + l_j.$$

Now let

$$G = Q_0 \cup \left(\bigcup_{j=1}^{\infty} Q_j \cup N_j^1 \cup N_j^2\right),$$

$$G' = Q_0 \cup \left(\bigcup_{j=1}^{\infty} Q_j \cup N_j^1 \cup N_j^3\right),$$

where $Q_0 = (0,1)^2$ and, for each $j \in \mathbf{N}$,

$$\begin{aligned}
Q_j &= (a_j, a_j + 2\varepsilon_j) \times (-l_j - 2\varepsilon_j, -l_j),\\
N_j^1 &= (a_j, a_j + \varepsilon_j) \times [-l_j, 0],\\
N_j^2 &= (a_j + \varepsilon_j, a_j + 2\varepsilon_j) \times [-l_j, 0],\\
N_j^3 &= \bigl\{(x,y) \in \mathbf{R}^2 : -l_j \le y \le 0,\ g_j(y) - \varepsilon_j < x < g_j(y)\bigr\}.
\end{aligned}$$

Theorem 3.6. *Both of the domains G and G' defined above satisfy a two-sided k-cap condition, and there is a quasiconformal mapping from G onto G'. However, G satisfies a one-sided wslice condition, while G' does not. Furthermore G is quasiconformally equivalent to a Hölder domain.*

Proof. Let us first make some definitions. Let z_j be the center of Q_j, let $N_j = N_j^1 \cup N_j^2$, and let $U_j = N_j \cup Q_j$. Let $\pi_k\colon \mathbf{R}^2 \to \mathbf{R}$ be projection on the kth coordinate, $k = 1, 2$, and define

$$\begin{aligned}
\widetilde{Q}_0 &= Q_0 \cup \left(\bigcup_{i=1}^{\infty} \bigl\{x \in N_j : \pi_2(x) > -\varepsilon_i\bigr\} \right),\\
\widetilde{Q}_j &= Q_j \cup \bigl\{x \in N_j : \pi_2(x) < -l_j + \varepsilon_j\bigr\},
\end{aligned}$$

for each $j \in \mathbf{N}$. Note that each $\widetilde{Q}_j$, $j \ge 0$, consists of a main square with (either two or infinitely many) smaller squares attached, and that every $\widetilde{Q}_j$, $j \ge 0$, is a uniform domain (with uniformity constant bounded independent of j). We say that two positive numbers A and B are *roughly comparable* if $A \le C(1+B)$ and $B \le C(1+A)$, for some universal constant C. We use $[u,v]$ to denote any quasihyperbolic geodesic segment between u and v, and we write $[v_1, \dots, v_m]$ for the path between v_1 and v_m formed by concatenating the geodesics segments $[v_i, v_{i+1}]$, $1 \le i < m$.

As well as the previously defined (inner) annular slices, there is one other type of wslices that we shall need. Suppose $u = (a,b)$ and $v = (a,c)$ are points in N_j, with a equal to either $a_j + \frac{1}{2}\varepsilon_j$ or $a_j + \frac{3}{2}\varepsilon_j$, and $b - c \ge 2\varepsilon_j$. We define *the box slices for u, v* to be the collection of open sets

$$S_i = \bigl\{z = (z_1, z_2) \in N_j : (i-1)\varepsilon_j < z_2 - c - \tfrac{1}{2}\varepsilon_j < i\varepsilon_j\bigr\}, \qquad 1 \le i \le (b-c-\varepsilon_j)/\varepsilon_j.$$

Note that $k_G(u,v) = 2(b-c)/\varepsilon_j$ is roughly comparable with the number of box slices.

We first show simultaneously that G satisfies a two-sided k-cap condition and a one-sided wslice condition for G (with basepoint z_0). Let $x, y \in G$ be arbitrary. We assume, as we may, that $\delta_G(x) \le \delta_G(y)$, and that $k_G(x,y) \ge 2$. If there is a uniform path for the points x, y, then (1.4) and (2.5) together imply a k-cap inequality for x, y, and similarly (1.4) and (3.2) imply a wslice inequality. Clearly there is such a path (with uniformly bounded C) if x, y lie in $\widetilde{Q}_j$ for the same value of $j \geq 0$.

In most other cases, we shall define a concatenated path $\gamma = [x, w_1, \ldots, w_m, y]$ connecting x and y, and we shall associate sets of wslices for x, y with each of the component segments. The cardinality of each wslice set will be roughly comparable with the quasihyperbolic length of the associated segment, and there will only be a bounded number of segments, so we deduce (W-3) by taking the wslice set associated with the quasihyperbolically longest of these segments. The k-cap inequality for these pair of points then follows by (3.4).

Suppose one of x, y lies in Q_0 and the other lies in Q_j for some fixed $j \in \mathbf{N}$; without loss of generality $x \in Q_0$ and $y \in Q_j$. We define points $u_j = \bigl(a_j + \frac{1}{2}\varepsilon_j, 0\bigr)$ and $v_j = \bigl(a_j + \frac{1}{2}\varepsilon_j, -l_j\bigr)$ and then let $\gamma = [x, u_j, v_j, y]$. We associate with $[x, u_j]$ the annular slices about x with maximum radius $|x - u_j|$, with $[u_j, v_j]$ the box slices for u_j, v_j, and with $[v_j, y]$ the annular slices about y with maximum radius $|y - v_j|$. It is easy to verify that each of these sets of slices satisfy the wslice condition for the points x, y, and that each has cardinality roughly comparable with the quasihyperbolic length of the associated segment. The k-cap inequality therefore follows for x, y in this case.

The case where $x \in Q_0$ and $y \in N_j^1 \setminus \widetilde{Q}_0$ is formally identical except for one change: v_j is replaced by the point $\bigl(a_j + \frac{1}{2}\varepsilon_j, \pi_2(y)\bigr)$. Note that the only difficulty in verifying that the various sets of slices form wslice sets is to verify that the annular slices satisfy (W-2), and this follows from the estimate $\operatorname{dist}(y, Q_0) \geq \varepsilon_j$, which in turn holds because $y \notin \widetilde{Q}_0$. Note also that the case $x \in Q_0$, $y \in \widetilde{Q}_0$ has already been covered.

The case where $x \in Q_0$, $y \in N_j^2 \setminus \widetilde{Q}_0$ is similar: u_j is replaced by $\bigl(a_j + \frac{3}{2}\varepsilon_j, 0\bigr)$, and v_j by $\bigl(a_j + \frac{3}{2}\varepsilon_j, \pi_2(y)\bigr)$. The cases where $x \in Q_j$ and y lies in either $N_j^1 \setminus \widetilde{Q}_j$ or $N_j^2 \setminus \widetilde{Q}_j$ are also similar and left to the reader.

By symmetry considerations, there remain only two cases to consider: the case $x \in U_i$, $y \in U_j$, with $j > i > 0$, and the case $x, y \in N_j$, $j > 0$. The former case actually splits into several subcases but all are handled like the earlier cases. For instance if $x \in Q_i$, $y \in Q_j$, we consider the path $[x, v_i, u_i, u_j, v_j, y]$. The sets of wslices associated with the component segments are as before, except for the segment $[u_i, u_j]$, with which we associate wslices given by the intersection with Q_0 of the annular slices about u_j with maximum radius $|u_i - u_j|$; intersecting with Q_0 ensures that (W-2) is satisfied.

Finally, let us tackle the case $x, y \in N_j$, $j > 0$. We may assume that $l_G(x, y) > 2\varepsilon_j$, since otherwise x and y can be connected by a uniform path. If x and y both lie in either N_j^1 or N_j^2, a wslice inequality (and so also a k-cap inequality) follows by considering annular and box slices as before. Suppose therefore that $x \in N_j^1$, $y \in N_j^2$. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we consider the path $[x, w_1, \ldots, w_4, y]$, where $w_1 = \big(a_j + \frac{1}{2}\varepsilon_j, x_2\big)$, $w_2 = \big(a_j + \frac{1}{2}\varepsilon_j, p\big)$, $w_3 = \big(a_j + \frac{3}{2}\varepsilon_j, p\big)$, and $w_4 = \big(a_j + \frac{3}{2}\varepsilon_j, y_2\big)$, where p is either 0 or $-l_j$, depending on which of the two choices minimizes the sum $|w_1 - w_2| + |w_3 - w_4|$ (and hence also the sum $k_G(w_1, w_2) + k_G(w_3, w_4)$).

The path $[w_2, w_3]$ is harmless as it has uniformly bounded quasihyperbolic length. As for the (straight line) segments $[x, w_1]$ and $[w_4, y]$, we associate with them inner annular slices about x and y, respectively, with maximum radius $\frac{1}{2}\varepsilon_j$ in both cases. Because $l_G(x, y) \geq \varepsilon_j$, these sets of inner annular slices are sets of wslices for x, y, and their cardinalities are roughly comparable to the quasihyperbolic lengths of the associated segments. A wslice condition for x, y (and hence the associated k-cap condition) therefore follows if $K_1 \equiv \mathrm{len}_{k;G}(\gamma)$ is roughly comparable to $K_2 \equiv \mathrm{len}_{k;G}[x, w_1] + \mathrm{len}_{k;G}[w_4, y]$. This rough comparability fails precisely when either or both of $K_3 \equiv \mathrm{len}_{k;G}[w_1, w_2]$ and $K_4 \equiv \mathrm{len}_{k;G}[w_3, w_4]$ is much larger than $K_2 + 1$. It is only because of this case that the two-sided wslice condition for G fails (we leave this failure as an exercise to the reader since it is irrelevant to our theorem).

It remains to prove a k-cap condition in the case where the two-sided wslice condition fails, i.e. when $K_3 \vee K_4$ is much larger than $K_2 + 1$, and so comparable with K_1. By symmetry it suffices to consider the case where $|w_1 - w_2| > |w_3 - w_4|$, and so $K_4 \lesssim K_3 \approx L/\varepsilon_j \approx K_1$, where $L = \big(\frac{1}{2}l_j\big) \wedge |w_1 - w_2|$. Let $u\colon G \to [0, 1]$ be the function which is constantly 1 on $G \setminus N_j^1$, and satisfies $u(z) = f\big(|\pi_2(z) - \pi_2(x)|\big)$ for all $z \in N_j^1$, where f is the piecewise linear interpolating function for the values $f(0) = f\big(\frac{1}{2}\varepsilon_j\big) = 0$, $f\big(L - \frac{1}{2}\varepsilon_j\big) = f(\infty) = 1$. A straightforward calculation shows that $\int_G |\nabla u|^2$ is roughly comparable with ε_j/L, and so with $1/K_1$. Thus we have proved a k-cap condition in all cases, together with a one-sided wslice condition.

It is easy to see that $G' = f(G)$ for some quasiconformal map f. For instance, we first define $h_j\colon N_j^2 \to N_j^3$ by the equation $h_j(x, y) = \big(x + g_j(y) - a_j - 2\varepsilon_j, y\big)$, so that each h_j is bilipschitz, with bilipschitz constant less than $\sqrt{5}$. We then define f to be the map which satisfies $f|_{N_j^2} = h_j$ for each $j \in \mathbf{N}$, and which is the identity map off the sets N_j^2. Then f is locally bilipschitz and so quasiconformal. The k-cap condition for G' follows by quasiconformal quasi-invariance from that for G.

On the other hand, we now show that for fixed $C \in [1, \infty)$ and sufficiently large j, the pair (z_0, z_j) fails to satisfy the C-wslice condition for G'. For a given number $j \in \mathbf{N}$, we suppose that $\mathscr{F}$ is a collection of wslices large enough to verify the C-wslice condition for the pair $z_0, z_j \in G'$. We define a pair of injective polygonal (i.e. piecewise straight) paths γ^1 and γ^3 that go through the points $u^1 = \big(a_j + \frac{1}{2}\varepsilon_j, 0\big)$, $v^1 = \big(a_j + \frac{1}{2}\varepsilon_j, -l_j\big)$, $u^3 = \big(a_j + \frac{3}{2}\varepsilon_j, 0\big)$, $v^3 = \big(a_j + \frac{3}{2}\varepsilon_j, -l_j\big)$, and $w^3 = \big(g_j\big(-\frac{1}{2}l_j\big) - \frac{1}{2}\varepsilon_j, -\frac{1}{2}l_j\big)$. Specifically, γ^1 is a polygonal path from z_0 to u^1 to v^1 to z_j, and γ^3 is a polygonal path from z_0 to u^3 to w^3 to v^3 to z_j;

the parametrizations are irrelevant. Note that $L_j^1 \equiv (\gamma^1)^* \cap N_j^1$ is the midline of N_j^1, and $L_j^3 \equiv (\gamma^3)^* \cap N_j^3$ is the "bent midline" of N_j^3. Let E consist of the union of the two segments of $(\gamma^1)^*$ from z_0 to u^1, and from v^1 to z_j, and the two segments of $(\gamma^3)^*$ from z_0 to u^3, and from v^3 to z_j. Clearly $\mathrm{len}_{k;G}(E) \approx j$, and so by (3.3), we have

$$N(\mathscr{F}, E, 1/2C) \le (C^2 + C)\,\mathrm{len}_{k;G}(E) \lesssim j. \tag{3.7}$$

However the quasihyperbolic distance from the top of N_j^1 (or N_j^3) to the bottom is approximately 3^j, and so $k_G(z_0, z_j) \approx 3^j$. Combining (W-3) and (3.7), we deduce that the cardinality of $\mathscr{F}' \equiv \mathscr{F} \setminus \mathscr{L}(\mathscr{F}, E, 1/2C)$ is comparable to 3^j, when j is sufficiently large (which we henceforth assume).

Next, subdivide L_j^1 into pieces

$$P_i = \bigl\{ z \in L_j^1 : 2^{i-1}\varepsilon_j \le \mathrm{dist}\bigl(z, L_j^3\bigr) < 2^i \varepsilon_j \bigr) \bigr\}, \quad i \in \mathbf{Z}.$$

It follows from the construction that P_i is empty for $i < 0$ and for $i > 1 + j\log_2 3$, and that $\mathrm{len}(\widetilde{P}_i) \lesssim 2^i\varepsilon_j$, where

$$\widetilde{P}_i = \bigl\{ z \in L_j^1 : \mathrm{dist}(z, L_j^3) < 2^i\varepsilon_j \bigr\}, \quad i \in \mathbf{Z}.$$

Applying (W-1) to γ^1 and γ^3, we deduce that

$$\left.\begin{aligned} \mathrm{len}(L_j^1 \cap S) &\ge \mathrm{dia}(S)/2C, \\ \mathrm{len}(L_j^3 \cap S) &\ge \mathrm{dia}(S)/2C, \end{aligned}\right\} \quad S \in \mathscr{F}'. \tag{3.8}$$

We partition the elements of $\mathscr{F}'$ into subsets $\mathscr{F}_i'$ by the rule $S \in \mathscr{F}_i'$ if S intersects P_i but not $P_{i'}$ for any $i' > i$. It follows that $\mathrm{dia}(S) \gtrsim 2^i\varepsilon_j$ for each $S \in \mathscr{F}_i'$, and hence from (3.8) and the length of $\widetilde{P}_i$ that each $\mathscr{F}_i'$ has bounded cardinality. Thus the cardinality of $\mathscr{F}'$ is at most comparable with j. This contradicts the earlier cardinality estimate for $\mathscr{F}'$ whenever j is sufficiently large. Consequently, the pair (z_0, z_j) fails to satisfy a C-wslice condition.

It remains to prove that G is quasiconformally equivalent to a Hölder domain H. To see this we define a quasiconformal mapping $f\colon G \to H = f(G)$ which is the identity map on Q_0, and such that for all $z = (a_j + h, y) \in U_j$, $j \in \mathbf{N}$, we have $f(z) = (u, v)$, where $u = a_j + \exp(\varepsilon_j^{-1} y)h$ and $v = \varepsilon_j\bigl(\exp(\varepsilon_j^{-1} y) - 1\bigr)$. Note that f transforms the elongated attachments $\{U_j\}_{j=1}^\infty$ into a sequence of truncated triangular regions. It is easily verified that H is a Hölder domain. □

Certain slice-type conditions that are strictly weaker than wslice conditions also imply a k-cap condition. In particular, it is clear that the required estimates for the construction in the paragraphs prior to Proposition 3.5 work also if, in place of a collection of wslices, we have a similar number of functions $\{u_i\}_{i=1}^m \subset W^{1,n}(G)$ each of which equals 0 on $B\big(x, \delta_G(x)/C\big)$ and 1 on $B\big(y, \delta_G(y)/C\big)$, and whose gradients have pairwise disjoint supports (modulo sets of measure zero) and satisfy $\|\nabla u\|_{L^n(G)} \le C$. Such a *capacitary-wslice condition* seems very similar to the k-cap condition. In particular, it is quasiconformally quasi-invariant for similar reasons, and so it would suffice as a tool in the proof of Theorem 2.1 in place of the k-cap condition. It implies the k-cap condition, but we do not know if they are equivalent.

Another slice-type condition is obtained when we replace the assumption (W-1) in the definition of a wslice condition by the weaker property

$$\text{(W-1a)} \qquad \operatorname{len}(\lambda^* \cap S) \ge |\bar{S}\,|^{1/n}/C \qquad \text{for all } \lambda \in \Gamma_G(x,y).$$

The resulting class of *one-sided area-wslice domains* strictly contains the usual class of one-sided wslice domains. It fact, it is straightforward to modify part of the proof of Theorem 3.6 to prove that G, G' are one-sided area-wslice domains.

It is clear that any area-wslice condition implies a capacitary-wslice condition. However, unlike the capacitary version, the one-sided area-wslice condition is not quasiconformally quasi-invariant. To see this, we begin with the domain G in Theorem 3.6. We then apply a quasiconformal mapping f to G' which is the identity map on each Q_j, $j \ge 0$, and N_j^3, $j \ge 1$, but which sends each N_j^1 onto a "pinched rectangle" consisting of an irregular hexagon $v_1 v_3 v_4 v_5 v_6 v_2$ with vertices

$$\begin{aligned}
v_1 &= (a_j, 0),\\
v_2 &= (a_j + \varepsilon_j, 0),\\
v_3 &= (a_j, -l_j),\\
v_4 &= (a_j + \varepsilon_j, -l_j),\\
v_5 &= \big(g_j\big(-\tfrac12 l_j\big) - \varepsilon_j, -\tfrac12 l_j\big),\\
v_6 &= \big(a_j + \varepsilon_j \exp(-l_j/\varepsilon_j), -\varepsilon_j\big).
\end{aligned}$$

We leave it to the reader to construct such a quasiconformal map f, and to verify that any C-area-wslice condition fails for points $f(z_0)$, $f(z_j)$, if j is sufficiently large. The proof of this last fact is similar to the proof that G' fails to satisfy a wslice condition; intuitively the reason is that if we compare the hyperbolically shortest path from $f(z_0)$ to $f(z_j)$ that passes through $f(N_j^1)$ with the corresponding path passing through $f(N_j^3)$, then the former is mostly contained in the strip $-2\varepsilon_j < x < 0$ (when measured by hyperbolic length), while only a small fraction of the latter is contained in that strip.

We now briefly discuss those types of wslice conditions in [BS1] and [BS2] that have not yet been mentioned. First there are the wslice$^+$ variants, and the variants with respect to metrics that lie between the Euclidean metric and inner

Euclidean metric. These variants give conditions stronger than the corresponding Euclidean wslice conditions so it follows *a fortiori* that these conditions imply, but are not implied by, the k-cap condition.

These earlier papers also allow an additional parameter α: our wslice conditions correspond to the $\alpha = 0$ case. However, unlike all the slice-type conditions that we considered above, the wslice conditions in the case $\alpha > 0$ do not imply a k-cap condition. This is not surprising because, while all the slice conditions we have considered up until now are associated with the quasihyperbolic metric in some sense, the ones corresponding to $\alpha > 0$ are instead associated with a subhyperbolic metric which is rather different from k_G.

For an explicit counterexample, consider the planar domain G which consists of the planar triangle $T = \{(x,y) : 0 < x < 1,\ |y| < x\}$ with 2^j equally spaced points are removed along each line $x = 2^{-j-1}$ for each $j \in \mathbf{N}$. It follows from [BS1, Proposition 4.5] that G is a two-sided (α, C)-wslice domain for each $\alpha > 0$.

On the other hand, we claim that G does not satisfy a one-sided k-cap condition. By Proposition 2.3, it suffices to show that G is a Trudinger domain but not a Hölder domain. Defining $z_i \equiv (3 \cdot 2^{-i-2}, 0)$, $i \geq 0$, it is straightforward to show that $k(z_j, z_0) \approx j^2$. Since $\log\bigl(1/\delta_G(z_j)\bigr) \approx j$, it follows that G is not a Hölder domain. Clearly T is a Hölder domain and so a Trudinger domain. Since $T \setminus G$ is countable, and countable sets are $W^{1,n}$-removable[2], it follows that G supports a Trudinger inequality.

Let us finish with some open questions and a conjecture. Recall that quasiconformal images of Hölder domains satisfy a one-sided k-cap condition.

Question. *Are there bounded domains $G \subsetneq \mathbf{R}^n$, $n \geq 2$, that satisfy a one-sided k-cap condition but are not quasiconformal images of Hölder domains*?

We note that there are certainly unbounded domains that satisfy a (one-sided or even two-sided) k-cap condition but are not the quasiconformal images of Hölder domains. For instance, it is well known that $\mathbf{R}^n \setminus \{0\}$ is not the quasiconformal image of any bounded domain (and Hölder domains are bounded by Lemma 1.5). Nevertheless $\mathbf{R}^n \setminus \{0\}$ satisfies a two-sided k-cap condition since it is a uniform domain. This example hints at the fact that it would be better to use the spherical metric σ instead of the Euclidean metric when considering this and related questions for unbounded domains. Basic concepts such as diameter, length, distance to the boundary, and derived concepts such as Hölder domains and the k-cap condition, should all be redefined in terms of σ. Doing this, $\mathbf{R}^n \setminus \{0\}$ becomes a perfectly good σ-Hölder domain, so the analogue of the above question is open for arbitrary proper subdomains of the Riemann sphere. More generally the same question could be asked for any incomplete rectifiably connected metric space (X, d), where we consider the metric completion $\overline{X}$ to be a superset of X

[2] This fact follows trivially from the ACL characterization of $W^{1,n}$, as stated in [Z2, 2.1.4].

and distance to the boundary means distance to $\overline{X}\setminus X$. For some related analysis in this metric space context, we refer the reader to [BB], [BHK], and [BS3].

Other open questions concern $p \neq n$ versions of Theorem 2.1. Using the results in [BK1], [BK2], and other papers cited therein, we know that a quasiconformal image of a uniform domain lies in a certain class $\mathscr{C}_p$ if and only if it satisfies a certain Sobolev-type imbedding I_p. Briefly, $\mathscr{C}_p$ is the class of John domains if $p < n$, the class of Hölder domains if $p = n$, and a p-dependent class of "weak cigar" (or "local Lipschitz") domains if $p > n$. The imbedding I_p is a Sobolev-Poincaré imbedding for $p < n$, Trudinger's inequality for $p = n$, and a certain Hölder continuity imbedding when $p > n$.

The class $\mathscr{C}_p$ in each case properly contains the class of uniform domains. Suppose G' is the quasiconformal image of a domain $G \in \mathscr{C}_p$, $p \geq 1$. One might hope that G' supports the imbedding I_p if and only if $G' \in \mathscr{C}_p$. By the results listed in [BK1] and [BK2], the "only if" part is the only implication that requires proof. Theorem 2.1 says that the answer is "yes" when $p = n$, but we do not know the answer when $p \neq n$. As a special case, we think that the following result is likely to be true.

Conjecture. *The quasiconformal image $G' \subsetneq \mathbf{R}^n$ of a John domain is a John domain if and only if it supports a Sobolev–Poincaré imbedding $W^{1,p}(G') \hookrightarrow L^p(G')$ for any (and hence all) $p \in (1,n)$.*

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