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MAPPINGS OF FINITE DISTORTION: GLOBAL HOMEOMORPHISM THEOREM

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Abstract. We study mappings of finite distortion between Riemannian manifolds. We establish Väisälä's inequality for moduli of path families and as an application we prove a version of the so-called global homeomorphism theorem for mappings of finite distortion under a subexponential integrability condition on the distortion.

1. Introduction

Zorich's theorem for quasiregular mappings states that every quasiregular local homeomorphism from $\mathbf{R}^n$ into $\mathbf{R}^n$ is a homeomorphism if $n \geq 3$; see [18]. A geometric version of this theorem, known as the Global homeomorphism theorem, states that every quasiregular local homeomorphism from an n-parabolic Riemannian n-manifold into a simply connected Riemannian n-manifold is an embedding if $n \geq 3$; see [19] and [1, 6.30].

Proofs of these results are based on geometrical properties of quasiregular mappings and especially on path family techniques like Poletsky's inequality. In [10] Koskela and Onninen proved that substitutes of the Poletsky inequality and its generalization, Väisälä's inequality, hold for mappings of (sub)exponentially integrable distortion. Thus these mappings have similar geometrical properties as quasiregular mappings in this sense. Using the Euclidean results of Koskela and Onninen we first generalize these inequalities for mappings of finite distortion between Riemannian manifolds. Then we combine Poletsky's inequality with methods introduced by Zorich to extend the Global homeomorphism theorem for mappings of finite distortion.

Let Γ_M^∞ be the family of all paths γ in a Riemannian manifold M whose locus $|\gamma|$ is not contained in any compact subset of M. We say that M is *ω-parabolic*, where $\omega\colon M \to [0,\infty]$ is measurable, if the ω-weighted n-modulus of Γ_M^∞ is zero; see Section 3 for details.

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Theorem 1 (Global homeomorphism theorem). *Let M be a $K^{n-1}(\cdot)$-parabolic Riemannian n-manifold, N a simply connected Riemannian n-manifold, both of dimension $n \geq 3$, and let $f\colon M \to N$ be a local homeomorphism of finite distortion $K(\cdot)$ satisfying the condition* (A). *Then f is a homeomorphism onto its image and the set $N \setminus fM$ has zero n-capacity.*

See Section 2 for the definition of a mapping of finite distortion between Riemannian manifolds.

The proof of the global homeomorphism theorem depends on the following two propositions.

We say that a point $q \in N$ is *an asymptotic limit of f* if there exists a path $\gamma\colon [a,b[\to M$ belonging to Γ_M^∞ such that $(f \circ \gamma)(t) \to q$ as $t \to b$. We denote by $E(f)$ the set of all asymptotic limits of f.

Proposition 2. *Let M be a $K(\cdot)^{n-1}$-parabolic Riemannian n-manifold, N a Riemannian n-manifold, with $n \geq 3$, and let $f\colon M \to N$ be a local homeomorphism of finite distortion $K(\cdot)$ satisfying* (A). *Then $E(f)$ is σ-compact, $\operatorname{cap}_n E(f) = 0$, and $N \setminus fM \subset E(f)$.*

The following proposition extends the well-known simply connectedness result [13, Lemma 3.3] and its generalization [12, Theorem 6.13] to the case of σ-compact sets on Riemannian manifolds.

Proposition 3. *Let N be a connected Riemannian manifold of dimension $n \geq 2$ and let E be a σ-compact subset of N such that $\mathscr{H}^{n-k}(E) = 0$ for some integer $k \in \{1, \dots, n\}$. Then the inclusion $N \setminus E \hookrightarrow N$ induces an isomorphism $\pi_m(N \setminus E) \to \pi_m(N)$ for all $0 \leq m \leq k-1$.*

2. Mappings of finite distortion on Riemannian manifolds

We assume throughout the paper that M and N are C^∞, oriented Riemannian n-manifolds without boundary. We denote by $\Gamma(X)$ the set of measurable vector fields on M and by $\Gamma^k(M)$ the set of C^k-smooth vector fields on M. The measure on M given by the Riemannian volume form will be denoted by m_M.

A locally integrable vector field $X \in \Gamma(M)$ is called *a weak gradient of a function $u \in L^1_{\mathrm{loc}}(M)$* if

$$\int_M \langle X, Y \rangle \, \mathrm{d}m_M = -\int_M u \operatorname{div} Y \, \mathrm{d}m_M$$

holds for every compactly supported vector field $Y \in \Gamma^1(M)$. We denote by ∇u the weak gradient of u. The Sobolev space $W^{1,p}_{\mathrm{loc}}(M)$, $1 \leq p < \infty$, consists of all functions $u \in L^p_{\mathrm{loc}}(M)$ whose weak gradient ∇u belongs to $L^p_{\mathrm{loc}}(M)$.

Let us recall the definition of a mapping of finite distortion of an open set $G \subset \mathbf{R}^n$. We say that $f\colon G \to \mathbf{R}^n$ is a *mapping of finite distortion* if the following conditions are satisfied:

(a) $f \in W^{1,1}_{\text{loc}}(G, \mathbf{R}^n)$.
(b) The Jacobian determinant $J(\,\cdot\,, f)$ of f is locally integrable.
(c) There exists a measurable function $K\colon G \to [1, \infty]$, finite almost everywhere, such that

$$\|Df(x)\|^n \le K(x)J(x, f) \qquad \text{a.e.}$$

Here $Df(x)$ is the formal derivative of f at x, that is, the linear map $\mathbf{R}^n \to \mathbf{R}^n$ defined by the partial derivatives $D_i f(x)$ as $Df(x)e_i = D_i f(x)$. Furthermore, $\|Df(x)\| = \max\{|Df(x)v|\colon v \in \mathbf{R}^n,\ |v| = 1\}$ and $J(x, f) = \det Df(x)$. If $K \in L^\infty(G)$, the condition (b) implies that $f \in W^{1,n}_{\text{loc}}(G)$ and we recover the class of *mappings of bounded distortion*, also called *quasiregular mappings*. It is a deep result of Reshetnyak that a non-constant quasiregular mapping is locally Hölder continuous, discrete, and open; see [14]. Recently these topological properties have been established for mappings of finite distortion under minimal integrability assumptions on the distortion function K; see e.g. [4], [6], and [8].

In this paper we are mainly interested in locally homeomorphic mappings of finite distortion between Riemannian manifolds. In particular, we assume that mappings in question are a priori *continuous*. Therefore we may use local representations to define mappings of finite distortion on Riemannian manifolds.

Definition 4. We say that a continuous mapping $f\colon M \to N$ has *finite distortion* if, for each $x \in M$, there exist orientation-preserving charts (U, φ) at x and (V, ψ) at $f(x)$ such that $fU \subset V$ and that $\psi \circ f \circ \varphi^{-1}\colon \varphi U \to \mathbf{R}^n$ is a mapping of finite distortion in $\varphi U \subset \mathbf{R}^n$.

If $f\colon M \to N$ is a mapping of finite distortion, we define for a.e. $x \in M$ a linear map $Df(x)\colon T_x M \to T_{f(x)} N$ by

$$Df(x) = D\psi^{-1}\big(f(x)\big) \circ D\big(\psi \circ f \circ \varphi^{-1}\big)\big(\varphi(x)\big) \circ D\varphi(x),$$

where φ and ψ are any orientation-preserving chart mappings at x and at $f(x)$, respectively. Observe that the definition of $Df(x)$ is independent of the choice of φ and ψ. Now there exists a measurable (distortion) function $K\colon M \to [1, +\infty]$, finite almost everywhere, such that

$$\|Df(x)\|^n \le K(x)J(x, f) \qquad \text{a.e.,} \tag{1}$$

where $\|Df(x)\| = \max\{|Df(x)v|\colon v \in T_x M,\ |v| = 1\}$ and $J(x, f) = \det Df(x)$. If we want to emphasize the role of the function K in (1), we call f *a mapping of finite distortion* K.

Our main objective in this section is to extend the basic topological and metric properties of mappings of finite distortion from the Euclidean case to the Riemannian setting. The results we consider are valid only under some additional assumptions on the distortion function K. To describe these assumptions, we consider infinitely differentiable and strictly increasing functions, called Orlicz-functions, $\mathscr{A}\colon [0,\infty[\to [0,\infty[$, with $\mathscr{A}(0)=0$ and $\lim_{s\to\infty}\mathscr{A}(s)=\infty$. We say that a mapping f of finite distortion K satisfies the condition (A) if

(A-0) $\exp\bigl(\mathscr{A}(K)\bigr)\in L^1_{\mathrm{loc}}(M)$,

where $\mathscr{A}$ is an Orlicz-function such that

(A-1) $\displaystyle\int_1^\infty \frac{\mathscr{A}'(t)}{t}\,\mathrm{d}t=\infty$, and

(A-2) $t\mathscr{A}'(t)$ increases to ∞ for sufficiently large t.

For the next lemma we recall that, for each $L>1$ and $x\in M$, the Riemannian normal coordinates at x when restricted to a sufficient small neighborhood U of x provide an L-bilipschitz chart $\varphi\colon U\to\mathbf{R}^n$.

Lemma 5. *Let $f\colon M\to N$ be a continuous mapping of finite distortion K. Suppose that (U,φ) and (V,ψ) are orientation-preserving L-bilipschitz charts on M and on N, respectively, such that $fU\subset V$. Then $h=\psi\circ f\circ\varphi^{-1}\colon\varphi U\to\mathbf{R}^n$ is a mapping of finite distortion $L^{4n}K\circ\varphi^{-1}$. Furthermore, if f satisfies* (A), *then so does h.*

Proof. The mapping h has finite distortion by our definition. Since φ and ψ are L-bilipschitz, they satisfy $\|D\psi(\,\cdot\,)\|\le L$, $\|D\varphi^{-1}(\,\cdot\,)\|\le L$, $J(\,\cdot\,,\psi)\ge L^{-n}$, and $J(\,\cdot\,,\varphi^{-1})\ge L^{-n}$. For $x\in\varphi U$, we write $y=\psi\bigl(f\bigl(\varphi^{-1}(x)\bigr)\bigr)$. Then by the chain rule, h satisfies a distortion inequality

$$\begin{aligned}\|Dh(x)\|^n&=\bigl\|D\psi(y)\circ Df\bigl(\varphi^{-1}(x)\bigr)\circ D\varphi^{-1}(x)\bigr\|^n\\&\le\|D\psi(y)\|^n\bigl\|Df\bigl(\varphi^{-1}(x)\bigr)\bigr\|^n\|D\varphi^{-1}(x)\|^n\\&\le L^{4n}K\bigl(\varphi^{-1}(x)\bigr)J(y,\psi)J(\varphi^{-1}(x),f)J(x,\varphi^{-1})\\&=L^{4n}K\bigl(\varphi^{-1}(x)\bigr)J(x,h)\end{aligned}$$

for a.e. $x\in\varphi U$.

Suppose that f satisfies (A) with an Orlicz-function $\mathscr{A}$. Then h satisfies the condition (A) with an Orlicz-function $\tilde{\mathscr{A}}(t)=\mathscr{A}(t/L^{4n})$ since $J(\,\cdot\,,\varphi^{-1})\ge L^{-n}$ in φU. □

Using Lemma 5 we obtain the results of [8, Theorem 1.3] and [9, Theorem 1.1] in our setting.

Theorem 6. *Let $f\colon M\to N$ be a continuous mapping of finite distortion satisfying* (A). *Then f is either constant or both open and discrete. Moreover, if f is non-constant, $J(x,f)>0$ for a.e. $x\in M$, $m_M(B_f)=0$, and f satisfies* Lusin *conditions (N) and (N^{-1}), that is, for every measurable $E\subset M$,*

$$m_M(E)=0\quad\textit{if and only if}\quad m_N(fE)=0.$$

Here B_f is the branch set of f, that is, the set of points $x\in M$ where f fails to be a local homeomorphism.

3. Väisälä's inequality

For the statement and proof of Väisälä's inequality we fix some terminology. Let I and J be intervals on $\mathbf{R}$. If $\alpha: J \to M$ is a subpath of $\beta: I \to M$, that is, $J \subset I$ and $\alpha = \beta \mid J$, we denote $\alpha \subset \beta$.

Let $f: M \to N$ be a continuous, discrete, and open mapping, and let $\beta: [a, b[\to N$ be a path, where $b \in [a, \infty]$. A *maximal f-lifting* of β starting at a point $x \in f^{-1}\big(\beta(a)\big)$ is a path $\alpha: [a, c[\to M$, $a \le c \le b$, such that

(i) $\alpha(a) = x$,
(ii) $f \circ \alpha = \beta \mid [a, c[$, and
(iii) if $c < c' \le b$, then there does not exists a path $\alpha': [a, c'[\to M$ such that $\alpha = \alpha' \mid [a, c[$ and $f \circ \alpha' = \beta \mid [a, c'[$.

Moreover, a maximal f-lifting is *total*, if $f \circ \alpha = \beta$. We use the same terminology for paths defined on compact intervals.

Let Γ be a path family in M, $1 \le p < \infty$, and let $\omega: M \to [0, +\infty]$ be measurable. The *weighted p-modulus of Γ, with the weight ω,* is defined by

$$\mathsf{M}_{p,\omega}(\Gamma) = \inf_{\varrho} \int_M \varrho^p \omega \, \mathrm{d}m_M, \tag{2}$$

where the infimum is taken over all admissible functions ϱ for the path-family Γ, that is, over all Borel functions $\varrho: M \to [0, \infty]$ such that

$$\int_\gamma \varrho \, \mathrm{d}s \ge 1$$

for every locally rectifiable $\gamma \in \Gamma$, see [10]. If $\omega \equiv 1$, (2) defines the usual p-modulus of Γ which we denote by $\mathsf{M}_p(\Gamma)$.

Theorem 7 (Väisälä's inequality). *Let $f: M \to N$ be a continuous, non-constant mapping of finite distortion K satisfying the condition* (A). *Let Γ be a path family in M, Γ' be a path family in N, and m be a positive integer such that the following is true*: *For every path $\beta: I \to N$ in Γ' there are paths $\alpha_1, \dots, \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ for all j and such that for every $x \in M$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices j. Then*

$$\mathsf{M}_n(\Gamma') \le \frac{\mathsf{M}_{n, K_I(\,\cdot\,, f)}(\Gamma)}{m}. \tag{3}$$

Here $i(x, f)$ is the local topological index of f at x; see, for example, [16] or [15]. The weight $K_I(\,\cdot\,, f)$ on the right-hand side of (3) is the *inner distortion function* of f defined by

$$K_I(x, f) = \begin{cases} \dfrac{\|D^\# f(x)\|^n}{J(x, f)^{n-1}}, & \text{if } J(x, f) \ne 0, \\ 1, & \text{if } J(x, f) = 0 \text{ and } \|D^\# f(x)\| = 0, \\ \infty, & \text{if } J(x, f) = 0 \text{ and } \|D^\# f(x)\| \ne 0, \end{cases}$$

where $D^{\#}f(x)$ is the matrix of cofactors of $Df(x)$. The function $K_I(\,\cdot\,,f)$ satisfies a point-wise inequality $K_I(x,f)\le K^{n-1}(x)$, [5, Section 6].

We need the following version of the change of variable formula; see [2] for a more general Euclidean version.

Lemma 8. *Let $f\colon M\to N$ be a continuous mapping of finite distortion satisfying the condition (N). Let $E\subset M$ be a measurable set and $u\colon fE\to[0,\infty]$ a measurable function. Then*

$$\int_E u\big(f(x)\big)J(x,f)\,\mathrm{d}m_M=\int_N u(y)N(y,f,E)\,\mathrm{d}m_N.$$

Proof. Let (U,φ) and (V,ψ) be orientation-preserving charts on M and N, respectively, such that $fU\subset V$. Let $F\subset E\cap U$ be a measurable set, $v\colon\psi(fF)\to\mathbf{R}$, $v(y)=u\big(\psi^{-1}(y)\big)$, and $h=\psi\circ f\circ\varphi^{-1}\colon\varphi U\to\psi V$. Then

$$\begin{aligned}\int_F u\big(f(x)\big)J(x,f)\,\mathrm{d}m_M&=\int_{\varphi F}u\big(f\big(\varphi^{-1}(x)\big)\big)J\big(\varphi^{-1}(x),f\big)J(x,\varphi^{-1})\,\mathrm{d}x\\&=\int_{\varphi F}v\big(h(x)\big)J\big(h(x),\psi^{-1}\big)J(x,h)\,\mathrm{d}x.\end{aligned}$$

Furthermore, by the (Euclidean) change of variables formula [2], we obtain

$$\begin{aligned}\int_{\varphi F}v\big(h(x)\big)J\big(h(x),\psi^{-1}\big)J(x,h)\,\mathrm{d}x&=\int_{\psi V}v(y)J(y,\psi^{-1})N(y,h,\varphi F)\,\mathrm{d}y\\&=\int_V v\big(\psi(y)\big)N\big(\psi(y),h,\varphi F\big)\,\mathrm{d}m_N\\&=\int_V u(y)N(y,f,F)\,\mathrm{d}m_N.\end{aligned}$$

Let $\{(U_i,\varphi_i)\}$ and $\{(V_j,\psi_j)\}$ be atlases of M and N, respectively, consisting of orientation-preserving mappings such that for every i there exists j_i, with $fU_i\subset V_{j_i}$. Let $E_1=E\cap U_1$ and $E_i=(E\cap U_i)\setminus E_{i-1}$ for $i\ge 2$. Then the sets E_i are disjoint and $E=\cup E_i$. The claim follows by applying the equations above to the sets E_i. □

Let us recall the definition of the absolutely precontinuity of a continuous, discrete, and open mapping $f\colon M\to N$ on a path (see [10, p. 18] and [15, p. 40]). Let $\beta\colon I_0\to N$ be a closed rectifiable path, and let $\alpha\colon I\to M$ be a path such that $f\circ\alpha\subset\beta$. Let $s_\beta\colon I_0\to[0,l(\beta)]$ be the length function of β. If s_β is constant on some interval $J\subset I$, β is also constant on J, and the discreteness of f implies that also α is constant on J. It follows that there exists a unique path $\alpha^*\colon s_\beta(I)\to M$ such that $\alpha=\alpha^*\circ(s_\beta\mid I)$. We say that f is absolutely precontinuous on α if α^* is absolutely continuous.

We have the following version of Poletsky's lemma.

Lemma 9. *Let $f: M \to N$ be a continuous mapping of finite distortion satisfying* (A). *Let Γ be a family of paths γ in M such that $f \circ \gamma$ is locally rectifiable and there exists a closed subpath of γ on which f is not absolutely precontinuous. Then $\mathsf{M}_n(f\Gamma) = 0$.*

Proof. Let $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ be smooth atlases of M and N, respectively, such that each φ_i and ψ_j is an orientation-preserving L-bilipschitz mapping, and that for every i there exists j_i satisfying $fU_i \subset V_{j_i}$. Since the modulus of a path family depends only on locally rectifiable paths, we may assume that there are only locally rectifiable paths in $f\Gamma$. Let $\widetilde{\Gamma}$ be the set of paths α in M such that α is a closed subpath of some $\gamma \in \Gamma$ and f is not absolutely precontinuous on α. Every path $\alpha \in \widetilde{\Gamma}$ has a subpath β such that f is not absolutely precontinuous on β and $|\beta| \subset U_i$ for some i. Let $\widetilde{\Gamma}_i$ be a set of those subpaths β of paths $\alpha \in \widetilde{\Gamma}$. Then the path family Γ is minorized by $\cup\widetilde{\Gamma}_i$, that is, every path in Γ has a subpath in $\cup\widetilde{\Gamma}_i$.

Since mappings φ_i and ψ_j are smooth L-bilipschitz mappings, we have that $f \circ \varphi_i^{-1}$ is not absolutely precontinuous on paths $\varphi_i \circ \gamma \in \varphi_i\widetilde{\Gamma}_i$, and furthermore $\psi_{j_i} \circ f \circ \varphi_i^{-1}$ is not absolutely precontinuous on paths $\varphi_i\widetilde{\Gamma}_i$. Thus, by Lemma 5 and [10, Lemma 4.3], we have

$$\mathsf{M}_n\big((\psi_{j_i} \circ f \circ \varphi_i^{-1})\varphi_i\widetilde{\Gamma}_i\big) = 0.$$

Since mappings ψ_{j_i} are L-bilipschitz, we have

$$\mathsf{M}_n(f\Gamma) \le \sum_i \mathsf{M}_n(f\widetilde{\Gamma}_i) \le \sum_i L^{2n}\mathsf{M}_n(\psi_{j_i} f\widetilde{\Gamma}_i) = 0. \quad \square$$

Now Väisälä's inequality can be proved exactly as in the Euclidean case, see [15, Theorem II.9.1]. For the reader's convenience we sketch the main steps.

Proof of Väisälä's inequality. Let $C \subset M$ be such that $m_M(C) = 0$, $B_f \subset C$ and f is differentiable in $M \setminus C$. The differentiability of f a.e. in M follows from [15, VI Lemma 4.4], Condition (A), and Lemma 5. As $J(x,f) > 0$ a.e. in M, we may assume that $J(x,f) > 0$ for every $x \in M \setminus C$. Since f satisfies the condition (N), we find a Borel set B of measure zero containing $f(C)$. Thus $B_f \subset f^{-1}(B)$ and f is differentiable at every $x \in M \setminus f^{-1}(B)$ with $J(x,f) > 0$.

We may assume that every $\beta \in \Gamma'$ is locally rectifiable and since $m_N(B) = 0$, we may also assume that

$$\int_\beta \chi_B \, \mathrm{d}s = 0$$

for every $\beta \in \Gamma'$. By Lemma 9, we may assume that if α is a path in Γ such that $f \circ \alpha \subset \beta \in \Gamma'$ then f is locally absolutely precontinuous on α.

Let ϱ be a non-negative Borel function $M \to \mathbf{R}$ such that

$$\int_\gamma \varrho \geq 1$$

for every $\gamma \in \Gamma$. Define $\sigma\colon M \to \mathbf{R}$ by

$$\sigma(x) = \begin{cases} \dfrac{\varrho(x)}{\min_{|v|=1} |Df(x)v|}, & \text{if } x \in M \setminus f^{-1}(B), \\ 0, & \text{if } x \in f^{-1}(B). \end{cases}$$

Let $\varrho'\colon N \to \mathbf{R}$ be the function

$$\varrho'(y) = \frac{1}{m} \chi_{fM}(y) \sup_A \sum_{x \in A} \sigma(x),$$

where A runs through over all subsets $A \subset f^{-1}(y)$ such that $\# A \leq m$.

To see that ϱ' is a non-negative Borel function, we apply the argument in the proof of [15, Theorem II.9.1]. Moreover, by the same argument as in the proofs of [15, Theorem II.9.1, Theorem II.8.1], we have that

$$\int_\beta \varrho' \, \mathrm{d}s \geq 1$$

for all $\beta \in \Gamma'$. Poletsky's Lemma 9 is employed also at this point.

Let (Ω_i) be an exhaustion of M by relatively compact sets Ω_i such that $\overline{\Omega}_i \subset \Omega_{i+1}$. We set

$$\begin{aligned} \varrho_i &= \varrho \chi_{\overline{\Omega}_i}\colon M \to \mathbf{R}, \\ \sigma_i &= \sigma \chi_{\overline{\Omega}_i}\colon M \to \mathbf{R}, \end{aligned}$$

and

$$\varrho_i'\colon M \to \mathbf{R}, \qquad \varrho_i'(y) = \frac{1}{m} \chi_{f\overline{\Omega}_i}(y) \sup_A \sum_{x \in A} \sigma_i(x). \tag{4}$$

Suppose $y_0 \in f\overline{\Omega}_i \setminus f(\overline{\Omega}_i \cap B_f)$ and let $k = \#\big(f^{-1}(y_0) \cap \overline{\Omega}_i\big)$. Then there is a connected neighborhood V of y_0 and k inverse mappings $g_\mu\colon V \to D_\mu$ with

$$\overline{\Omega}_i \cap f^{-1}V = \bigcup \{ \overline{\Omega}_i \cap D_\mu : 1 \leq \mu \leq k \}.$$

For each $y \in V$ we define $L_y \subset J := \{1, \ldots, k\}$ as follows. If $k \leq m$, then $L_y = J$. If $k > m$, then $\# L_y = m$, and for each $\mu \in L_y$ and $\nu \in J \setminus L_y$ either

$\sigma_i\big(g_\mu(y)\big) > \sigma_i\big(g_\nu(y)\big)$ or $\sigma_i\big(g_\mu(y)\big) = \sigma_i\big(g_\nu(y)\big)$ and $\mu > \nu$. Thus the sum over the set L_y gives the supremum in the formula (4). Then

$$\begin{aligned}\varrho_i'(y) &= \frac{1}{m}\chi_{f\,\overline{\Omega}_i}(y)\sup_A \sum_{x\in A}\sigma_i(x) = \frac{1}{m}\chi_{f\,\overline{\Omega}_i}(y)\sum_{\mu\in L_y}\sigma_i\big(g_\mu(y)\big)\\ &= \frac{1}{m}\sum_{\mu\in L_y}\sigma_i\big(g_\mu(y)\big)\end{aligned}$$

for $y \in V$. Furthermore, for $L \subset J$, the sets $V_L = \{y \in V : L_y = L\}$ are pairwise disjoint Borel sets. By Hölder's inequality,

$$\varrho_i'(y)^n \le \frac{1}{m}\sum_{\mu\in L_y}\sigma_i\big(g_\mu(y)\big)^n.$$

Now

$$\int_{V_L}\varrho_i'(y)^n\,\mathrm{d}m_N(y) \le \frac{1}{m}\sum_{\mu\in L}\int_{V_L}(\sigma_i\circ g_\mu)^n\,\mathrm{d}m_N.$$

By Lemma 8 we have the inequality

$$\int_{V_L}\varrho_i'(y)^n\,\mathrm{d}m_N(y) \le \frac{1}{m}\sum_{\mu\in V_L}\int_{g_\mu V_L}\sigma_i(x)^n J(x,f)\,\mathrm{d}m_M(x).$$

Since $J(x,f) > 0$ for a.e. $x \in g_\mu(V_L)$,

$$K_I(x,f) = \frac{\|D^\# f(x)\|^n}{J(x,f)^{n-1}} = \frac{J(x,f)}{\min_{|v|=1}|Df(x)v|^n},$$

for a.e. $x \in g_\mu(V_L)$. Thus

$$\int_{V_L}\varrho_i'(y)^n\,\mathrm{d}m_N(y) \le \frac{1}{m}\sum_{\mu\in L}\int_{g_\mu V_L}\varrho_i(x)^n K_I(x,f)\,\mathrm{d}m_M(x).$$

As in [15, pp. 51–52], we conclude that

$$\int_N\varrho_i'(y)^n\,\mathrm{d}m_N(y) \le \frac{1}{m}\int_M\varrho_i(x)^n K_I(x,f)\,\mathrm{d}m_M(x).$$

Since (ϱ_i) and (ϱ_i') are increasing sequences tending to ϱ and ϱ', respectively, we have

$$\int_N\varrho'(y)^n\,\mathrm{d}m_N(y) \le \frac{1}{m}\int_M\varrho(x)^n K_I(x,f)\,\mathrm{d}m_M(x).$$

Thus $\mathsf{M}_n(\Gamma') \le \mathsf{M}_{n,K_I(\,\cdot\,,f)}(\Gamma)/m$. □

As a direct consequence of Väisälä's inequality we obtain a version of Poletsky's inequality.

Corollary 10 (Poletsky's inequality). *Let $f\colon M \to N$ be a continuous mapping of finite distortion K satisfying the condition* (A). *Let Γ be a path family on M. Then*

$$\mathsf{M}_n(f\Gamma) \le \mathsf{M}_{n,K^{n-1}(\,\cdot\,)}(\Gamma).$$

4. Parabolic manifolds and Global homeomorphism theorem

Let X and Y be topological spaces. As in the introduction we denote by

$$\Gamma_X^\infty = \{\gamma : |\gamma| \not\subset C \text{ for any compact } C \subset X\}$$

the family of all paths γ in X tending to infinity. If $A \subset X$ and $f\colon A \to Y$, we say that a point $y \in Y$ is an *asymptotic limit of f at the infinity of X* if there exists a path $\gamma\colon [a,b[\to A$ belonging to the family Γ_X^∞ such that $\lim_{t\to b} f\circ\gamma(t) = y$. We denote by $E(f;X)$ the set of asymptotic limits of f at the infinity of X. If $A = X$, we omit the phrase "at the infinity of X", and write $E(f) = E(f;X)$.

As an application of Poletsky's inequality we prove the Global homeomorphism theorem (Theorem 1). In the proof we employ Propositions 2 and 3 that were announced in the introduction. The proofs of these propositions are presented in the following Subsections 4.1 and 4.2.

4.1. The proof of Proposition 2. The proof of Proposition 2 is based on the Euclidean localization method presented in [19]. We formulate this localization as follows.

Proposition 11. *Let M, N, and $f\colon M \to N$ be as in Proposition 2. Suppose that $q_0 \in fM$ and (U,φ) is a chart at q_0 such that φU contains the closed unit ball $\overline{B}^n$ and $\varphi(q_0) = 0$. Then for every $p_0 \in f^{-1}(q_0)$ the set $E(\varphi\circ f \mid V; M)\cap B^n$ has zero n-capacity, where V is the p_0-component of the set $(\varphi\circ f)^{-1}B^n$.*

The proof of Proposition 11 is divided into the following lemmata. For any sets C_1 and C_2 in a topological space D we denote by $\Delta(C_1,C_2;D)$ the family of all paths γ in D such that $|\gamma|\cap C_i \neq \emptyset$, $i = 1,\ 2$.

Lemma 12 ([19, Lemma 2]). *Let $n \ge 3$, $T \subset \mathbf{R}$ be measurable, $r\colon T \to \mathbf{R}$ be a positive bounded measurable function, and $x_t \in S^{n-2}\big(0, r(t)\big)\times\{t\}$ for every $t \in T$. Then, for every $R > 0$,*

$$\mathsf{M}_n\biggl(\bigcup_{t\in T}\Delta(\{(0,t)\},\{x_t\};B^{n-1}\big(0,r(t)\big)\times\{t\}\biggr) \ge c_n\frac{\mathscr{H}^1(T_R)}{R},$$

where $T_R = r^{-1}\big(]0,R]\big)$ and c_n is a constant depending only on n.

Proof. Let us denote $\Gamma_t = \Delta\big(\{(0,t)\},\{x_t\};B^{n-1}(0,r(t))\times\{t\}\big)$ for every $t \in E$ and $\Gamma' = \Delta\big(\{\mathbf{0}\},\{e_1\};B^{n-1}\times\{0\}\big)$. It is well known that $\mathsf{M}_n\big(\Gamma';\mathbf{R}^{n-1}\times\{0\}\big) > 0$, where the n-modulus is taken with respect to the $(n-1)$-dimensional Lebesgue measure of $\mathbf{R}^{n-1}\times\{0\}\subset\mathbf{R}^n$. Using the rotational symmetry and the dilatation

$$(x_1,\dots,x_{n-1}) \mapsto \left(\frac{x_1}{r(t)},\dots,\frac{x_{n-1}}{r(t)}\right), \qquad (x_1,\dots,x_{n-1})\in\mathbf{R}^{n-1},$$

we obtain

$$\mathsf{M}_n\big(\Gamma_t;\mathbf{R}^{n-1}\times\{t\}\big)=\frac{\mathsf{M}_n\big(\Gamma';\mathbf{R}^{n-1}\times\{0\}\big)}{r(t)}$$

for every $t\in T$. We set $c_n=\mathsf{M}_n\big(\Gamma';\mathbf{R}^{n-1}\times\{0\}\big)$.

Let ϱ be an admissible function for the path family $\bigcup_{t\in E}\Gamma_t$. Then

$$\int_{\mathbf{R}^n}\varrho^n\,\mathrm{d}x\ge\int_T\bigg(\int_{\mathbf{R}^{n-1}}\varrho^n\,\mathrm{d}y\bigg)\,\mathrm{d}t\ge\int_T\frac{c_n}{r(t)}\,\mathrm{d}t\ge c_n\frac{\mathscr{H}^1(T_R)}{R}. \square$$

For the proof of Proposition 11 let us introduce some notation. We denote $h=\varphi\circ f\mid V\colon V\to\varphi U$. For every $y\in S^{n-1}$, define $\gamma_y\colon[0,1]\to\overline{B}^n$ by $\gamma_y(t)=ty$, and let $\tilde\gamma_y$ be the maximal h-lifting of γ_y starting at p_0. For each $y\in S^{n-1}$, let $\lambda(y)$ be the length of the maximal interval where $\tilde\gamma_y$ is defined, that is, the maximal interval is either $[0,\lambda(y)[$ or $[0,\lambda(y)]$. Furthermore, we denote

$$\begin{aligned}
&E=\{y\in S^{n-1}:\lambda(y)<1\},\\
&C=\{\lambda(y)y\in B^n:y\in E\},\\
&F=\{ty\in\overline{B}^n:\lambda(y)\le t\le 1,\ y\in E\},\\
&G=B^n\setminus F,
\end{aligned}$$

and

$$\Gamma=\{\gamma_y:y\in E\}.$$

Finally, we denote by G' the p_0-component of $h^{-1}G$.

Lemma 13. *The set G is star-shaped and open, and $E\subset S^{n-1}$ is closed.*

Proof. Clearly G is star-shaped since

$$G=\{ty\in B^n:0\le t<\lambda(y),\ y\in S^{n-1}\}.$$

The openness of G follows from the fact that h is a local homeomorphism and therefore homeomorphic in a neighborhood of $\tilde\gamma_y\big([0,c]\big)$, where $c<\lambda(y)$ for every y; see [18, Remark 1]. The same reasoning shows that $S^{n-1}\setminus E$ is open in S^{n-1}, and hence E is closed. $\square$

The following lemmata are discussed in [19, Section 3]; see also [13]. We say that a subset A of a topological space X is *relatively locally connected at* $x\in\bar A$ if every neighborhood U of x contains a neighborhood W of x such that $W\cap A$ is connected.

Lemma 14. (a) *The mapping $h\mid G'\colon G'\to G$ is a homeomorphism.*

(b) *Let $x\in(\partial G')\cap V$ be such that G is relatively locally connected at $h(x)\in\partial G$. Then there are arbitrary small neighborhoods W of x such that $h(W\cap G')=hW\cap G$ and $h\mid W\colon W\to hW$ is a homeomorphism.*

(c) *If G is relatively locally connected at $q \in B^n \cap \partial G$, then for every pair of paths α, $\beta: [a,b] \to G \cup \{q\}$ such that $\alpha\bigl([a,b[\bigr) \subset G$, $\beta\bigl([a,b[\bigr) \subset G$, $\alpha(a) = \beta(a) = 0$, and that $\alpha(b) = \beta(b) = q$, the path α has a total h-lifting starting at p_0 if and only if the path β has a total h-lifting starting at p_0. Moreover, if $\tilde\alpha$ and $\tilde\beta$ are the total h-liftings of α and β, respectively, then $\tilde\alpha(b) = \tilde\beta(b)$.*

(d) *If $q \in C$ and $\gamma: [a,b] \to G \cup \{q\}$ is a path such that $\gamma\bigl([a,b[\bigr) \subset G$, $\gamma(a) = 0$, and $\gamma(b) = q$, then the maximal h-lifting of γ starting at p_0 belongs to the path family Γ_M^∞.*

Proof. (a) The formula $x \mapsto \tilde\gamma_y(t)$, where $x = ty \in G$, with $t \ge 0$ and $y \in S^{n-1}$, gives the inverse mapping. Since h is a local homeomorphism, the continuity of the inverse mapping follows.

(b) It follows from the assumptions that there exists a neighborhood W of x such that $h \mid W: W \to hW$ is a homeomorphism and $hW \cap G$ is connected. Furthermore, W can be chosen to be arbitrarily small. Since $x \in \partial G'$, the set $W \cap G'$ is non-empty, and therefore $G' \cap (h \mid W)^{-1}(hW \cap G) \neq \emptyset$. Now $(h \mid W)^{-1}(hW \cap G) \subset G'$ since G' is the p_0-component of $h^{-1}G$ and $(h \mid W)^{-1}(hW \cap G)$ is connected. Thus $hW \cap G = h\bigl((h \mid W)^{-1}(hW \cap G) \cap W\bigr) \subset h(W \cap G')$. The inclusion $h(W \cap G') \subset hW \cap G$ holds trivially.

(c) It is sufficient to show that if α has the total h-lifting $\tilde\alpha$ starting at p_0, then the maximal h-lifting of β, say $\tilde\beta$, starting at p_0 is total and $\tilde\beta(b) = \tilde\alpha(b)$.

First we observe that $\tilde\beta$ is given on $[a,b[$ by $\tilde\beta \mid \bigl[a,b\bigr[= (h \mid G')^{-1} \circ \beta \mid [a,b[$ since $h \mid G': G' \to G$ is a homeomorphism. By (b), there exists a neighborhood W of $\tilde\alpha(b)$ such that $h(W \cap G') = hW \cap G$. Thus there exists $c \in]a,b[$ such that $\beta\bigl([c,b[\bigr) \subset hW \cap G = h(W \cap G')$ since hW is a neighborhood of q. It follows that $\tilde\beta\bigl([c,b[\bigr) \subset W \cap G'$. Since W can be chosen to be arbitrarily small, $\tilde\beta(t) \to \tilde\alpha(b)$ as $t \to b$. Hence $\tilde\beta$ is total and $\tilde\beta(b) = \tilde\alpha(b)$.

(d) Let $\gamma: [a,b] \to G \cup \{q\}$ be as in the claim and let $\tilde\gamma$ be the maximal h-lifting $\tilde\gamma$ of γ starting at p_0. Suppose that $\tilde\gamma \notin \Gamma_M^\infty$. Then $\tilde\gamma$ is total and $\tilde\gamma(b) \in \partial G'$. Let $z = q/|q|$. We may assume that $a = 0$ and $b = \lambda(z)$. Let W be a neighborhood of $\tilde\gamma(b)$ such that $h \mid W: W \to hW$ is a homeomorphism and hW is an open ball around q. Then there exists $c \in]0,b[$ such that $\gamma_z\bigl([c,b[\bigr) \subset hW \cap G$. Since G is open, there exists, for every $t \in [c,b[$, an open ball $B\bigl(\gamma_z(t), r_t\bigr) \subset hW \cap G$. Furthermore, since G is star-shaped with respect to 0, there exists $b_t \in]0,b[$ such that the ray between 0 and $\gamma(t')$ intersects the ball $B\bigl(\gamma_z(t), r_t\bigr)$ for all $t' \in [b_t, b[$. Thus we can connect the points $\gamma(t')$ and $\gamma_z(t)$ by a path in $hW \cap G$. Since W can be chosen to be an arbitrarily small neighborhood of $\tilde\gamma(b)$, we conclude that $\lim_{t\to b} \tilde\gamma_z(t) = \tilde\gamma(b)$. This leads to a contradiction since $\tilde\gamma_z \in \Gamma_M^\infty$. Hence $\tilde\gamma \in \Gamma_M^\infty$. □

Lemma 15. *The topological dimension of E is zero and $\mathscr{H}^{n-1}(E) = 0$. Furthermore,*

$$G \subset hV \subset \overline{G} = \overline{B}^n \tag{5}$$

and

$$G' \subset V \subset \overline{G}'. \tag{6}$$

Proof. Suppose that E contains a non-degenerate continuum K. By applying an additional rotation, we may assume that there exists $a \in]0,1[$ such that $K \cap (\mathbf{R}^{n-1} \times \{t\}) \neq \emptyset$ for every $t \in [-a,a]$. Let $\delta \in]0,1[$ be such that $B^n(0,\delta) \subset G$, let

$$D = \bigl\{(z_1,\dots,z_n) \in B^n : |z_n|/|z| < a \text{ or } |z_n| < \delta a\bigr\}$$

and let $g\colon D \to B^n$ be the mapping that is the identity in $D \cap B^n(\delta)$ and $g(z_1,\dots,z_n) = (z_1,\dots,z_{n-1},\delta z_n/|z|)$ in $D \setminus B^n(\delta)$. Then g is L-bilipschitz and for every $z \in K$ it maps the ray between $\lambda(z)z$ and z to the ray between $\bigl(\lambda(z)z_1,\dots,\lambda(z)z_{n-1},\delta z_n/|z|\bigr)$ and $(z_1,\dots,z_{n-1},\delta z_n/|z|)$.

Define $r\colon [-\delta a,\delta a] \to [\delta,1]$ by

$$r(t) = \sup\bigl\{s\colon B^{n-1}(0,s) \times \{t\} \subset g(G \cap D)\bigr\}.$$

By the openness of G, r is lower semi-continuous. Thus r is a measurable function and for every $t \in [\delta a,\delta a]$ there exists $x_t \in S^{n-2}\bigl(0,r(t)\bigr) \times \{t\}$ such that $x_t \in g(D \cap C)$.

We set

$$\Gamma = \bigcup_{t \in [-\delta a,\delta a]} \Delta\bigl(\{(0,t)\},\{x_t\};B^{n-1}(0,r(t)) \times \{t\}\bigr).$$

By Lemma 12,

$$\mathsf{M}_n(\Gamma) \geq 2c_n\delta a,$$

where c_n is a positive constant depending only on n. On the other hand,

$$\mathsf{M}_n(\Gamma) \leq \mathsf{M}_n\bigl((g \circ h)\Gamma_M^\infty\bigr)$$

since $\Gamma \subset (g \circ h)\Gamma_M^\infty$ by Lemma 14. Furthermore, since g is L-bilipschitz, we have

$$\mathsf{M}_n\bigl((g \circ h)\Gamma_M^\infty\bigr) \leq L^{2n}\mathsf{M}_n(h\Gamma_M^\infty).$$

Finally, applying Poletsky's inequality (Corollary 10) we obtain

$$\mathsf{M}_n(h\Gamma_M^\infty) \leq \mathsf{M}_{n,K^{n-1}(\,\cdot\,)}(\Gamma_M^\infty) = 0$$

since M is $K^{n-1}(\,\cdot\,)$-parabolic. Hence $\mathsf{M}_n(\Gamma) = 0$ which is a contradiction. Thus the components of E are singletons.

To prove that $\mathscr{H}^{n-1}(E)=0$, consider the path family $\Gamma=\{\gamma_z|[\delta,1]:z\in E\}$. Then by [17, 7.5] and Poletsky's inequality,

$$\mathscr{H}^{n-1}(E)=\left(\log\frac{1}{\delta}\right)^{n-1}\mathsf{M}_n(\Gamma)\le\left(\log\frac{1}{\delta}\right)^{n-1}\mathsf{M}_{n,K^{n-1}}(\Gamma_M^\infty)=0.$$

Clearly $G\subset hV\subset\overline{B}^n$ and $\overline{G}\subset\overline{B}^n$. Thus (5) follows if we show that $\overline{B}^n\subset\overline{G}$. Suppose that there exists a point $q\in\overline{B}^n\setminus\overline{G}$. Then there is an open ball $B^n(q,r)\subset\mathbf{R}^n\setminus\overline{G}$. In particular, $B^n(q/|q|,r)\cap S^{n-1}\subset E$, which is a contradiction since E is totally disconnected.

The inclusion $G'\subset V$ in (6) is clear, so it remains to show that $V\subset\overline{G}'$. Suppose that there is a point $x\in V\setminus\overline{G}'$. Then x has a neighborhood $U\subset V\setminus\overline{G}'$. Since h is an open mapping, $hU\subset B^n$ is open. Therefore $hU\cap G\neq\emptyset$ since $\overline{G}=\overline{B}^n$. This is a contradiction. □

Proof of Proposition 11. Here we use the notation introduced before Lemma 13. By Lemma 15, the topological dimension of E is zero, and therefore E does not locally separate the sphere S^{n-1}, see [3, Corollary IV.4]. Hence every point $x\in S^{n-1}$ has arbitrary small neighborhoods $U\subset S^{n-1}$ of x such that each $(S^{n-1}\setminus E)\cap U$ is path-connected. Recall that E is closed. Thus G is relatively locally connected at every point $q\in\overline{B}^n$. Furthermore, for every $q\in\overline{B}^n$, there exists a path $\gamma:[0,1]\to G\cup\{q\}$ such that $\gamma([0,1[)\subset G$, $\gamma(0)=0$, and $\gamma(1)=q$. Such a path can be constructed piece-wise by connecting points $q_i\in W_i\cap G$ and $q_{i+1}\in W_{i+1}\cap G$ by a path in $W_i\cap G$, where (W_i) is a nested sequence of neighborhoods of q shrinking to q such that each $W_i\cap G$ is connected.

Let $\Omega\subset B^n$ be the set of all points $y\in B^n$ such that, for every path $\gamma:[0,1]\to G\cup\{y\}$, with $\gamma([0,1[)\subset G$, $\gamma(0)=0$, and $\gamma(1)=y$, the unique maximal h-lifting $\tilde\gamma$ starting at p_0 is total. In particular, $\tilde\gamma(1)\in V$ is defined, and therefore $y=h(\tilde\gamma(1))\in hV$. We claim that $\Omega=hV$. It suffices to prove that $hV\setminus G\subset\Omega$. Fix $y\in hV\setminus G$ and let $\gamma:[0,1]\to G\cup\{y\}$ be a path such that $\gamma([0,1[)\subset G$, $\gamma(0)=0$, and $\gamma(1)=y$. As in the proof of Lemma 14(c), the maximal h-lifting of γ, say $\tilde\gamma$, starting at p_0 is given on $[0,1[$ by $\tilde\gamma\mid[0,1[=(h\mid G')^{-1}\circ\gamma\mid[0,1[$. Let $x\in h^{-1}(y)$. Then $x\in\partial G'$ since h is an open mapping. Let W be a neighborhood of x given by Lemma 14(b). Then $\gamma([c,1[)\subset hW\cap G=h(W\cap G')$ for some $c\in]0,1[$. As in the proof of Lemma 14(c), we conclude that $\tilde\gamma$ is a total h-lifting of γ starting at p_0. Hence $y\in\Omega$ and therefore $\Omega=hV$.

Next we show that $E(h;M)\cap B^n=B^n\setminus hV$ which then implies that $E(h;M)\cap B^n$ is σ-compact. Since $hV=\Omega$, the inclusion $E(h;M)\cap B^n\subset B^n\setminus hV$ follows immediately from the definition of Ω. On the other hand, for every point $y\in B^n\setminus\Omega$, there exists a path $\gamma:[0,1]\to G\cup\{y\}$, with $\gamma([0,1[)\subset G$, $\gamma(0)=0$, and $\gamma(1)=y$, whose maximal h-lifting $\tilde\gamma$ starting at p_0 is not total. If $\tilde\gamma\notin\Gamma_M^\infty$, there is an increasing sequence of positive numbers t_i such that $t_i\to 1$

and $\tilde{\gamma}(t_i) \to x \in M$. Since f is a local homeomorphism at x and $\gamma(t_i) \to y \in B^n$, there is a connected neighborhood U of x such that $(\varphi \circ f)U \subset B^n$. Hence $x \in V$ and, furthermore, $y \in hV = \Omega$. We conclude that $\tilde{\gamma} \in \Gamma_M^\infty$, and so $y \in E(h; M)$. We have proved that $E(h; M) \cap B^n$ is σ-compact.

To show that $E(h; M) \cap B^n$ has zero n-capacity, we study the family Γ_0 of all paths γ such that $|\gamma| \cap E(h; M) \cap B^n \neq \emptyset$. Since $E(h; M) \cap B^n \subset F$ and the set F has zero measure by Fubini's theorem and Lemma 15, it is sufficient to consider only paths γ satisfying $|\gamma| \cap G \neq \emptyset$. Let $\gamma \in \Gamma_0$. We may assume that γ is defined on the unit interval and $\gamma(1) \in E(h; M)$. We show that $\gamma \in h\Gamma_M^\infty$. If $|\gamma| \cap F = \{\gamma(1)\}$, we have by Lemma 14(c) that $\gamma \in h\Gamma_M^\infty$. Assume that $|\gamma| \cap F \neq \{\gamma(1)\}$. If $\gamma \notin h\Gamma_M^\infty$, it has a total h-lifting, say $\tilde{\gamma}$, starting at $h^{-1}\bigl(\gamma(0)\bigr)$. Let $\{W_i\}$ be a finite open cover of $|\tilde{\gamma}|$ such that $h \mid W_i$ is a homeomorphism, and $hW_i \cap G$ is connected. Then there is a path $\beta\colon [0,1] \to B^n$ such that $\beta(0) = \gamma(0)$, $\beta\bigl([0,1[\bigr) \subset \bigl(\bigcup_i hW_i\bigr) \cap G$, and $\beta(1) = \gamma(1)$. Then β has a total h-lifting starting at $h^{-1}\bigl(\gamma(0)\bigr)$. This contradicts Lemma 14(c), since $\beta(1) = \gamma(1) \in E(h; M) \cap B^n$. Thus the maximal h-lifting of γ is not total and $\gamma \in h\Gamma_M^\infty$. By Poletsky's inequality, $\mathsf{M}_n(\Gamma_0) = 0$, and so $E(h; M) \cap B^n$ has zero n-capacity. □

In the proof of Proposition 2 we apply the following lemma.

Lemma 16 ([11, 2.3]). *Let M be a Riemannian manifold, $C \subset M$ compact and $L > 1$. Then there exists $R = R(C, L) > 0$ such that for every $x \in C$ there exist a neighborhood V of x and an L-bilipschitz diffeomorphism $\psi\colon V \to B^n(R)$ with $\psi(x) = 0$.*

Proof of Proposition 2. Let (Ω_i) be an increasing sequence of relatively compact open subsets of N and, for every i, let $\{q_{i,j}\}$ be a countable dense subset of Ω_i. Since f is a local homeomorphism, $f^{-1}(\{q_{i,j}\})$ is a countable dense subset of M.

For every i, let $R_i > 0$ be such that there exists a 2-bilipschitz diffeomorphism $\psi_{i,j}\colon V_{i,j} \to B^n(R_i)$, where $V_{i,j}$ is a neighborhood of $q_{i,j}$ as in Lemma 16.

Suppose that $\tilde{\gamma}\colon [a, b[\to M$ belongs to the family Γ_M^∞ and that the limit $\lim_{t\to b}(f \circ \tilde{\gamma})(t) = q \in N$ exists. Let $\gamma = f \circ \tilde{\gamma}$. Then, by the density of $\{q_{i,j}\}$, there exists $c \in]a, b[$ such that $\gamma\bigl([c, b[\bigr) \subset V_{i,j}$ for some i and j. Thus, for some i, j, and $p \in f^{-1}(q_{i,j})$, we have $\tilde{\gamma}\bigl([c, b[\bigr) \subset \widetilde{V}_{i,j}$, where $\widetilde{V}_{i,j}$ is the p-component of $f^{-1}(V_{i,j})$. It follows that $q \in E(f \mid \widetilde{V}_{i,j}; M)$.

Since $\psi_{i,j}^{-1}\bigl(E(\psi_{i,j} \circ f \mid \widetilde{V}_{i,j}; M) \cap B^n(R_i)\bigr) \subset E(f)$, we have

$$E(f) = \bigcup_{i,j} \psi_{i,j}^{-1}\bigl(E(\psi_{i,j} \circ f \mid \widetilde{V}_{i,j}; M) \cap B^n(R_i)\bigr).$$

Therefore $E(f)$ is σ-compact and has zero n-capacity since it is a countable union of σ-compact sets of zero n-capacity by Proposition 11.

It remains to prove that $N \setminus fM \subset E(f)$. We first show that $\partial(N \setminus fM) \subset E(f)$. Let $q \in \partial(N \setminus fM)$. Since f is an open mapping, $q \notin fM$. Choose i and j such that $q \in V_{i,j}$. Fix $p \in f^{-1}(q_{i,j})$ and let $\widetilde{V}_{i,j}$ be the p-component of $f^{-1}V_{i,j}$ and $h_{i,j} = (\psi_{i,j} \circ f) \mid \widetilde{V}_{i,j}$. Let $\gamma\colon [0,1] \to B^n(R_i)$ be a path such that $\gamma(0) = 0$ and $\gamma(1) = \psi_{i,j}(q)$. Then the maximal $h_{i,j}$-lifting of γ is not total since $\psi_{i,j}(q) \notin h_{i,j}\widetilde{V}_{i,j}$. Thus $\psi_{i,j}(q) \in E(h_{i,j}; M)$, and so $q \in E(f)$. Since $E(f)$ has zero n-capacity, the topological dimension of $\partial(N \setminus fM)$ is zero. Thus $N \setminus fM$ has no interior points and $N \setminus fM = \partial(N \setminus fM) \subset E(f)$. □

4.2. The proof of Proposition 3. Let X be a compact topological space. We define the distance of continuous mappings $F\colon X \to \mathbf{R}^n$ and $H\colon X \to \mathbf{R}^n$ by

$$d(F,H) = \max_{x \in X} |F(x) - H(x)|.$$

In the proof of Proposition 3 we use the method introduced in [13]. In order to obtain the necessary approximation results, we prove a quantitative version of [13, Lemma 3.3] for all homotopy groups in question. The closed unit interval is denoted by I in this section.

Proposition 17. *Let n, k, and m be integers such that $1 \le m \le k \le n$. Let $\Omega \subset \mathbf{R}^n$ be a domain, $E \subset \Omega$ be a compact set such that $\mathscr{H}^{n-k}(E) = 0$ and let $F\colon I^m \to \Omega$ and $f\colon \partial I^m \to \Omega \setminus E$ be continuous mappings satisfying*

$$tf(x) + (1-t)F(x) \in \Omega \tag{7}$$

for every $x \in \partial I^m$, and every $t \in I$.

Then, for every $\varepsilon > 0$, there exists a continuous mapping $H\colon I^m \to \Omega \setminus E$ satisfying $H \mid \partial I^m = f$ and

$$d(F,H) < d(f, F \mid \partial I^m) + \varepsilon.$$

The proof of Proposition 17 is based on the following lemma.

Lemma 18. *Let $A \subset \mathbf{R}^n$ be such that $\mathscr{H}^{n-k}(A) = 0$ and let $P \subset \mathbf{R}^n$ be a set that has a countable covering by k-dimensional hyperplanes. Then $(P + y) \cap A = \emptyset$ for a.e. $y \in \mathbf{R}^n$.*

Proof. The proof is almost verbatim to the proof in the 2-dimensional case presented in [13, Lemma 3.2] and is omitted. □

Proof of Proposition 17. We proceed using the proof of [13, Lemma 3.3] as our guideline. To avoid technical difficulties we prove the claim for continuous mappings $F\colon J^m \to \Omega$ and $f\colon \partial J^m \to \Omega \setminus E$, satisfying the condition (7) for every $y \in \partial J^m$ and $t \in I$, where $J = [-1,1]$.

Let us simplify the notation by introducing following auxiliary functions. Let ϱ be a metric in J^m given by the l_∞-norm in $\mathbf{R}^m$, that is,

$$\varrho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_m - y_m|\}$$

for every $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ in J^m.

For every $r \in]0,1[$, we define $\lambda_r \colon J^m \to J^m$ by

$$\lambda_r(x) = \begin{cases} \dfrac{x}{r}, & \varrho(x,0) \le r, \\ \dfrac{x}{\varrho(x,0)}, & \varrho(x,0) > r. \end{cases}$$

Then λ_r maps the set rJ^m homeomorphically onto J^m and the set $J^m \setminus rJ^m$ onto the boundary ∂J^m keeping the boundary fixed.

Let $\varepsilon > 0$. By continuity, there exists $r > 0$ such that the mapping $F_0 \colon J^m \to \mathbf{R}^n$,

$$F_0(x) = \begin{cases} F\big(\lambda_r(x)\big), & \varrho(x,0) \le r, \\ \left(\dfrac{\varrho(x,0) - r}{1-r}\right) f\big(\lambda_r(x)\big) + \left(\dfrac{1 - \varrho(x,0)}{1-r}\right) F\big(\lambda_r(x)\big), & \varrho(x,0) > r, \end{cases}$$

is well-defined and satisfies conditions $F_0 \mid \partial J^m = f$ and

$$d(F_0, F) < d(f, F \mid \partial J^m) + \tfrac{1}{4}\varepsilon. \tag{8}$$

Moreover, $F_0(J^m) \subset \Omega$ by (7).

Let $\delta = \operatorname{dist}\big(f(\partial J^m), E \cup \partial\Omega\big) > 0$. Then there exist a triangulation of J^m and a simplicial approximation of F_0, say F_1, with respect to the triangulation such that

$$d(F_1, F_0) < \min\big\{\delta, \tfrac{1}{4}\varepsilon\big\}. \tag{9}$$

By Lemma 18, there exists $y \in \mathbf{R}^n$ such that $F_1(J^m) + y \subset \Omega \setminus E$ and

$$|y| < \min\big\{\delta - d(F_1, F_0), \tfrac{1}{4}\varepsilon\big\}. \tag{10}$$

Define $F_2 \colon J^m \to \Omega \setminus E$ by $F_2(x) = F_1(x) + y$. Then

$$tf(x) + (1-t)F_2(x) \in \Omega \setminus E$$

for every $x \in \partial J^m$ and $t \in I$. Thus there exists $r' \in]0,1[$ such that the continuous mapping $H \colon J^m \to \Omega \setminus E$,

$$H(x) = \begin{cases} F_2\big(\lambda_{r'}(x)\big), & \varrho(x,0) \le r', \\ \left(\dfrac{\varrho(x,0) - r}{1-r}\right) f\big(\lambda_{r'}(x)\big) + \left(\dfrac{1 - \varrho(x,0)}{1-r}\right) F_2\big(\lambda_{r'}(x)\big), & \varrho(x,0) > r', \end{cases}$$

is well-defined and satisfies conditions $H|\partial J^m = f$ and

$$d(H, F_2) < \tfrac{1}{4}\varepsilon. \tag{11}$$

The claim follows now from estimates (8), (9), (10), and (11). ▫

Proposition 19. *Let n, k, and m be integers such that $1 \le m \le k \le n$. Let $\Omega \subset \mathbf{R}^n$ be a domain, $E \subset \Omega$ be a σ-compact set such that $\mathscr{H}^{n-k}(E) = 0$, and let $F\colon I^m \to \Omega$ and $f\colon \partial I^m \to \Omega \setminus E$ be continuous mappings satisfying* (7) *for every $x \in \partial I^m$ and $t \in I$.*

Then for every $\varepsilon > 0$ there exists a continuous mapping $H\colon I^m \to \Omega \setminus E$ such that $H \mid \partial I^m = f$ and $d(H,F) < d(f, F \mid \partial I^m) + \varepsilon$.

Proof. Let $\varepsilon > 0$ and let $E_0 \subset E_1 \subset \ldots \subset E$ be an increasing sequence of compact subsets exhausting E. Thus, using Proposition 17 iteratively, there exist mappings $F_i\colon I^m \to \Omega \setminus E_i$, $i = 0, 1, \ldots$, such that $F_i \mid \partial I^m = f$ and

$$d(F_{i+1}, F_i) < \min\bigl\{\delta_0, \ldots, \delta_i, \tfrac{1}{2}\varepsilon\bigr\}/2^i,$$

where $\delta_i = \operatorname{dist}\bigl(F_i(I^m), E_i \cup (\mathbf{R}^n \setminus \Omega)\bigr) > 0$. Moreover, we may assume that $d(F, F_0) < d(f, F \mid \partial I^m) + \frac{1}{2}\varepsilon$.

Since (F_i) is a uniformly convergent sequence of continuous mappings, the limit mapping $H\colon I^m \to \mathbf{R}^n$, $H(x) = \lim_{i\to\infty} F_i(x)$, is continuous. Since $H \mid \partial I^m = f$ and $d(H,F) < d(f, F \mid \partial I^m) + \varepsilon$, it suffices to show that $H(I^m) \subset \Omega \setminus E$.

For this it is sufficient to show that $\operatorname{dist}\bigl(H(I^m), E_k \cup \partial\Omega)\bigr) > 0$ for every k. Fix $k \ge 0$. By the triangle inequality, we have

$$\delta_k = \operatorname{dist}\bigl(F_k(I^m), E_k \cup \partial\Omega\bigr) \le d(F_k, F_i) + \operatorname{dist}\bigl(F_i(I^m), E_k \cup \partial\Omega\bigr)$$

for every $i \ge k$. Since $d(F_k, F_i) \le \frac{1}{2}\delta_k$,

$$\operatorname{dist}\bigl(F_i(I^m), E_k \cup \partial\Omega\bigr) \ge \tfrac{1}{2}\delta_k > 0,$$

for every $i \ge k$. Thus $\operatorname{dist}\bigl(H(I^m), E_k \cup \partial\Omega\bigr) > 0$ for every k. □

Proposition 20. *Let N be a Riemannian n-manifold and $E \subset N$ a σ-compact subset of N such that $\mathscr{H}^{n-k}(E) = 0$ for some integer $k \in \{1, \ldots, n\}$. Let $1 \le m \le k$ and $F\colon I^m \to N$ be a continuous mapping such that $F(\partial I^m) \subset N \setminus E$. Then there exists a continuous mapping $H\colon I^m \to N \setminus E$ such that $H \mid \partial I^m = F \mid \partial I^m$. Moreover, given $\varepsilon > 0$, the mapping H can be chosen such that $d(H,F) < \varepsilon$.*

Proof. Let us introduce some notation. For given integer $l \ge 1$, let

$$V_l = \left\{ \frac{1}{l}(v_1, \ldots, v_m) \in I^m : v_i \in \{0, \ldots, l\} \text{ for all } i \right\} \subset \frac{1}{l}\mathbf{Z}^m.$$

We say that vertices v and w of V_l are neighbors in V_l if $|v - w| = 1/l$. If v and w are neighbors in V_l, we define

$$[v,w] = \{tv + (1-t)w : t \in [0,1]\}.$$

If $w_1, \ldots, w_j$ are neighbors of v in V_l for some $j \geq 2$, we define

$$[v, w_1, \ldots, w_j] = [v, w_1] \times \cdots \times [v, w_j]. \tag{12}$$

For completeness we define $[v] = \{v\}$ for every $v \in V_l$. The sets $[v, w_1, \ldots, w_j]$ defined in (12) are called j-faces. We denote by C_j^l the set of all j-faces in I^m. The set $S_j^l = \cup\{v : v \in C_j^l\}$ is called the j-skeleton of I^m. The boundary of an j-face v with respect to the j-skeleton S_j^l is denoted by $\partial_s v$.

We construct a sequence $F_0, \ldots, F_m$ of mappings such that $F_m(I^m) \subset N \setminus E$ and $F_m \mid \partial I^m = F \mid \partial I^m$. By the compactness of $F(I^m)$ there exists a finite collection $\{(U_\alpha, \varphi_\alpha)\}$ of L-bilipschitz charts covering $F(I^m)$. Fix an integer $l > 0$ such that for every $v \in C_m^l$ there is α such that $F(v) \subset U_\alpha$. Let $\delta > 0$ be such that $\operatorname{dist}\big(F(v), N \setminus U_\alpha\big) > \delta$ if $v \in C_j^l$ is contained in U_α. Since the collection of all faces is finite such δ exists. If $\varepsilon > 0$ is given, we assume that $\delta < \varepsilon$.

Let us construct the mapping $F_0 : \partial I^m \cup S_0^l \to N \setminus E$. Since $\mathscr{H}^n(E) = 0$, there exists $x_v \in B\big(F(v), \delta\big) \setminus E$ for every vertex $v \in V_l \cap]0,1[^m$. We define F_0 using formulas $F_0 \mid \partial I^m = F \mid \partial I^m$ and $F_0(v) = x_v$ for every $v \in V_l \cap]0,1[^m$.

Suppose that for some $j \leq m-1$ we have constructed continuous mappings $F_0, \ldots, F_j$ satisfying properties: $F_i : \partial I^m \cup S_i^l \to N \setminus E$, $F_i \mid (\partial I^m \cup S_{i-1}^l) = F_{i-1}$, and $d(F \mid v, F_i \mid v) < \delta$ for every i-face v.

Next we construct the mapping F_{j+1}. We define $F_{j+1} \mid \partial I^m = F \mid \partial I^m$. For every $v \in C_{j+1}^l$ not contained in ∂I^m we do the following construction. Let Ω_v be the δ-neighborhood of $F(v)$, that is,

$$\Omega_v = \big\{q \in N : \operatorname{dist}\big(q, F(v)\big) < \delta\big\}.$$

Then $\Omega_v \subset U_\alpha$ for every U_α containing $F(v)$. Fix one such α. Then, by Lemma 19, there exists a continuous mapping $\widetilde{F}_v : v \to \varphi_\alpha(\Omega_v \setminus E)$ such that

$$\widetilde{F}_v \mid \partial_s v = \varphi_\alpha \circ F_j \mid \partial_s v$$

and

$$\begin{aligned} d(\widetilde{F}_v, \varphi_\alpha \circ F \mid v) &< d\big(\varphi_\alpha \circ F_j \mid \partial_s v, \varphi_\alpha \circ F \mid \partial_s v\big) \\ &\quad + \big(\delta/L - d(\varphi_\alpha \circ F_j \mid \partial_s v, \varphi_\alpha \circ F \mid \partial_s v)\big) \\ &= \delta/L. \end{aligned}$$

We define $F_{j+1} \mid v = \varphi_\alpha^{-1} \circ \widetilde{F}_v$. The mapping $F_{j+1} : \partial I^m \cup S_{j+1}^l \to N \setminus E$ is well-defined and continuous. Moreover, $F_{j+1} \mid \partial I^m \cup S_j^l = F_j$, and $d(F \mid v, F_{j+1} \mid v) < \delta$ for every $v \in C_{j+1}^l$. This concludes the induction step and the proof. □

Proof of Proposition 3. Let us first consider the injectivity of the induced mapping $\pi_m(N\setminus E) \to \pi_m(N)$. Let f and h be continuous mappings $I^m \to N\setminus E$ such that $f(\partial I^m) = h(\partial I^m) = \{q\}$ for some $q \in N \setminus E$ and that there exists a homotopy $F\colon I^m \times I \to N$ such that $F : f \simeq h \operatorname{rel} \partial I^m$. Since $m+1 \le k$, there exists by Proposition 20 a continuous mapping $H\colon I^{m+1} \to N \setminus E$ such that $H \mid \partial I^{m+1} = F \mid \partial I^{m+1}$. Thus H is a homotopy $f \simeq h \operatorname{rel} \partial I^m$ in $N \setminus E$. Hence the induced mapping $\pi_m(N \setminus E) \to \pi_m(N)$ is injective.

Let us show the surjectivity of the induced mapping. Let $f\colon I^m \to N$ be a representative of some homotopy class of $\pi_m(N)$. We may assume that $f(\partial I^m) = \{q\}$, where $q \in N \setminus E$. We have to show that there exists $h\colon I^m \to N \setminus E$ such that $h \simeq f \operatorname{rel} \partial I^m$ in N.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a finite collection of L-bilipschitz charts covering $f(I^m)$. Let C_j^l and S_j^l be as in the proof of Proposition 20 for every $l \ge 1$ and $0 \le j \le m$. We fix l and choose $\delta > 0$ as in the proof of Proposition 20 as follows. Let $l \ge 1$ be such that for every $v \in C_m^l$ there exists α such that $f(v) \subset U_\alpha$. Let $\delta > 0$ be such that for all $0 \le j \le m$ and $v \in C_j^l$, and for all β such that $f(v) \subset U_\beta$, we have $d\big(f(v), N \setminus U_\beta\big) > \delta/L^2$. By Proposition 20, there exists a continuous mapping $h\colon I^m \to N \setminus E$ such that $h \mid \partial I^m = f \mid \partial I^m$ and $d(h,f) < \delta/L^2$.

For every $0 \le j \le m$ and every $v \in C_j^l$, we construct a mapping $F_v\colon v \times I \to N$ as follows. Let α be such that $f(v) \subset U_\alpha$. It follows from $d(f,h) < \delta/L^2$ that $h(v) \subset U_\alpha$ and the mapping $F_v\colon v \times I \to N$,

$$(x,t) \mapsto \varphi_\alpha^{-1}\big(t\varphi_\alpha(h(x)) + (1-t)\varphi_\alpha(f(x))\big),$$

is well-defined and continuous. Moreover, it is a homotopy $f \mid v \simeq h \mid v$ in U_α.

By construction, $F_v \mid v \times \{0,1\} = F_w \mid w \times \{0,1\}$ for every $v \in C_j^l$ and $w \in C_{j-1}^l$ such that $w \subset \partial_s v$. Moreover, since $d(F_v \mid w, F_w) < \delta$ for such v and w, we may assume that $F_v \mid w = F_w$. Indeed, since $F_v(v \times I)$ and $F_w(w \times I)$ are contained in the same chart U_α for some α, we may use the same technique as in the proof of Proposition 17 to modify the boundary values of the mapping F_v. Since the mappings F_v and $F_{v'}$ agree on the set $v \cap v'$ for every $v \in C_j^l$, we may define a continuous mapping $H\colon I^m \times I \to N$ by $H \mid (v \times I) = F_v$. The mapping H is a homotopy $f \simeq h \operatorname{rel} \partial I^m$ in N. □

4.3. The proof of the Global homeomorphism theorem. The last observation used in the proof of the Global homeomorphism theorem is the following homotopy lifting lemma.

Lemma 21. *Let X, Y, and Z be topological spaces and let $f\colon X \to Y$ be a local homeomorphism. Then for every pair of continuous functions $h\colon Z \to X$ and $F\colon Z \times I \to Y \setminus E(f)$ satisfying*

(1) $(f \circ h)Z \subset Y \setminus E(f)$, *and*

(2) $F \mid (Z \times \{0\}) = f \circ h$

there exists a continuous function $H: Z \times I \to X$ *such that* $f \circ H = F$ *and* $H \mid (Z \times \{0\}) = h$.

Proof. For every $z \in Z$ there exists a unique total f-lifting of the path $t \mapsto F(z,t)$ starting at $h(z)$, say γ_z. Thus we can define $H(z,t) = \gamma_z(t)$ for every $(z,t) \in Z \times I$. Since f is a local homeomorphism, the mapping H is continuous. ▫

Proof of the Global homeomorphism theorem. By Proposition 2, the set $N \setminus fM$ has zero n-capacity. Thus it is sufficient to consider only the injectivity of f. We construct a certain mapping $g: fM \setminus E(f) \to M$ by using Lemma 21 and show that it is the inverse of f.

By Propositions 2 and 3, $fM \setminus E(f) = N \setminus E(f)$ is non-empty and simply connected. Fix $q \in fM \setminus E(f)$ and $p \in f^{-1}(q)$. Let $\alpha: I \to fM \setminus E(f)$ and $\beta: I \to fM \setminus E(f)$ be any paths such that $\alpha(0) = \beta(0) = q$ and $\alpha(1) = \beta(1)$. Both paths have total f-liftings, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, starting at p. Since $fM \setminus E(f)$ is simply connected, there exists a homotopy $F: I^2 \to fM \setminus E(f)$ such that $F: \alpha \simeq \beta \text{ rel}\{0,1\}$. By Lemma 21, there exists a homotopy $H: I^2 \to M$ such that $H \mid I \times \{0\} = \tilde{\alpha}$ and $f \circ H = F$. By the uniqueness of path lifting, $H: \tilde{\alpha} \simeq \tilde{\beta}$. The path $I \to M$, $t \mapsto H(1,t)$, is a constant path since it is an f-lifting of the constant path $I \to N$, $t \mapsto \alpha(1)$, and f is a local homeomorphism. Thus $\tilde{\beta}(1) = \tilde{\alpha}(1)$. We define $g: fM \setminus E(f) \to M$ by $g(y) = \tilde{\alpha}(1)$, where $\tilde{\alpha}: I \to M$ is an f-lifting starting at p of some path $\alpha: I \to fM \setminus E(f)$, with $\alpha(0) = q$ and $\alpha(1) = y$. By the argument above, the mapping g is well-defined. Let us denote $X = g\big(fM \setminus E(f)\big)$. Clearly g is the inverse of $f \mid X$. Thus g is a local homeomorphism and $f \mid X$ is injective.

We show that $\overline{X}$ is both open and closed. Fix a point $x \in \overline{X}$. Let U be a connected neighborhood of x such that $f \mid U$ is a homeomorphism. We show that $U \subset \overline{X}$. Since $E(f)$ has zero n-capacity and f is an open mapping, the set $f^{-1}E(f)$ has no interior points. Thus it is sufficient to show that $U \setminus f^{-1}E(f) \subset X$. By Proposition 3, $fU \setminus E(f)$ is path-connected. Since $U \cap X \neq \emptyset$, there exists a point $x' \in U \cap X$. Let $x'' \in U \setminus f^{-1}E(f)$. Then there exists a path $\gamma: I \to fU \setminus E(f)$ such that $\gamma(0) = f(x')$ and $\gamma(1) = f(x'')$. Observe that $f(x') \in fU \setminus E(f)$, since $x' \in U \cap X$. Thus $\tilde{\gamma} = (f \mid U)^{-1} \circ \gamma$ is an f-lifting of γ starting at x'. Since the f-lifting of γ is total, there exists a path connecting q and $f(x'')$ such that it has a total f-lifting from p to x''. Therefore $x'' \in X$. Thus $U \setminus f^{-1}E(f) \subset X$. Hence the set $\overline{X}$ is both open and closed. Since M is connected, $M = \overline{X}$.

We show that f is injective on M. Suppose that there exist two points x and x' in M such that $f(x) = f(x')$. Let U and U' be disjoint neighborhoods of x and x'. Then there exist points $z \in U \cap X$ and $z' \in U' \cap X$ such that $f(z) = f(z')$. This is a contradiction with the injectivity of $f \mid X$. Thus the mapping f is injective. ▫

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