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THE BOUNDEDNESS OF CLASSICAL OPERATORS ON VARIABLE L^p SPACES

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Abstract. We show that many classical operators in harmonic analysis—such as maximal operators, singular integrals, commutators and fractional integrals—are bounded on the variable Lebesgue space $L^{p(\cdot)}$ whenever the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}$. Further, we show that such operators satisfy vector-valued inequalities. We do so by applying the theory of weighted norm inequalities and extrapolation.

As applications we prove the Calderón–Zygmund inequality for solutions of $\triangle u = f$ in variable Lebesgue spaces, and prove the Calderón extension theorem for variable Sobolev spaces.

1. Introduction

Given an open set $\Omega \subset \mathbf{R}^n$, we consider a measurable function $p\colon \Omega \longrightarrow [1,\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalize the standard L^p spaces: if $p(x) = p_0$ is

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constant, then $L^{p(\cdot)}(\Omega)$ equals $L^{p_0}(\Omega)$. (Here and below we write $p(\cdot)$ instead of p to emphasize that the exponent is a function and not a constant.) They have many properties in common with the standard L^p spaces.

These spaces, and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\Omega)$, are of interest in their own right, and also have applications to partial differential equations and the calculus of variations. (See, for example, [1], [12], [15], [19], [30], [39], [46] and their references.)

In many applications, a crucial step has been to show that one of the classical operators of harmonic analysis—e.g., maximal operators, singular integrals, fractional integrals—is bounded on a variable L^p space. Many authors have considered the question of sufficient conditions on the exponent function $p(\cdot)$ for given operators to be bounded: see, for example, [13], [15], [27], [28], [29], [40].

Our approach is different. Rather than consider estimates for individual operators, we apply techniques from the theory of weighted norm inequalities and extrapolation to show that the boundedness of a wide variety of operators follows from the boundedness of the maximal operator on variable L^p spaces, and from known estimates on weighted Lebesgue spaces. In order to provide the foundation for stating our results, we discuss each of these ideas in turn.

The maximal operator. In harmonic analysis, a fundamental operator is the Hardy–Littlewood maximal operator. Given a function f, we define the maximal function, Mf, by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes containing x. It is well known that M is bounded on L^p, $1 < p < \infty$, and it is natural to ask for which exponent functions $p(\cdot)$ the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$. For conciseness, define $\mathscr{P}(\Omega)$ to be the set of measurable functions $p\colon \Omega \longrightarrow [1,\infty)$ such that

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \qquad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Let $\mathscr{B}(\Omega)$ be the set of $p(\cdot) \in \mathscr{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Theorem 1.1. *Given an open set $\Omega \subset \mathbf{R}^n$, and $p(\cdot) \in \mathscr{P}(\Omega)$, suppose that $p(\cdot)$ satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad x, y \in \Omega, \quad |x-y| \le 1/2, \tag{1.1}$$

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \quad x, y \in \Omega, \quad |y| \ge |x|. \tag{1.2}$$

Then $p(\cdot) \in \mathscr{B}(\Omega)$, that is, the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

Theorem 1.1 is independently due to Cruz-Uribe, Fiorenza and Neugebauer [10] and to Nekvinda [35]. (In fact, Nekvinda replaced (1.2) with a slightly more general condition.) Earlier, Diening [12] showed that (1.1) alone is sufficient if Ω is bounded. Examples show that the continuity conditions (1.1) and (1.2) are in some sense close to necessary: see Pick and Růžička [37] and [10]. See also the examples in [33]. The condition $p_- > 1$ is necessary for M to be bounded; see [10].

Very recently, Diening [14], working in the more general setting of Musielak–Orlicz spaces, has given a necessary and sufficient condition on $p(\cdot)$ for M to be bounded on $L^{p(\cdot)}(\mathbf{R}^n)$. His exact condition is somewhat technical and we refer the reader to [14] for details.

Because our proofs rely on duality arguments, we will not need that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$ but on its associate space $L^{p'(\cdot)}(\Omega)$, where $p'(\cdot)$ is the conjugate exponent function defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad x \in \Omega.$$

Since

$$|p'(x) - p'(y)| \le \frac{|p(x) - p(y)|}{(p_- - 1)^2},$$

it follows at once that if $p(\cdot)$ satisfies (1.1) and (1.2), then so does $p'(\cdot)$—i.e., if these two conditions hold, then M is bounded on $L^{p(\cdot)}(\Omega)$ and $L^{p'(\cdot)}(\Omega)$. Furthermore, Diening's characterization of variable L^p spaces on which the maximal operator is bounded has the following important consequence (see [14, Theorem 8.1]).

Theorem 1.2. *Let $p(\cdot) \in \mathscr{P}(\mathbf{R}^n)$. Then the following conditions are equivalent:*

(a) $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$.
(b) $p'(\cdot) \in \mathscr{B}(\mathbf{R}^n)$
(c) $p(\cdot)/q \in \mathscr{B}(\mathbf{R}^n)$ *for some* $1 < q < p_-$.
(d) $\big(p(\cdot)/q\big)' \in \mathscr{B}(\mathbf{R}^n)$ *for some* $1 < q < p_-$.

Weights and extrapolation. By a weight we mean a non-negative, locally integrable function w. There is a vast literature on weights and weighted norm inequalities; here we will summarize the most important aspects, and we refer the reader to [17], [21] and their references for complete information.

Central to the study of weights are the so-called A_p weights, $1 \le p \le \infty$. When $1 < p < \infty$, we say $w \in A_p$ if for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)^{p-1} \le C < \infty.$$

We say that $w \in A_1$ if $Mw(x) \le Cw(x)$ for a.e. x. If $1 \le p < q < \infty$, then $A_p \subset A_q$. We let A_∞ denote the union of all the A_p classes, $1 \le p < \infty$.

Weighted norm inequalities are generally of two types. The first is

$$\int_{\mathbf{R}^n} |Tf(x)|^{p_0} w(x)\,dx \le C \int_{\mathbf{R}^n} |f(x)|^{p_0} w(x)\,dx, \tag{1.3}$$

where T is some operator and $w \in A_{p_0}$, $1 < p_0 < \infty$. (In other words, T is defined and bounded on $L^{p_0}(w)$.) The constant is assumed to depend only on the A_{p_0} constant of w. The second type is

$$\int_{\mathbf{R}^n} |Tf(x)|^{p_0} w(x)\,dx \le C \int_{\mathbf{R}^n} |Sf(x)|^{p_0} w(x)\,dx, \tag{1.4}$$

where S and T are operators, $0 < p_0 < \infty$, $w \in A_\infty$, and f is such that the left-hand side is finite. The constant is assumed to depend only on the A_∞ constant of w. Such inequalities are known for a wide variety of operators and pairs of operators. (See [17], [21].)

Corresponding to these types of inequalities are two extrapolation theorems. Associated with (1.3) is the classical extrapolation theorem of Rubio de Francia [38] (also see [17], [21]). He proved that if (1.3) holds for some operator T, a fixed value p_0, $1 < p_0 < \infty$, and every weight $w \in A_{p_0}$, then (1.3) holds with p_0 replaced by any p, $1 < p < \infty$, whenever $w \in A_p$. Recently, the analogous extrapolation result for inequalities of the form (1.4) was proved in [11]: if (1.4) holds for some p_0, $0 < p_0 < \infty$ and every $w \in A_\infty$, then it holds for every p, $0 < p < \infty$. (More general versions of these results will be stated in Section 6 below.)

1.1. Main results. The proofs of the above extrapolation theorems depend not on the properties of the operators, but rather on duality, the structure of A_p weights, and norm inequalities for the Hardy–Littlewood maximal operator. These ideas can be extended to the setting of variable L^p spaces to yield our main result, which can be summarized as follows: If an operator T, or a pair of operators (T, S), satisfies weighted norm inequalities on the classical Lebesgue spaces, then it satisfies the corresponding inequality in a variable L^p space on which the maximal operator is bounded.

To state and prove our main result, we will adopt the approach taken in [11]. There it was observed that since nothing is assumed about the operators involved (e.g., linearity or sublinearity), it is better to replace inequalities (1.3) and (1.4) with

$$\int_{\mathbf{R}^n} f(x)^{p_0} w(x)\,dx \le C \int_{\mathbf{R}^n} g(x)^{p_0} w(x)\,dx, \tag{1.5}$$

where the pairs (f, g) are such that the left-hand side of the inequality is finite. One important consequence of adopting this approach is that vector-valued inequalities follow immediately from extrapolation.

Hereafter $\mathscr{F}$ will denote a family of ordered pairs of non-negative, measurable functions (f, g). Whenever we say that an inequality such as (1.5) holds for any $(f, g) \in \mathscr{F}$ and $w \in A_q$ (for some q, $1 \le q \le \infty$), we mean that it holds for any pair in $\mathscr{F}$ such that the left-hand side is finite, and the constant C depends only on p_0 and the A_q constant of w.

Finally, note that in the classical Lebesgue spaces we can work with L^p where $0 < p < 1$. (Thus, in (1.4) or (1.5) we can take $p_0 < 1$.) We would like to consider analogous spaces with variable exponents. Define $\mathscr{P}^0(\Omega)$ to be the set of measurable functions $p\colon \Omega \longrightarrow (0, \infty)$ such that

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 0, \qquad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

Given $p(\,\cdot\,) \in \mathscr{P}^0(\Omega)$, we can define the space $L^{p(\,\cdot\,)}(\Omega)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0} \in L^{q(\,\cdot\,)}(\Omega)$, where $0 < p_0 < p_-$ and $q(x) = p(x)/p_0 \in \mathscr{P}(\Omega)$. We can define a quasi-norm on this space by

$$\|f\|_{p(\,\cdot\,),\Omega} = \big\||f|^{p_0}\big\|_{q(\,\cdot\,),\Omega}^{1/p_0}.$$

We will not need any other properties of these spaces, so this definition will suffice for our purposes.

Theorem 1.3. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, suppose that for some p_0, $0 < p_0 < \infty$, and for every weight $w \in A_1$,*

$$\int_\Omega f(x)^{p_0} w(x)\,dx \le C_0 \int_\Omega g(x)^{p_0} w(x)\,dx, \qquad (f, g) \in \mathscr{F}, \tag{1.6}$$

where C_0 depends only on p_0 and the A_1 constant of w. Let $p(\,\cdot\,) \in \mathscr{P}^0(\Omega)$ be such that $p_0 < p_-$, and $\big(p(\,\cdot\,)/p_0\big)' \in \mathscr{B}(\Omega)$. Then for all $(f, g) \in \mathscr{F}$ such that $f \in L^{p(\,\cdot\,)}(\Omega)$,

$$\|f\|_{p(\,\cdot\,),\Omega} \le C\,\|g\|_{p(\,\cdot\,),\Omega}, \tag{1.7}$$

where the constant C is independent of the pair (f, g).

We want to call attention to two features of Theorem 1.3. First, the conclusion (1.7) is an *a priori* estimate: that is, it holds for all $(f, g) \in \mathscr{F}$ such that $f \in L^{p(\,\cdot\,)}(\Omega)$. In practice, when applying this theorem in conjunction with inequalities of the form (1.3) to show that an operator is bounded on variable L^p we will usually need to work with a collection of functions f which satisfy the given weighted Lebesgue space inequality and are dense in $L^{p(\,\cdot\,)}(\Omega)$. When working with inequalities of the form (1.3) the final estimate will hold for a suitable family of "nice" functions.

Second, the family $\mathscr{F}$ in the hypothesis of and conclusion of Theorem 1.7 is the same, so the goal is to find a large, reasonable family $\mathscr{F}$ such that (1.6) holds with a constant depending only on p_0 and the A_1 constant of w.

Remark 1.4. In Theorem 1.3, (1.7) holds if $p(\cdot)$ satisfies (1.1) and (1.2). By Theorem 1.1, setting $q(x) = p(x)/p_0$ we have that $q(\cdot) \in \mathscr{P}(\Omega)$ and

$$|q'(x) - q'(y)| \le \frac{|p(x) - p(y)|}{p_0(p_-/p_0 - 1)^2}.$$

Remark 1.5. When $\Omega = \mathbf{R}^n$, if $1 \le p_0 < p_-$, then by Theorem 1.2 the hypothesis that $\big(p(\cdot)/p_0\big)' \in \mathscr{B}(\mathbf{R}^n)$ is equivalent to assuming that $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. As we will see below, this will allow us to conclude that a variety of operators are bounded on $L^{p(\cdot)}(\mathbf{R}^n)$ whenever the Hardy–Littlewood maximal operator is.

Remark 1.6. Our approach using pairs of functions leads to an equivalent formulation of Theorem 1.3 in which the exponent p_0 does not play a role. This can be done by defining a new family $\mathscr{F}_{p_0}$ consisting of the pairs (f^{p_0}, g^{p_0}) with $(f,g) \in \mathscr{F}$. Notice that in this case (1.6) is satisfied by $\mathscr{F}_{p_0}$ with $p_0 = 1$. Thus, the case $p_0 = 1$ will imply that if $1 < p_-$ and $p(\cdot)' \in \mathscr{B}(\Omega)$ then (1.7) holds. Therefore, if we define $r(x) = p(x)\,p_0$, we have that $r(\cdot) \in \mathscr{P}^0(\Omega)$, $p_0 < r_-$, $(r(\cdot)/p_0)' \in \mathscr{B}(\Omega)$ and (1.7) holds with $r(\cdot)$ in place of $p(\cdot)$. But this is exactly the conclusion of Theorem 1.3.

Remark 1.7. We believe that a more general version of Theorem 1.3 is true, one which holds for larger classes of weights and yields inequalities in weighted variable L^p spaces. However, proving such a result will require a weighted version of Theorem 1.1, and even the statement of such a result has eluded us. For such a weighted extrapolation result the appropriate class of weights is no longer A_1, but A_p (as in [38]) or A_∞ (as in [11]). We emphasize, though, that the class A_1, which is the smallest among the A_p classes, is the natural one to consider when attempting to prove unweighted estimates.

Theorem 1.3 can be generalized to give "off-diagonal" results. In the classical setting, the extrapolation theorem of Rubio de Francia was extended in this manner by Harboure, Macías and Segovia [24].

Theorem 1.8. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, assume that for some p_0 and q_0, $0 < p_0 \le q_0 < \infty$, and every weight $w \in A_1$,*

$$\left(\int_\Omega f(x)^{q_0} w(x)\,dx\right)^{1/q_0} \le C_0 \left(\int_\Omega g(x)^{p_0} w(x)^{p_0/q_0}\,dx\right)^{1/p_0}, \quad (f,g) \in \mathscr{F}. \tag{1.8}$$

Given $p(\cdot) \in \mathscr{P}^0(\Omega)$ such that $p_0 < p_- \le p_+ < p_0\,q_0/(q_0 - p_0)$, define the function $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \qquad x \in \Omega. \tag{1.9}$$

If $\big(q(x)/q_0\big)' \in \mathscr{B}(\Omega)$, then for all $(f,g) \in \mathscr{F}$ such that $f \in L^{q(\cdot)}(\Omega)$,

$$\|f\|_{q(\cdot),\Omega} \le C\|g\|_{p(\cdot),\Omega}. \tag{1.10}$$

Remark 1.9. As before, (1.10) holds if $p(\cdot)$ satisfies (1.1) and (1.2).

We can generalize Theorem 1.3 by combining it with the two extrapolation theorems discussed above. This is possible since $A_1 \subset A_p$, $1 < p \leq \infty$. This has two advantages. First, it makes clear that the hypotheses which must be satisfied correspond to those of the known weighted norm inequalities; see, in particular, the applications discussed in Section 2 below. Second, as in [11], we are able to prove vector-valued inequalities in variable L^p spaces with essentially no additional work. All such inequalities are new.

Corollary 1.10. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, assume that for some p_0, $0 < p_0 < \infty$, and for every $w \in A_\infty$,*

$$\int_\Omega f(x)^{p_0} w(x)\,dx \leq C_0 \int_\Omega g(x)^{p_0} w(x)\,dx, \qquad (f,g) \in \mathscr{F}. \tag{1.11}$$

Let $p(\cdot) \in \mathscr{P}^0(\Omega)$ be such that there exists $0 < p_1 < p_-$ with $\big(p(\cdot)/p_1\big)' \in \mathscr{B}(\Omega)$. Then for all $(f,g) \in \mathscr{F}$ such that $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot),\Omega} \leq C\,\|g\|_{p(\cdot),\Omega}. \tag{1.12}$$

Furthermore, for every $0 < q < \infty$ and sequence $\{(f_j, g_j)\}_j \subset \mathscr{F}$,

$$\bigg\|\Big(\sum_j (f_j)^q\Big)^{1/q}\bigg\|_{p(\cdot),\Omega} \leq C \bigg\|\Big(\sum_j (g_j)^q\Big)^{1/q}\bigg\|_{p(\cdot),\Omega}. \tag{1.13}$$

Corollary 1.11. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, assume that* (1.11) *holds for some $1 < p_0 < \infty$, for every $w \in A_{p_0}$ and for all $(f,g) \in \mathscr{F}$. Let $p(\cdot) \in \mathscr{P}(\Omega)$ be such that there exists $1 < p_1 < p_-$ with $\big(p(\cdot)/p_1\big)' \in \mathscr{B}(\Omega)$. Then* (1.12) *holds for all $(f,g) \in \mathscr{F}$ such that $f \in L^{p(\cdot)}(\Omega)$. Furthermore, for every $1 < q < \infty$ and $\{(f_j, g_j)\}_j \in \mathscr{F}$, the vector-valued inequality* (1.13) *holds.*

The rest of this paper is organized as follows. To illustrate the power of our results, we first consider some applications. In Section 2 we give a number of examples of operators which are bounded on $L^{p(\cdot)}$. These results are immediate consequences of the above results and the theory of weighted norm inequalities. Some of these have been proved by others, but most are new. We also prove vector-valued inequalities for these operators, all of which are new results. In Section 3 we present an application to partial differential equations: we extend the Calderón–Zygmund inequality (see [5], [22]) to solutions of $\triangle u = f$ with $f \in L^{p(\cdot)}(\Omega)$. In Section 4 we give an application to the theory of Sobolev spaces: we show that the Calderón extension theorem (see [2], [4]) holds in variable Sobolev spaces. In Section 5 we prove Theorems 1.3 and 1.8. Our proof is adapted from the arguments given in [11]. Finally, in Section 6 we prove Corollaries 1.10 and 1.11.

Throughout this paper, we will make use of the basic properties of variable L^p spaces, and will state some results as needed. For a detailed discussion of these spaces, see Kováčik and Rákosník [30]. As we noted above, in order to emphasize that we are dealing with variable exponents, we will always write $p(\cdot)$ instead of p to denote an exponent function. Throughout, C will denote a positive constant whose exact value may change at each appearance.

2. Applications: Estimates for classical operators on $L^{p(\cdot)}$

In this section we give a number of applications of Theorems 1.3 and 1.8, and Corollaries 1.10 and 1.11, to show that a wide variety of classical operators are bounded on the variable L^p spaces. In the following applications we will impose different conditions on the exponents $p(\cdot)$ to guarantee the corresponding estimates. In most of the cases, it will suffice to assume that $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, or in particular that $p(\cdot)$ satisfies (1.1) and (1.2).

As we noted in the remarks following Theorem 1.3, to prove these applications we will need to use density arguments. In doing so we will use the following facts:

(1) L_c^∞, bounded functions of compact support, and C_c^∞, smooth functions of compact support, are dense in $L^{p(\cdot)}(\Omega)$. See Kováčik and Rákosník [30].
(2) If $p_+ < \infty$ and $f \in L^{p_+}(\Omega) \cap L^{p_-}(\Omega)$, then $f \in L^{p(\cdot)}(\Omega)$. This follows from the fact that $|f(x)|^{p(x)} \le |f(x)|^{p_+}\chi_{\{|f(x)|\ge 1\}} + |f(x)|^{p_-}\chi_{\{|f(x)|<1\}}$.

2.2. The Hardy–Littlewood maximal function. It is well known that for $1 < p < \infty$ and for $w \in A_p$,

$$\int_{\mathbf{R}^n} Mf(x)^p w(x)\,dx \le C \int_{\mathbf{R}^n} f(x)^p w(x)\,dx.$$

From Corollary 1.11 with the pairs $(Mf, |f|)$, we get vector-valued inequalities for M on $L^{p(\cdot)}$, provided there exists $1 < p_1 < p_-$ with $\big(p(\cdot)/p_1\big)' \in \mathscr{B}(\mathbf{R}^n)$; by Theorem 1.2, this is equivalent to $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. To apply Corollary 1.11 we need to restrict the pairs to functions $f \in L_c^\infty$, but since these form a dense subset we get the desired estimate for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$.

Corollary 2.1. *If $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, then for all $1 < q < \infty$,*

$$\Big\| \Big(\sum_j (Mf_j)^q \Big)^{1/q} \Big\|_{p(\cdot),\mathbf{R}^n} \le C \Big\| \Big(\sum_j |f_j|^q \Big)^{1/q} \Big\|_{p(\cdot),\mathbf{R}^n}.$$

Remark 2.2. From Corollary 1.11 we also get one of the implications of Theorem 1.2: if $\big(p(\cdot)/p_1\big)' \in \mathscr{B}(\mathbf{R}^n)$ then $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. It is very tempting to speculate that all of Theorem 1.2 can be proved via extrapolation, but we have been unable to do so.

2.2. The sharp maximal operator. Given a measurable function f and a cube Q, define

$$f_Q = \frac{1}{|Q|} \int_Q f(y)\,dy,$$

and the sharp maximal operator by

$$M^\# f(x) = \sup_{x \ni Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy.$$

The sharp maximal operator was introduced by Fefferman and Stein [20], who showed that for all p, $0<p<\infty$, and $w\in A_\infty$,

$$\int_{\mathbf{R}^n} Mf(x)^p w(x)\,dx \le C\int_{\mathbf{R}^n} M^{\#}f(x)^p w(x)\,dx.$$

(Also see Journé [26].) Therefore, by Corollary 1.10 with the pairs $(Mf, M^{\#}f)$, $f\in L_c^\infty(\mathbf{R}^n)$, and by Theorem 1.2 we have the following result.

Corollary 2.3. *Let $p(\cdot)\in\mathscr{P}^0(\mathbf{R}^n)$ be such that there exists $0<p_1<p_-$ with $p(\cdot)/p_1\in\mathscr{B}(\mathbf{R}^n)$. Then,*

$$\|Mf\|_{p(\cdot),\mathbf{R}^n} \le C\,\|M^{\#}f\|_{p(\cdot),\mathbf{R}^n}, \tag{2.1}$$

and for all $0<q<\infty$,

$$\Big\|\Big(\sum_j (Mf_j)^q\Big)^{1/q}\Big\|_{p(\cdot),\mathbf{R}^n} \le C\Big\|\Big(\sum_j (M^{\#}f_j)^q\Big)^{1/q}\Big\|_{p(\cdot),\mathbf{R}^n}. \tag{2.2}$$

Remark 2.4. Corollary 2.3 generalizes results due to Diening and Růžička [15, Theorem 3.6] and Diening [14, Theorem 8.10], who proved (2.1) with Mf replaced by f on the left-hand side and under the assumptions that $p(\cdot)$ and $p'(\cdot)\in\mathscr{B}(\mathbf{R}^n)$ with $1<p_-\le p_+<\infty$ in the first paper and $p(\cdot)\in\mathscr{B}(\mathbf{R}^n)$ in the second. Notice that our result is more general since we allow $p(\cdot)$ to go below 1 and we only need $\big(p(\cdot)/p_1\big)'\in\mathscr{B}(\mathbf{R}^n)$ for some small value $0<p_1<p_-$. Furthermore, we automatically obtain the vector-valued inequalities given in (2.2).

2.3. Singular integral operators. Given a locally integrable function K defined on $\mathbf{R}^n\setminus\{0\}$, suppose that the Fourier transform of K is bounded, and K satisfies

$$|K(x)|\le \frac{C}{|x|^n},\quad |\nabla K(x)|\le \frac{C}{|x|^{n+1}},\quad x\ne 0. \tag{2.3}$$

Then the singular integral operator T, defined by $Tf(x)=K*f(x)$, is a bounded operator on weighted L^p. More precisely, given $1<p<\infty$, if $w\in A_p$, then

$$\int_{\mathbf{R}^n}|Tf(x)|^p w(x)\,dx \le C\int_{\mathbf{R}^n}|f(x)|^p w(x)\,dx. \tag{2.4}$$

(For details, see [17], [21].)

From Corollary 1.11, we get that T is bounded on variable L^p provided there exists $1<p_1<p_-$ with $\big(p(\cdot)/p_1\big)'\in\mathscr{B}(\mathbf{R}^n)$; by Theorem 1.2 this is equivalent to $p(\cdot)\in\mathscr{B}(\mathbf{R}^n)$. Again, to apply the corollary we need to restrict ourselves to a suitable dense family of functions. We use the fact that C_c^∞ is dense in $L^{p(\cdot)}(\mathbf{R}^n)$, and the fact that if $f\in C_c^\infty$, then $Tf\in\bigcap_{1<p<\infty}L^p\subset L^{p(\cdot)}(\mathbf{R}^n)$.

Corollary 2.5. *If* $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$*, then*

$$\|Tf\|_{p(\cdot),\mathbf{R}^n} \le C\,\|f\|_{p(\cdot),\mathbf{R}^n}, \tag{2.5}$$

and for all $1 < q < \infty$*,*

$$\left\|\left(\sum_j |Tf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbf{R}^n} \le C\left\|\left(\sum_j |f_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbf{R}^n}. \tag{2.6}$$

Remark 2.6. We can get estimates on sets Ω in the following way: observe that (2.4) implies that for any $\Omega \subset \mathbf{R}^n$ we have

$$\begin{aligned}\int_\Omega |Tf(x)|^p w(x)\,dx &\le \int_{\mathbf{R}^n} |Tf(x)|^p w(x)\,dx \\ &\le C\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx = C\int_\Omega |f(x)|^p w(x)\,dx\end{aligned}$$

for all f such that $\operatorname{supp}(f) \subset \Omega$ and for all $w \in A_p$. Thus, we can apply Corollary 1.11 on Ω and in particular, if $p(\cdot) \in \mathscr{P}(\Omega)$ satisfies (1.1) and (1.2), then

$$\|Tf\|_{p(\cdot),\Omega} \le C\|f\|_{p(\cdot),\Omega}.$$

We will use this observation below.

Singular integrals satisfy another inequality due to Coifman and Fefferman [7]:

$$\int_{\mathbf{R}^n} |Tf(x)|^p w(x)\,dx \le C\int_{\mathbf{R}^n} |Mf(x)|^p w(x)\,dx,$$

for all $0 < p < \infty$ and $w \in A_\infty$ and f such that the left-hand side is finite. In particular, if $w \in A_1 \subset A_p$, then the left-hand side is finite for all $f \in L_c^\infty(\mathbf{R}^n)$. Thus, by applying Corollary 1.10 we can prove the following.

Corollary 2.7. *Let* $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$ *be such that there exists* $0 < p_1 < p_-$ *with* $p(\cdot)/p_1 \in \mathscr{B}(\mathbf{R}^n)$*. Then*

$$\|Tf\|_{p(\cdot),\mathbf{R}^n} \le C\,\|Mf\|_{p(\cdot),\mathbf{R}^n}, \tag{2.7}$$

and for all $0 < q < \infty$*,*

$$\left\|\left(\sum_j |Tf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbf{R}^n} \le C\left\|\left(\sum_j |Mf_j|^q\right)^{1/q}\right\|_{p(\cdot),\mathbf{R}^n}. \tag{2.8}$$

Remark 2.8. Inequality (2.5) was proved by Diening and Růžička [15, Theorem 4.8] using (2.1) and assuming that $p(\cdot), \big(p(\cdot)/s\big)' \in \mathscr{B}(\mathbf{R}^n)$ for some $0 < s < 1$. More recently, Diening [14] showed that it was enough to assume $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. Note that our technique provides an alternative proof which also yields vector-valued inequalities. A weighted version of (2.5) was proved by Kokilashvili and Samko [28].

Remark 2.9. These results can be generalized to the so-called Calderón–Zygmund operators of Coifman and Meyer. Also, the same estimates hold for T_*, the supremum of the truncated integrals. We refer the reader to [17], [26] for more details.

Similar inequalities hold for homogeneous *singular integral operators with "rough" kernels*. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$, and suppose

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n}, \tag{2.9}$$

where $\Omega \in L^r(S^{n-1})$, for some r, $1 < r \le \infty$, and $\int_{S^{n-1}} \Omega(y)\,dy = 0$. Then, if $r' < p < \infty$ and $w \in A_{p/r'}$, inequality (2.4) holds. (See Duoandikoetxea [16] and Watson [44].) To apply Theorem 1.3 we restate these weighted norm estimates as

$$\int_{\mathbf{R}^n} \big(|Tf(x)|^{r'}\big)^s w(x)\,dx \le \int_{\mathbf{R}^n} \big(|f(x)|^{r'}\big)^s w(x)\,dx$$

for every $1 < s < \infty$ and all $w \in A_s$. We consider the family of pairs $\big(|Tf|^{r'}, |f|^{r'}\big)$ which satisfy the hypotheses of Corollary 1.11. Then for $s(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ such that $\big(s(\cdot)/s_1\big)' \in \mathscr{B}(\mathbf{R}^n)$ for some $1 < s_1 < s_-$, we have

$$\big\| |Tf|^{r'} \big\|_{s(\cdot),\mathbf{R}^n} \le C \, \big\| |f|^{r'} \big\|_{s(\cdot),\mathbf{R}^n}.$$

By Theorem 1.2, the assumptions on $s(\cdot)$ are equivalent to $s(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. If we let $p(x) = s(x)r'$, then we see that T is bounded $L^{p(\cdot)}(\mathbf{R}^n)$ for all $p(\cdot)$ such that $p(\cdot)/r' \in \mathscr{B}(\mathbf{R}^n)$. In the same way we can prove l^q-valued inequalities as (2.6) for all $r' < q < \infty$. Note in particular that all of these estimates hold if $p_- > r'$ and $p(\cdot)$ satisfies (1.1) and (1.2).

Similar inequalities also hold for *Banach space valued singular integrals*, since such operators satisfy weighted norm inequalities with A_p weights. For further details, we refer the reader to [21]. Here we note one particular application. Let $\varphi \in L^1$ be a non-negative function such that

$$|\varphi(x-y) - \varphi(x)| \le \frac{C|y|}{|x|^{n+1}}, \quad |x| > 2|y| > 0.$$

Let $\varphi_t(x) = t^{-n}\varphi(x/t)$, and define the maximal operator M_φ by

$$M_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|.$$

If $1 < p < \infty$ and $w \in A_p$, then $\|M_\varphi f\|_{L^p(w)} \le C\|f\|_{L^p(w)}$. (In the unweighted case, this result is originally due to Zo [48].) Therefore, by Corollary 1.11, M_φ is bounded on $L^{p(\cdot)}$ for $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$. In particular, it is bounded if $p(\cdot)$ satisfies (1.1) and (1.2); this gives a positive answer to a conjecture made in [9].

2.4. Commutators. Given a Calderón–Zygmund singular integral operator T, and a function $b \in \mathrm{BMO}$, define the commutator $[b, T]$ to be the operator

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

These operators were shown to be bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, by Coifman, Rochberg and Weiss [8]. In [36] it was shown that for all $0 < p < \infty$ and all $w \in A_\infty$,

$$\int_{\mathbf{R}^n} \big|[b, T]f(x)\big|^p w(x)\, dx \le C \int_{\mathbf{R}^n} M^2 f(x)^p w(x)\, dx, \tag{2.10}$$

where $M^2 = M \circ M$. Hence, if $1 < p < \infty$ and $w \in A_p$, then $[b, T]$ is bounded on $L^p(w)$. Thus, we can apply Corollaries 1.10 and 1.11 and Theorem 1.2 to get the following.

Corollary 2.10. *Let* $p(\cdot) \in \mathscr{P}^0(\mathbf{R}^n)$.

(a) *If there exists* $0 < p_1 < p_-$ *with* $p(\cdot)/p_1 \in \mathscr{B}(\mathbf{R}^n)$, *then*

$$\big\|[T, b]f\big\|_{p(\cdot),\mathbf{R}^n} \le C\, \|M^2 f\|_{p(\cdot),\mathbf{R}^n},$$

and for all $0 < q < \infty$,

$$\bigg\|\Big(\sum_j \big|[T, b]f_j\big|^q\Big)^{1/q}\bigg\|_{p(\cdot),\mathbf{R}^n} \le C \bigg\|\Big(\sum_j |M^2 f_j|^q\Big)^{1/q}\bigg\|_{p(\cdot),\mathbf{R}^n}.$$

(b) *If* $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$, *then*

$$\big\|[T, b]f\big\|_{p(\cdot),\mathbf{R}^n} \le C\, \|f\|_{p(\cdot),\mathbf{R}^n},$$

and for all $1 < q < \infty$,

$$\bigg\|\Big(\sum_j \big|[T, b]f_j\big|^q\Big)^{1/q}\bigg\|_{p(\cdot),\mathbf{R}^n} \le C \bigg\|\Big(\sum_j |f_j|^q\Big)^{1/q}\bigg\|_{p(\cdot),\mathbf{R}^n}.$$

Very recently, the boundedness of commutators on variable L^p spaces was proved by Karlovich and Lerner [27].

2.5. Multipliers. Given a bounded function m, define the operator T_m, (initially on $C_c^\infty(\mathbf{R}^n)$) by $\widehat{T_m f} = m\hat{f}$. The function m is referred to as a multiplier. Here we consider two important results: the multiplier theorems of Marcinkiewicz and Hörmander.

On the real line, if m has uniformly bounded variation on each dyadic interval in $\mathbf{R}$, then for $1 < p < \infty$ and $w \in A_p$,

$$\int_{\mathbf{R}} |T_m f(x)|^p w(x)\,dx \le C \int_{\mathbf{R}} |f(x)|^p w(x)\,dx. \tag{2.11}$$

(See Kurtz [31].) Therefore, by Corollary 1.11, if $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$,

$$\|T_m f\|_{p(\cdot),\mathbf{R}} \le C\|f\|_{p(\cdot),\mathbf{R}};$$

we also get the corresponding vector-valued inequalities with $1 < q < \infty$.

In higher dimensions (i.e., $n \ge 2$) let $k = [n/2] + 1$ and suppose that m satisfies $|D^\beta m(x)| \le C|x|^{-|\beta|}$ for $x \ne 0$ and every multi-index β with $|\beta| \le k$. If $n/k < p < \infty$ and $w \in A_{pk/n}$ then T_m is bounded on $L^p(w)$. (See Kurtz and Wheeden [32].) Proceeding as in the case of the singular integral operators with "rough" kernels we obtain that if $p(\cdot)/(n/k) \in \mathscr{B}(\mathbf{R}^n)$, then

$$\|T_m f\|_{p(\cdot),\mathbf{R}^n} \le C\|f\|_{p(\cdot),\mathbf{R}^n},$$

with constant C independent of $f \in C_c^\infty(\mathbf{R}^n)$. We also get l^q-valued inequalities with $n/k < q < \infty$ in the same way.

Remark 2.11. Weighted inequalities also hold for Bochner–Riesz multipliers, so from these we can deduce results on variable L^p spaces. For details, see [17] and the references it contains.

2.6. Square functions. Let φ be a Schwartz function such that $\int \varphi(x)\,dx = 0$, and for $t > 0$ let $\varphi_t(x) = t^{-n}\varphi(x/t)$. Given a locally integrable function f, we define two closely related functions: the area integral,

$$S_\varphi f(x) = \left(\int_{|x-y|<t} |\varphi_t * f(y)|^2 \frac{dt\,dy}{t^{n+1}} \right)^{1/2},$$

and for $1 < \lambda < \infty$ the Littlewood–Paley function

$$g_\lambda^* f(x) = \left(\int_0^\infty \int_{\mathbf{R}^n} |\varphi_t * f(y)|^2 \left(\frac{t}{t + |x-y|} \right)^{n\lambda} \frac{dy\,dt}{t^{n+1}} \right)^{1/2}.$$

In the classical case, we take φ to be the derivative of the Poisson kernel.

Given p, $1 < p < \infty$, and $w \in A_p$, the area integral is bounded on $L^p(w)$. In the classical case, this is due to Gundy and Wheeden [23]; in the general case it is due to Strömberg and Torchinsky [43]. Therefore, for all $p(\cdot) \in \mathscr{B}(\mathbf{R}^n)$,

$$\|S_\varphi f\|_{p(\cdot),\mathbf{R}^n} \le C\|f\|_{p(\cdot),\mathbf{R}^n}.$$

The same inequality is true for g_λ^* if $\lambda \ge 2$. If $1 < \lambda < 2$, then for $2/\lambda < p < \infty$ and $w \in A_{\lambda p/2}$, g_λ^* is bounded on $L^p(w)$. In the classical case, this is due to Muckenhoupt and Wheeden [34]; in the general case it is due to Strömberg and Torchinsky [43]. Therefore, arguing as before, if $p(\cdot)/(2/\lambda) \in \mathscr{B}(\mathbf{R}^n)$, then

$$\|g_\lambda^* f\|_{p(\cdot),\mathbf{R}^n} \le C\|f\|_{p(\cdot),\mathbf{R}^n},$$

with constant C independent of $f \in C_c^\infty(\mathbf{R}^n)$. For both kinds of square functions we also get the corresponding vector-valued inequalities.

2.7. Fractional integrals and fractional maximal operators. Given $0 < \alpha < n$, define the fractional integral operator I_α (also known as the Riesz potential), by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Define the associated fractional maximal operator, M_α, by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy.$$

Both operators satisfy weighted norm inequalities. To state them, we need a different class of weights: given p, q such that $1 < p < n/\alpha$ and

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n},$$

we say that $w \in A_{p,q}$ if for all cubes Q,

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-p'/q}\, dx \right)^{q/p'} \le C < \infty.$$

Note that this is equivalent to $w \in A_r$, where $r = 1 + q/p'$, so in particular, if $w \in A_1$, then $w \in A_{p,q}$. Muckenhoupt and Wheeden 34] showed that if $w \in A_{p,q}$ then

$$\left(\int_{\mathbf{R}^n} |I_\alpha f(x)|^q w(x)\, dx \right)^{1/q} \le C \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)^{p/q}\, dx \right)^{1/p},$$

$$\left(\int_{\mathbf{R}^n} M_\alpha f(x)^q w(x)\, dx \right)^{1/q} \le C \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)^{p/q}\, dx \right)^{1/p}.$$

(These results are usually stated with the class $A_{p,q}$ defined slightly differently, with w replaced by w^q. Our formulation, though non-standard, is better for our purposes.)

As in Remark 2.6, these estimates hold with the integrals restricted to any $\Omega \subset \mathbf{R}^n$. Thus Theorems 1.8 and 1.2 immediately yield the following results in variable L^p spaces.

Corollary 2.12. *Let $p(\cdot)$, $q(\cdot) \in \mathscr{P}(\Omega)$ be such that $p_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \qquad x \in \Omega.$$

If there exists q_0, $n/(n-\alpha) < q_0 < \infty$, such that $q(x)/q_0 \in \mathscr{B}(\Omega)$, then

$$\|I_\alpha f\|_{q(\cdot),\Omega} \le \|f\|_{p(\cdot),\Omega} \tag{2.12}$$

and

$$\|M_\alpha f\|_{q(\cdot),\Omega} \leq \|f\|_{p(\cdot),\Omega}. \tag{2.13}$$

Corollary 2.12 follows automatically from Theorem 1.8 applied to the pairs $(|I_\alpha f|, |f|)$ and $(M_\alpha f, |f|)$, since the estimates of Muckenhoupt and Wheeden above give (1.8) for all $1 < p_0 < n/\alpha$ and $n/(n-\alpha) < q_0 < \infty$ with

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}.$$

Remark 2.13. When $\Omega = \mathbf{R}^n$, the condition on $q(\cdot)$ is equivalent to saying that $q(\cdot)\,(n-\alpha)/n \in \mathscr{B}(\mathbf{R}^n)$. If there exists q_0 as above such that $q(\cdot)/q_0 \in \mathscr{B}(\mathbf{R}^n)$, then

$$\frac{q(x)}{n/(n-\alpha)} = \frac{q(x)}{q_0}\frac{q_0}{n/(n-\alpha)} \in \mathscr{B}(\mathbf{R}^n),$$

since the second ratio is greater than one. (Given $r(\cdot) \in \mathscr{B}(\mathbf{R}^n)$ and $\lambda > 1$, then by Jensen's inequality, $r(\cdot)\,\lambda \in \mathscr{B}(\mathbf{R}^n)$.)

On the other hand, by Theorem 1.2, if $q(\cdot)\,(n-\alpha)/n \in \mathscr{B}(\mathbf{R}^n)$ then there is $\lambda > 1$ such that $q(\cdot)\,(n-\alpha)/(n\,\lambda) \in \mathscr{B}(\mathbf{R}^n)$. Taking $q_0 = n\lambda/(n-\alpha)$ we have that $q_0 > n/(n-\alpha)$ and $q(\cdot)/q_0 \in \mathscr{B}(\mathbf{R}^n)$ as desired.

Inequality (2.12) extends several earlier results. Samko [40] proved (2.12) assuming that Ω is bounded, $p(\cdot)$ satisfies (1.1), and the maximal operator is bounded. (Note that given Theorem 1.1, his second hypothesis implies his third.) Diening [13] proved it on unbounded domains with (1.2) replaced by the stronger hypothesis that $p(\cdot)$ is constant outside of a large ball. Kokilashvili and Samko [29] proved it on $\mathbf{R}^n$ with $L^{q(\cdot)}$ replaced by a certain weighted variable L^p space. (They actually consider a more general operator $I_{\alpha(\cdot)}$ where the constant α in the definition of I_α is replaced by a function $\alpha(\cdot)$.) Implicit in these results are norm inequalities for M_α in the variable L^p spaces, since $M_\alpha f(x) \leq C I_\alpha(|f|)(x)$. This is made explicit by Kokilashvili and Samko [29].

Inequality (2.13) was proved directly by Capone, Cruz-Uribe and Fiorenza [6]; as an application they used it to prove (2.12) and to extend the Sobolev embedding theorem to variable L^p spaces. (Other authors have considered this question; see [6] and its references for further details.)

3. The Calderón–Zygmund inequality

In this section we consider the behavior of the solution of Poisson's equation,

$$\triangle u(x) = f(x), \quad \text{a.e. } x \in \Omega,$$

when $f \in L^{p(\cdot)}(\Omega)$, $p(\cdot) \in \mathscr{P}(\Omega)$. We restrict ourselves to the case $\Omega \subset \mathbf{R}^n$, $n \geq 3$.

We begin with a few definitions and a lemma. Given $p(\cdot) \in \mathscr{P}(\Omega)$ and a natural number k, define the variable Sobolev space $W^{k,p(\cdot)}(\Omega)$ to be the set of all functions $f \in L^{p(\cdot)}(\Omega)$ such that

$$\sum_{|\alpha|\le k} \|D^\alpha f\|_{p(\cdot),\Omega} < +\infty,$$

where the derivatives are understood in the sense of distributions.

Given a function f which is twice differentiable (in the weak sense), we define for $i = 1, 2$,

$$D^i f = \Big(\sum_{|\alpha|=i} (D^\alpha f)^2 \Big)^{1/2}.$$

We need the following auxiliary result whose proof can be found in [30].

Lemma 3.1. *If $\Omega \subset \mathbf{R}^n$ is a bounded domain, and if $p(\cdot), q(\cdot) \in \mathscr{P}(\Omega)$ are such that $p(x) \le q(x)$, $x \in \Omega$, then $\|f\|_{p(\cdot),\Omega} \le (1 + |\Omega|)\|f\|_{q(\cdot),\Omega}$.*

Theorem 3.2. *Given an open set $\Omega \subset \mathbf{R}^n$, $n \ge 3$, suppose $p(\cdot) \in \mathscr{P}(\Omega)$ with $p_+ < n/2$ satisfies (1.1) and (1.2). If $f \in L^{p(\cdot)}(\Omega)$, then there exists a function $u \in L^{q(\cdot)}(\Omega)$, where*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{2}{n}, \tag{3.1}$$

such that

$$\triangle u(x) = f(x), \quad \text{a.e. } x \in \Omega. \tag{3.2}$$

Furthermore,

$$\|D^2 u\|_{p(\cdot),\Omega} \le C\, \|f\|_{p(\cdot),\Omega}, \tag{3.3}$$

$$\|D^1 u\|_{r(\cdot),\Omega} \le C\, \|f\|_{p(\cdot),\Omega}, \tag{3.4}$$

$$\|u\|_{q(\cdot),\Omega} \le C\, \|f\|_{p(\cdot),\Omega}, \tag{3.5}$$

where

$$\frac{1}{p(x)} - \frac{1}{r(x)} = \frac{1}{n}.$$

In particular, if Ω is bounded, then $u \in W^{2,p(\cdot)}(\Omega)$.

Proof. Our proof roughly follows the proof in the setting of Lebesgue spaces given by Gilbarg and Trudinger [22], but also uses this result in key steps.

Fix $f \in L^{p(\cdot)}(\Omega)$; without loss of generality we may assume that $\|f\|_{p(\cdot),\Omega} = 1$. Decompose f as

$$f = f_1 + f_2 = f\,\chi_{\{x:|f(x)|>1\}} + f\,\chi_{\{x:|f(x)|\le 1\}}.$$

Note that $|f_i(x)| \leq |f(x)|$ and so $\|f_i\|_{p(\cdot),\Omega} \leq 1$. Further, we have that $f_1 \in L^{p_-}(\Omega)$ and $f_2 \in L^{p_+}(\Omega)$ since, by the definition of the norm in $L^{p(\cdot)}(\Omega)$ and since $\|f\|_{p(\cdot),\Omega} = 1$,

$$\int_\Omega f_1(x)^{p_-}\,dx = \int_{\{x\in\Omega:|f(x)|>1\}} |f(x)|^{p_-}\,dx \leq \int_\Omega |f(x)|^{p(x)}\,dx \leq 1,$$
$$\int_\Omega f_2(x)^{p_+}\,dx = \int_{\{x\in\Omega:|f(x)|\leq 1\}} |f(x)|^{p_+}\,dx \leq \int_\Omega |f(x)|^{p(x)}\,dx \leq 1.$$

Thus, we can solve Poisson's equation with f_1 and f_2 (see [22]): more precisely, define

$$u_1(x) = (\Gamma * f_1)(x), \qquad u_2 = (\Gamma * f_2)(x),$$

where Γ is the Newtonian potential,

$$\Gamma(x) = \frac{1}{n\,(2-n)\,\omega_n}\,|x|^{2-n},$$

and ω_n is the volume of the unit ball in $\mathbf{R}^n$. Since p_- and q_- also satisfy (3.1), by the Calderón–Zygmund inequality on classical Lebesgue spaces, $u_1 \in L^{q_-}(\Omega)$. Similarly, since p_+ and q_+ satisfy (3.1), $u_2 \in L^{q_+}(\Omega)$. Let $u = u_1 + u_2$; then $u \in L^{q_-}(\Omega) + L^{q_+}(\Omega)$. Since u_1 and u_2 are solutions of Poisson's equation,

$$\triangle u(x) = \triangle u_1(x) + \triangle u_2(x) = f_1(x) + f_2(x) = f(x), \qquad \text{a.e. } x \in \Omega.$$

We show that $u \in L^{q(\cdot)}(\Omega)$ and that (3.5) holds: by inequality (2.12),

$$\begin{aligned}
\|u\|_{q(\cdot),\Omega} &\leq \|u_1\|_{q(\cdot),\Omega} + \|u_2\|_{q(\cdot),\Omega} \\
&= \frac{1}{n\,(2-n)\,\omega_n}\big(\|I_2 f_1\|_{q(\cdot),\Omega} + \|I_2 f_2\|_{q(\cdot),\Omega}\big) \\
&\leq C\big(\|f_1\|_{p(\cdot),\Omega} + \|f_2\|_{p(\cdot),\Omega}\big) \\
&\leq C = C\|f\|_{p(\cdot),\Omega};
\end{aligned}$$

the last equality holds since $\|f\|_{p(\cdot),\Omega} = 1$.

Similarly, a direct computation shows that for any multi-index α, $|\alpha| = 1$,

$$|D^\alpha \Gamma(x)| \leq \frac{1}{n\,\omega_n}\,|x|^{1-n}.$$

Therefore,

$$\begin{aligned}
|D^\alpha u(x)| &\leq |D^\alpha(\Gamma * f_1)(x)| + |D^\alpha(\Gamma * f_2)(x)| \\
&= |(D^\alpha\Gamma * f_1)(x)| + |(D^\alpha\Gamma * f_2)(x)| \\
&\leq \frac{1}{n\,\omega_n}\big(I_1(|f_1|)(x) + I_1(|f_2|)(x)\big).
\end{aligned}$$

So again by inequality (2.12) we get

$$\|D^\alpha u\|_{r(\cdot),\Omega} \le C\left(\|f_1\|_{p(\cdot),\Omega} + \|f_2\|_{p(\cdot),\Omega}\right) \le C,$$

which yields inequality (3.4).

Given a multi-index α, $|\alpha| = 2$, another computation shows that $D^\alpha\Gamma$ is a singular convolution kernel which satisfies (2.3). Therefore, the operator

$$T_\alpha g(x) = (D^\alpha\Gamma * g)(x) = D^\alpha(\Gamma * g)(x)$$

is singular integral operator, and as before (3.3) follows from inequality (2.5) and Remark 2.6 applied to f_1 and f_2.

Finally, if Ω is bounded, since $p(x) \le q(x)$ and $p(x) \le r(x)$, $x \in \Omega$, by Lemma 3.1 we have that $u \in W^{2,p(\cdot)}(\Omega)$. □

Remark 3.3. In the previous estimates we could have worked directly with f. Had we done so, however, we would have had to check that all the integrals appearing were absolutely convergent. The advantage of decomposing f as f_1+f_2 is that we did not need to pay attention to this since $f_1 \in L^{p_-}(\Omega)$, $f_2 \in L^{p_+}(\Omega)$.

We also want to stress that u_1 and u_2, as solutions of Poisson's equation with $f_1 \in L^{p_-}(\Omega)$ and $f_2 \in L^{p_+}(\Omega)$, satisfy Lebesgue space estimates. For instance, as noted above, $u \in L^{q_-}(\Omega) + L^{q_+}(\Omega)$. However, we have actually proved more, since $L^{q(\cdot)}(\Omega)$ is a smaller space. Similar remarks hold for the first and second derivatives of u.

4. The Calderón extension theorem

In this section we state and prove the Calderón extension theorem for variable Sobolev spaces. Our proof follows closely the proof of the result in the classical setting; see, for example, R. Adams [2] or Calderón [4]. First, we give two definitions and a lemma.

Definition 4.1. Given a point $x \in \mathbf{R}^n$, a finite cone with vertex at x, C_x, is a set of the form

$$C_x = B_1 \cap \{x + \lambda(y-x) : y \in B_2,\ \lambda > 0\},$$

where B_1 is an open ball centered at x, and B_2 is an open ball which does not contain x.

Definition 4.2. An open set $\Omega \subset \mathbf{R}^n$ has the uniform cone property if there exists a finite collection of open sets $\{U_j\}$ (not necessarily bounded) and an associated collection $\{C_j\}$ of finite cones such that the following hold:

(1) there exists $\delta > 0$ such that

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\} \subset \bigcup_j U_j;$$

(2) for every index j and every $x \in \Omega \cap U_j$, $x + C_j \subset \Omega$.

An example of a set Ω with the uniform cone property is any bounded set whose boundary is locally Lipschitz. (See Adams [2].)

Finally, in giving extension theorems for variable L^p spaces, we must worry about extending the exponent function $p(\cdot)$. The following result shows that this is always possible, provided that $p(\cdot)$ satisfies (1.1) and (1.2).

Lemma 4.3. *Given an open set $\Omega \subset \mathbf{R}^n$ and $p(\cdot) \in \mathscr{P}(\Omega)$ such that* (1.1) *and* (1.2) *hold, there exists a function $\tilde{p}(\cdot) \in \mathscr{P}(\mathbf{R}^n)$ such that*:

(1) *$\tilde{p}$ satisfies* (1.1) *and* (1.2);
(2) $\tilde{p}(x) = p(x)$, $x \in \Omega$;
(3) $\tilde{p}_- = p_-$ *and* $\tilde{p}_+ = p_+$.

Remark 4.4. Diening [13] proved an extension theorem for exponents $p(\cdot)$ which satisfy (1.1), provided that Ω is bounded and has Lipschitz boundary. It would be interesting to determine if every exponent $p(\cdot) \in \mathscr{B}(\Omega)$ can be extended to an exponent function in $\mathscr{B}(\mathbf{R}^n)$.

Proof. Since $p(\cdot)$ is bounded and uniformly continuous, by a well-known result it extends to a continuous function on $\overline{\Omega}$. Straightforward limiting arguments show that this extension satisfies (1), (2) and (3).

The extension of $p(\cdot)$ on $\overline{\Omega}$ to $\tilde{p}(\cdot)$ defined on all of $\mathbf{R}^n$ follows from a construction due to Whitney [45] and described in detail in Stein [42, Chapter 6]. For ease of reference, we will follow Stein's notation. We first consider the case when $\overline{\Omega}$ is unbounded; the case when Ω is bounded is simpler and will be sketched below.

When $\overline{\Omega}$ is unbounded, (1.2) is equivalent to the existence of a constant p_∞, $p_- \le p_\infty \le p_+$, such that for all $x \in \overline{\Omega}$,

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)}.$$

Define a new function $r(\cdot)$ by $r(x) = p(x) - p_\infty$. Then $r(\cdot)$ is still bounded (though no longer necessarily positive), still satisfies (1.1) on $\overline{\Omega}$ and satisfies

$$|r(x)| \le \frac{C}{\log(e + |x|)}. \tag{4.1}$$

We will extend r to all of $\mathbf{R}^n$. If we define $\omega(t) = 1/\log(e/2t)$, $0 < t \le 1/2$, and $\omega(t) = 1$ for $t \ge 1/2$, then a straightforward calculation shows that $\omega(t)/t$ is a decreasing function and $\omega(2t) \le C\,\omega(t)$. Further, since $\log(e/2t) \approx \log(1/t)$, $0 < t < 1/2$, and since r is bounded, $|r(x) - r(y)| \le C\omega(|x - y|)$ for all $x, y \in \overline{\Omega}$. Therefore, by Corollary 2.2.3 in Stein [42, p. 175], there exists a function $\tilde{r}(\cdot)$ on $\mathbf{R}^n$ such that $\tilde{r}(x) = r(x)$, $x \in \overline{\Omega}$, and such that $\tilde{r}(\cdot)$ satisfies (1.1). For $x \in \mathbf{R}^n \setminus \overline{\Omega}$, $\tilde{r}(x)$ is defined by the sum

$$\tilde{r}(x) = \sum_k r(p_k)\varphi_k^*(x),$$

where $\{Q_k\}$ are the cubes of the Whitney decomposition of $\mathbf{R}^n \setminus \overline{\Omega}$, $\{\varphi_k^*\}$ is the partition of unity subordinate to this decomposition, and each point $p_k \in \overline{\Omega}$ is such that $\operatorname{dist}(p_k, Q_k) = \operatorname{dist}(\overline{\Omega}, Q_k)$.

It follows immediately from this definition that for all $x \in \mathbf{R}^n$, $r_- \le \tilde{r}(x) \le r_+$. However, $\tilde{r}(\cdot)$ need not satisfy (4.1) so we must modify it slightly. To do so we need the following observation: if f_1, f_2 are functions such that $|f_i(x) - f_i(y)| \le C\omega(|x-y|)$, $x, y \in \mathbf{R}^n$, $i = 1, 2$, then $\min(f_1, f_2)$ and $\max(f_1, f_2)$ satisfy the same inequality. The proof of this observation consists of a number of very similar cases. For instance, suppose $\min\bigl(f_1(x), f_2(x)\bigr) = f_1(x)$ and $\min\bigl(f_1(y), f_2(y)\bigr) = f_2(y)$. Then

$$\begin{aligned} f_1(x) - f_2(y) &\le f_2(x) - f_2(y) \le C\omega(|x-y|), \\ f_2(y) - f_1(x) &\le f_1(y) - f_1(x) \le C\omega(|x-y|). \end{aligned}$$

Hence,

$$\bigl|\min\bigl(f_1(x), f_2(x)\bigr) - \min\bigl(f_1(y), f_2(y)\bigr)\bigr| = |f_1(x) - f_2(y)| \le C\omega(|x-y|).$$

It follows immediately from this observation that

$$s(x) = \max\bigl(\min(\tilde{r}(x), C/\log(e+|x|)), -C/\log(e+|x|)\bigr)$$

satisfies (1.1) and (4.1). Therefore, if we define

$$\tilde{p}(x) = s(x) + p_\infty,$$

then (1), (2) and (3) hold.

Finally, if Ω is bounded, we define $r(x) = p(x) - p_+$ and repeat the above argument essentially without change. □

Theorem 4.5. *Given an open set $\Omega \subset \mathbf{R}^n$ which has the uniform cone property, and given $p(\cdot) \in \mathscr{P}(\Omega)$ such that* (1.1) *and* (1.2) *hold, then for any natural number k there exists an extension operator*

$$E_k\colon W^{k,p(\cdot)}(\Omega) \to W^{k,p(\cdot)}(\mathbf{R}^n),$$

such that $E_k u(x) = u(x)$, a.e. $x \in \Omega$, and

$$\|E_k u\|_{p(\cdot),\mathbf{R}^n} \le C(p(\cdot), k, \Omega)\|u\|_{p(\cdot),\Omega}.$$

The proof of Theorem 4.5 in variable Sobolev spaces is nearly identical to that in the classical setting. (See Adams [2].) The proof, beyond calculations, requires the following facts which our hypotheses insure are true.

- By Lemma 4.3, $p(\cdot)$ immediately extends to an exponent function on $\mathbf{R}^n$.
- Functions in $C^\infty(\Omega)$ are dense in $W^{k,p(\cdot)}(\Omega)$. By our hypotheses, the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, and the density of $C^\infty(\Omega)$ follows from this by the standard argument (cf. Ziemer [47]). For more details, see Diening [12] or Cruz-Uribe and Fiorenza [9].
- If φ is a smooth function on $\mathbf{R}^n \setminus \{0\}$ with compact support, and if there exists $\varepsilon > 0$ such that on $B_\varepsilon(0)$, φ is a homogeneous function of degree k, $k > -n$, then $\|\varphi * f\|_{p(\cdot),\Omega} \le C\big(p(\cdot),\varphi\big)\|f\|_{p(\cdot),\Omega}$. This again follows from the fact that the maximal operator is bounded on $L^{p(\cdot)}(\Omega)$, and from the well-known inequality $|\varphi * f(x)| \le CMf(x)$. For more details, see Cruz-Uribe and Fiorenza [9].
- Singular integral operators with kernels of the form

$$K(x) = \frac{G(x)}{|x|^n},$$

where G is bounded on $\mathbf{R}^n \setminus \{0\}$, has compact support, is homogeneous of degree zero on $B_R(0) \setminus \{0\}$ for some $R > 0$, and has $\int_{S_R} G\,dx = 0$, are bounded on $L^{p(\cdot)}(\Omega)$. Such kernels are essentially the same as those given by (2.9), and as discussed above, our hypotheses imply that they are bounded.

Remark 4.6. If $p(\cdot)$ satisfies (1.1), then $C_c^\infty(\mathbf{R}^n)$ is dense in $W^{k,p(\cdot)}(\mathbf{R}^n)$. (See [9], [41].) Hence, if the hypotheses of Theorem 4.5 hold, then it follows immediately that the set $\{u\chi_\Omega : u \in C_c^\infty(\mathbf{R}^n)\}$ is dense in $W^{k,p(\cdot)}(\Omega)$. However this result is true under much weaker hypotheses; see [9], [18], [19], [25], [46] for details.

5. Proof of Theorems 1.3 and 1.8

Since Theorem 1.3 is a particular case of Theorem 1.8 with $p_0 = q_0$, it suffices to prove the second result.

We need two facts about variable L^p spaces. First, if $p(\cdot), q(\cdot) \in \mathscr{P}^0(\Omega)$ and $p(x)/q(x) = r$, then it follows from the definition of the norm that

$$\|f\|_{p(\cdot),\Omega}^r = \big\||f|^r\big\|_{q(\cdot),\Omega}. \tag{5.1}$$

Second, given $p(\cdot) \in \mathscr{P}(\Omega)$, we have the generalized Hölder's inequality

$$\int_\Omega |f(x)g(x)|\,dx \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|f\|_{p(\cdot),\Omega}\|g\|_{p'(\cdot),\Omega}, \tag{5.2}$$

and the "duality" relationship

$$\|f\|_{p(\cdot),\Omega} \le \sup_g \left|\int_\Omega f(x)g(x)\,dx\right| \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right)\|f\|_{p(\cdot),\Omega}, \tag{5.3}$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ such that $\|g\|_{p'(\cdot),\Omega} = 1$. For proofs of these results, see Kováčik and Rákosník [30].

The proof of Theorem 1.8 begins with a version of a construction due to Rubio de Francia [38] (also see [11], [21]). Fix $p(\cdot) \in \mathscr{P}^0(\Omega)$ such that $p_- > p_0$, and let $\bar{p}(x) = p(x)/p_0$. Define $q(\cdot)$ as in (1.9), and let $\bar{q}(x) = q(x)/q_0$. By assumption, the maximal operator is bounded on $L^{\bar{q}'(\cdot)}(\Omega)$, so there exists a positive constant B such that

$$\|Mf\|_{\bar{q}'(\cdot),\Omega} \le B\|f\|_{\bar{q}'(\cdot),\Omega}.$$

Define a new operator $\mathscr{R}$ on $L^{\bar{q}'(\cdot)}(\Omega)$ by

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k\, B^k},$$

where, for $k \ge 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes k iterations of the maximal operator, and M^0 is the identity operator. It follows immediately from this definition that:

(a) if h is non-negative, $h(x) \le \mathscr{R}h(x)$;
(b) $\|\mathscr{R}h\|_{\bar{q}'(\cdot),\Omega} \le 2\,\|h\|_{\bar{q}'(\cdot),\Omega}$;
(c) for every $x \in \Omega$, $M(\mathscr{R}h)(x) \le 2\,B\,\mathscr{R}h(x)$, so $\mathscr{R}h \in A_1$ with an A_1 constant that does not depend on h.

We can now argue as follows: by (5.1) and (5.3),

$$\|f\|_{q(\cdot),\Omega}^{q_0} = \|f^{q_0}\|_{\bar{q}(\cdot),\Omega} \le \sup \int_\Omega f(x)^{q_0} h(x)\,dx,$$

where the supremum is taken over all non-negative $h \in L^{\bar{q}'(\cdot)}(\Omega)$ with $\|h\|_{\bar{q}'(\cdot),\Omega} = 1$. Fix any such function h; it will suffice to show that

$$\int_\Omega f(x)^{q_0} h(x)\,dx \le C\,\|g\|_{p(\cdot),\Omega}^{q_0}$$

with the constant C independent of h. First note that by (a) above we have that

$$\int_\Omega f(x)^{q_0} h(x)\,dx \le \int_\Omega f(x)^{q_0} \mathscr{R}h(x)\,dx. \tag{5.4}$$

By (5.2), (b), and since $f \in L^{q(\cdot)}(\Omega)$,

$$\begin{aligned}\int_\Omega f(x)^{q_0} \mathscr{R}h(x)\,dx &\le C\,\|f^{q_0}\|_{\bar{q}(\cdot),\Omega}\|\mathscr{R}h\|_{\bar{q}'(\cdot),\Omega}\\ &\le C\,\|f\|_{q(\cdot),\Omega}^{q_0}\|h\|_{\bar{q}'(\cdot),\Omega}\\ &\le C\,\|f\|_{q(\cdot),\Omega}^{q_0} < \infty.\end{aligned}$$

Therefore, we can apply (1.8) to the right-hand side of (5.4) and again apply (5.2), this time with exponent $\bar{p}(\cdot)$:

$$\begin{aligned}\int_\Omega f(x)^{q_0}\mathscr{R}h(x)\,dx &\le C\left(\int_\Omega g(x)^{p_0}\mathscr{R}h(x)^{p_0/q_0}\,dx\right)^{q_0/p_0}\\ &\le C\,\|g^{p_0}\|_{\bar{p}(\cdot),\Omega}^{q_0/p_0}\|(\mathscr{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot),\Omega}^{q_0/p_0}\\ &= C\,\|g\|_{p(\cdot),\Omega}^{q_0}\|(\mathscr{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot),\Omega}^{q_0/p_0}.\end{aligned}$$

To complete the proof, we need to show that $\|(\mathscr{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot),\Omega}^{q_0/p_0}$ is bounded by a constant independent of h. But it follows from (1.9) that for all $x \in \Omega$,

$$\bar{p}'(x) = \frac{p(x)}{p(x)-p_0} = \frac{q_0}{p_0}\frac{q(x)}{q(x)-q_0} = \frac{q_0}{p_0}\bar{q}'(x).$$

Therefore,

$$\|(\mathscr{R}h)^{p_0/q_0}\|_{\bar{p}'(\cdot),\Omega}^{q_0/p_0} = \|\mathscr{R}h\|_{\bar{q}'(\cdot),\Omega} \le C\,\|h\|_{\bar{q}'(\cdot),\Omega} = C.$$

This completes our proof. □

6. Proof of Corollaries 1.10 and 1.11

The proofs of Corollaries 1.10 and 1.11 require the more general versions of the extrapolation theorems discussed in the introduction. For the convenience of the reader we state them both here.

Theorem 6.1. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, assume that for some p_0, $0 < p_0 < \infty$, and for every $w \in A_\infty$,*

$$\int_\Omega f(x)^{p_0}w(x)\,dx \le C_0\int_\Omega g(x)^{p_0}w(x)\,dx, \quad (f,g)\in\mathscr{F}. \tag{6.1}$$

Then for all $0<p<\infty$ and $w\in A_\infty$,

$$\int_\Omega f(x)^{p}w(x)\,dx \le C_0\int_\Omega g(x)^{p}w(x)\,dx, \quad (f,g)\in\mathscr{F}. \tag{6.2}$$

Furthermore, for every $0<p,q<\infty$, $w\in A_\infty$, and sequence $\{(f_j,g_j)\}_j\subset\mathscr{F}$,

$$\left\|\left(\sum_j (f_j)^q\right)^{1/q}\right\|_{L^p(w,\Omega)} \le C\left\|\left(\sum_j (g_j)^q\right)^{1/q}\right\|_{L^p(w,\Omega)}. \tag{6.3}$$

Theorem 6.2. *Given a family $\mathscr{F}$ and an open set $\Omega \subset \mathbf{R}^n$, assume that for some p_0, $1<p_0<\infty$, and for every $w\in A_{p_0}$, (6.1) holds. Then for every $1<p<\infty$ and $w\in A_p$, (6.2) holds. Furthermore, for every $1<p,q<\infty$, $w\in A_p$, and sequence $\{(f_j,g_j)\}_j\subset\mathscr{F}$, (6.3) holds.*

Theorem 6.1 is proved in [11]. The original statement of Theorem 6.2 is only for pairs of the form $(|Tf|, f)$, and does not include the vector-valued estimate (6.3). (See [17], [21], [38].) However, an examination of the proofs shows that they hold without change when applied to pairs $(f,g) \in \mathscr{F}$. Furthermore, as we noted before, this approach immediately yields the vector-valued inequalities: given a family $\mathscr{F}$ and $1 < q < \infty$, define the new family $\mathscr{F}_q$ to consist of the pairs (F_q, G_q), where

$$F_q(x) = \Bigl(\sum_j (f_j)^q\Bigr)^{1/q}, \quad G_q(x) = \Bigl(\sum_j (g_j)^q\Bigr)^{1/q}, \quad \{(f_j, g_j)\}_j \subset \mathscr{F}.$$

Clearly, inequality (6.1) holds for $\mathscr{F}_q$ when $p_0 = q$, so by extrapolation we get (6.3).

Corollary 1.10 follows immediately from Theorems 1.3 and 6.1. Since (1.11) holds for some p_0, by Theorem 6.1 it holds for all $0 < p < \infty$ and for all $w \in A_\infty$. Therefore, we can apply Theorem 1.3 with p_1 in place of p_0 to obtain (1.12). To prove the vector-valued inequality (1.13), note that by (6.3) we can apply Theorem 1.3 to the family $\mathscr{F}_q$ defined above, again with p_1 in place of p_0.

In exactly the same way, Corollary 1.11 follows from Theorems 1.3 and 6.2.

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