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DUALITY OF A LARGE FAMILY OF ANALYTIC FUNCTION SPACES

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Abstract. We study the predual spaces of a large family of analytic function spaces and thereby extend the recently obtained results by Pavlović and Xiao.

1. Introduction

We denote by $H(\mathbf{D})$ the space of analytic functions on the unit disk $\mathbf{D}$. A general family of analytic function spaces, called the $F(p,\alpha,\beta)$-spaces, with $p \in (1,\infty)$, $\alpha \in (-2,\infty)$ and $\beta \in [0,\infty)$, were introduced in the dissertation paper of Zhao [Z1] and consist of functions in $H(\mathbf{D})$ such that

$$\|g\|_{F(p,\alpha,\beta)} := \sup_{a\in\mathbf{D}} \left(\int_{\mathbf{D}} |g'(z)|^p (1-|z|^2)^\alpha (1-|\sigma_a(z)|^2)^\beta \, dA(z) \right)^{\frac{1}{p}} < \infty.$$

Here $\sigma_a(z) = (a-z)/(1-\overline{a}z)$ is the automorphism of $\mathbf{D}$ that changes 0 and a, while dA denotes the Lebesgue area measure on the plane, normalized so that $A(\mathbf{D}) = 1$. One way to make these spaces Banach spaces is to endow them with the norm $|g(0)| + \|g\|_{F(p,\alpha,\beta)}$. However, in this paper we will always assume that $g(0) = 0$ and use the norm $\|\cdot\|_{F(p,\alpha,\beta)}$ defined above. The closed subspace $F_0(p,\alpha,\beta)$ of $F(p,\alpha,\beta)$ consists of those functions $g \in F(p,\alpha,\beta)$ such that

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |g'(z)|^p (1-|z|^2)^\alpha (1-|\sigma_a(z)|^2)^\beta \, dA(z) = 0,$$

where we define $F_0(p,\alpha,0) := F(p,\alpha,0)$.

Throughout the paper we will always assume that $p \in (1,\infty)$, $\alpha \in (-2,\infty)$ and $\beta \in [0,\infty)$ unless otherwise indicated. Furthermore, we will also assume that $\alpha + \beta > -1$ since otherwise $F(p,\alpha,\beta)$ reduces to the space of constant functions.

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The interest in the $F(p,\alpha,\beta)$-spaces arises from the fact they cover a lot of well-known function spaces, which can be seen from the following list:

$$
\begin{array}{llcll}
\bullet\ Q_\beta\text{-spaces:} & F(2,0,\beta) & = & Q_\beta & \\
 & F_0(2,0,\beta) & = & Q_{\beta,0} & \\
\quad -\ \text{The Bloch space:} & F(2,0,\beta) & = & \mathscr{B} & (\beta>1) \\
\quad -\ \text{The little Bloch space:} & F_0(2,0,\beta) & = & \mathscr{B}_0 & (\beta>1) \\
\quad -\ \text{BMOA:} & F(2,0,1) & = & \text{BMOA} & \\
\quad -\ \text{VMOA:} & F_0(2,0,1) & = & \text{VMOA} & \\
\bullet\ \text{Bloch-type spaces:} & F(p,\alpha p-2,\beta) & = & \mathscr{B}^\alpha & (\beta>1) \\
 & F_0(p,\alpha p-2,\beta) & = & \mathscr{B}^\alpha_0 & (\beta>1) \\
\bullet\ \text{BMOA-type spaces:} & F(p,\alpha p-2,1) & = & \text{BMOA}^\alpha_p & \\
 & F_0(p,\alpha p-2,1) & = & \text{VMOA}^\alpha_p & \\
\bullet\ \text{Besov-type spaces:} & F(p,\alpha p-2,0) & = & B^\alpha_p & \\
\bullet\ \text{Bergman spaces:} & F(p,p,0) & = & A^p & \\
\bullet\ \text{Weighted Dirichlet spaces:} & F(2,\alpha,0) & = & \mathscr{D}^\alpha & \\
\bullet\ \text{A Hardy space:} & F(2,1,0) & = & H^2 &
\end{array}
\tag{1}
$$

Note that $\mathscr{B}^1=\mathscr{B}$, $\mathscr{B}^1_0=\mathscr{B}_0$, $\text{BMOA}^1_2=\text{BMOA}$ and $\text{VMOA}^1_2=\text{VMOA}$. Similarly, $B^1_p=B_p$ (the Besov space) and $\mathscr{D}^0=\mathscr{D}$ (the Dirichlet space). Finally, we want to mention that $F_0(p,\alpha,\beta)$ contains all the polynomials. All these facts about the $F(p,\alpha,\beta)$-spaces can be found in [Z1] (see also [R] and [Z2]).

The Q_β-spaces were introduced by Aulaskari, Xiao and Zhao in [AXZ] and have been studied intensively ever since. A good source for these spaces are the Springer Lecture Notes by Xiao [X] and the related references therein. To see that there are predual spaces of the Q_β-spaces is easy (see for example Corollary 2 in [LMT]). However, to get a characterization of these spaces for $\beta\in(0,1)$ as Banach spaces of analytic functions on the unit disk has been an open problem until recently. The problem was solved by Pavlović and Xiao in [PX] (see also [ACS]).

The main aim of this paper is to extend the results in [PX] to the class of $F(p,\alpha,\beta)$-spaces. The paper is divided into several sections. We begin, in Section 2, by giving some preliminary results about the $F(p,\alpha,\beta)$-spaces and by introducing the $R(p,\alpha,\beta)$-spaces, while we in Section 3 show that the dual of $F_0(p,\alpha,\beta)$ is isomorphic to $R(p,\alpha,\beta)$. In Section 4 we show that $E(p,\alpha,\beta)$, a closed subspace of the dual space of $F(p,\alpha,\beta)$, is the unique isometric predual of $F(p,\alpha,\beta)$ and that $E(p,\alpha,\beta)$ is isomorphic to $R(p,\alpha,\beta)$. Moreover, we show that the bidual of $F_0(p,\alpha,\beta)$ is isometrically isomorphic to $F(p,\alpha,\beta)$. Finally, in Section 5, we characterize the bounded multiplication operators on $R(p,\alpha,\beta)$.

2. Preliminaries

We will use the following notations throughout the paper: $A\lesssim B$ means that there is a positive constant c such that $A\le c\,B$, while $A\approx B$ means that there are positive constants c_1 and c_2, such that $c_1A\le B\le c_2A$. In both cases, the constants do not depend on crucial properties of A and B (which will be clear from the context). Also, if X and Y are Banach spaces, we will write $X \cong Y$ in the meaning that X is isomorphic to Y.

Given a Banach space X, B_X will denote the closed unit ball of X, while X^* is the Banach space of all bounded linear functionals on X. Furthermore, *co* will denote the compact-open topology, while $\partial\mathbf{D}$ and $\overline{\mathbf{D}}$ will denote the unit circle and the closed unit disk, respectively. For $f \in H(\mathbf{D})$ and for $r \in (0,1)$, we denote by f_r the function defined by $f_r(z) := f(rz)$. Finally, the multiplication operator M_ψ, induced by $\psi \in H(\mathbf{D})$, is the linear map on $H(\mathbf{D})$ defined by $M_\psi f = \psi f$.

We will frequently use the following easily verified equality (without any further reference):

$$1 - |\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}.$$

We will also use the well-known integral formula (see for example Theorem 1.7 in [HKZ]), which states that for $a \in \mathbf{D}$, $c \in \mathbf{R}$ and $t > -1$,

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}}\, dA(z) \lesssim \begin{cases} 1 & \text{for } c < 0, \\ -\log(1-|a|^2) & \text{for } c = 0, \\ (1-|a|^2)^{-c} & \text{for } c > 0. \end{cases} \tag{2}$$

The $F(p,\alpha,\beta)$-spaces. We will now state some preliminary results about the $F(p,\alpha,\beta)$-spaces that we will need later. Using the integral formula (2), we see that for any $f \in H(\mathbf{D})$ and for every $r \in (0,1)$, we have that $f_r \in F_0(p,\alpha,\beta)$.

Lemma 2.1. *$\big(B_{F(p,\alpha,\beta)}, co\big)$ is compact.*

Proof. By [Z1] we know that the norm-topology of $F(p,\alpha,\beta)$ is finer than the compact-open topology. Indeed, we have that $\|g\|_{\mathscr{B}^{\frac{\alpha+2}{p}}} \lesssim \|g\|_{F(p,\alpha,\beta)}$ and hence, for $g \in F(p,\alpha,\beta)$ and for all $z \in \mathbf{D}$ (see for example pp. 191–192 in [OSZ]),

$$|g(z)| \lesssim C(z)\|g\|_{F(p,\alpha,\beta)}, \quad \text{where } C(z) = \begin{cases} 1 & \text{for } p > \alpha+2, \\ -\log(1-|z|) & \text{for } p = \alpha+2, \\ (1-|z|)^{1-\frac{\alpha+2}{p}} & \text{for } p < \alpha+2. \end{cases}$$

Thus, Montel's theorem states that $B_{F(p,\alpha,\beta)}$ is relatively compact with respect to the compact-open topology. If $\{g_n\}$ is a sequence in $B_{F(p,\alpha,\beta)}$, we conclude from Fatou's lemma that

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \lim_{n\to\infty} |g_n'(z)|^p (1-|z|^2)^\alpha (1-|\sigma_a(z)|^2)^\beta\, dA(z) \le \liminf_{n\to\infty} \|g_n\|^p_{F(p,\alpha,\beta)} \le 1.$$

That is, $B_{F(p,\alpha,\beta)}$ is *co*-closed and therefore also *co*-compact. □

Lemma 2.2. *Let $\mu \in (0,1)$. If $g \in H(\mathbf{D})$ is such that*

$$I_\mu(a) := \int_{\mu<|z|<1} |g'(z)|^p (1-|z|^2)^\alpha (1-|\sigma_a(z)|^2)^\beta\, dA(z) < \infty \quad \textit{for all } a \in \mathbf{D},$$

then $I_\mu(a)$ is a continuous function of $a \in \mathbf{D}$.

Proof. Fix $a \in \mathbf{D}$ and take a sequence $\{a_n\} \subset \mathbf{D}$ such that $a_n \to a$. Define

$$G_n(z) := |g'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_{a_n}(z)|^2)^\beta,$$
$$G(z) := |g'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta.$$

Then G_n converges pointwise to G. Since $a_n \to a$, we can choose n_0 such that for $n \geq n_0$, $1-|a_n| \geq (1-|a|)/2$. A straightforward computation shows that for all n,

$$\frac{1-|\sigma_{a_n}(z)|^2}{1-|\sigma_a(z)|^2} \leq \frac{4}{(1-|a_n|)(1-|a|)}.$$

Using this we get that for $n \geq n_0$,

$$G_n(z) \leq \frac{8^\beta}{(1-|a|)^{2\beta}} G(z).$$

Lebesgue's dominated convergence theorem implies now that $\lim_{n\to\infty} I_\mu(a_n) = I_\mu(a)$. □

The following result is known for $p = 2$ and $\alpha = 0$ (see Proposition 2.3 in [ACS]), but for completeness and for the convenience of the reader, we will give a proof.

Proposition 2.3. *Let $\alpha \geq 0$ and $g \in F(p,\alpha,\beta)$. Then $\|g - g_r\|_{F(p,\alpha,\beta)} \to 0$ as $r \to 1$ if and only if $g \in F_0(p,\alpha,\beta)$.*

Proof. Assume that $\|g - g_r\|_{F(p,\alpha,\beta)} \to 0$ as $r \to 1$. Clearly $g_r \in F_0(p,\alpha,\beta)$ and since $F_0(p,\alpha,\beta)$ is a closed subspace of $F(p,\alpha,\beta)$ the assumption implies that $g \in F_0(p,\alpha,\beta)$.

Conversely, assume that $g \in F_0(p,\alpha,\beta)$. For any $\mu \in (0,1)$ and any $\delta \in (0,1)$,

$$\begin{aligned}
\|g-g_r\|^p_{F(p,\alpha,\beta)} &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |g'(z)-g_r'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z) \\
&\leq \sup_{\delta<|a|<1} \int_{\mathbf{D}} |g'(z)-g_r'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z) \qquad (3)\\
&\quad + \sup_{|a|\leq\delta} \int_{\mu<|z|<1} |g'(z)-g_r'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z) \qquad (4)\\
&\quad + \sup_{|a|\leq\delta} \int_{|z|\leq\mu} |g'(z)-g_r'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z). \qquad (5)
\end{aligned}$$

Given $\varepsilon > 0$ we need to show that we can choose μ and δ in $(0,1)$, so that

$$\lim_{r\to1}((3)+(4)+(5)) \lesssim \varepsilon.$$

We immediately notice that since $g \in F_0(p,\alpha,\beta)$, we can choose $\delta \in (0,1)$ such that (3) $\lesssim \varepsilon$ for r close enough to 1.

Let $I_\mu(a)$ be defined as in Lemma 2.2. For $a \in \mathbf{D}$ and $\mu \in (0,1)$, we have that $I_\mu(a) \leq \|g\|^p_{F(p,\alpha,\beta)} < \infty$. For every $a \in \mathbf{D}$, we can choose $\mu_a \in (0,1)$ so that $I_{\mu_a}(a) < \varepsilon$. By the continuity of I_{μ_a} there is a neighborhood $U(a) \subset \mathbf{D}$ of a so that $I_{\mu_a}(b) \lesssim \varepsilon$ for all $b \in U(a)$. Since $\{a : |a| \leq \delta\}$ is a compact set we can find

$\mu \in (0,1)$ such that $\sup_{|a|\leq\delta} I_\mu(a) \lesssim \varepsilon$. Thus, with the chosen constants δ and μ, we have that $\lim\limits_{r\to 1}(4) \lesssim \varepsilon$.

Finally, we have that $\lim\limits_{r\to 1}(5) = 0$ for the δ and μ chosen above. $\square$

The $R(p,\alpha,\beta)$-spaces. We will now introduce the $R(p,\alpha,\beta)$-spaces. Let $E_{k,j}$ be the pairwise disjoint sets given by

$$E_{k,j} := \left\{ z \in \mathbf{D} : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}\,,\ \frac{\pi j}{2^{k+1}} \leq \arg z < \frac{\pi(j+1)}{2^{k+1}} \right\},$$

where $k = 0,1,2,\dots$ and $j = 0,1,2,\dots,2^{k+2}-1$, so that

$$\bigcup_{k=0}^{\infty} \bigcup_{j=0}^{2^{k+2}-1} E_{k,j} = \mathbf{D}.$$

For simplicity we will rename these sets as E_m, where $m \in \mathbf{N}$. More precisely, we denote $m := j - 1 + \sum_{i=0}^{k} 2^{i+1}$ so that

$$E_1 = E_{0,0},\ \dots,\ E_4 = E_{0,3},\ E_5 = E_{1,0},\ \dots,\ E_{12} = E_{1,7},\ E_{13} = E_{2,0},\ \dots .$$

Furthermore, let a_m denote the center of E_m (see Figure 1). Then $R(p,\alpha,\beta)$ consists of those functions $f \in H(\mathbf{D})$ for which $f(z) = \sum_{m=1}^{\infty} f_m(z)$, where each $f_m \in H(\mathbf{D})$ and

$$\sum_{m=1}^{\infty} \left(\int_{\mathbf{D}} |f_m(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}} dA(z) \right)^{\frac{p-1}{p}} < \infty.$$

The norm of $R(p,\alpha,\beta)$ is given by

$$\|f\|_{R(p,\alpha,\beta)} := \inf \sum_{m=1}^{\infty} \left(\int_{\mathbf{D}} |f_m(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}} dA(z) \right)^{\frac{p-1}{p}},$$

where the infimum is taken over all such representations of f.

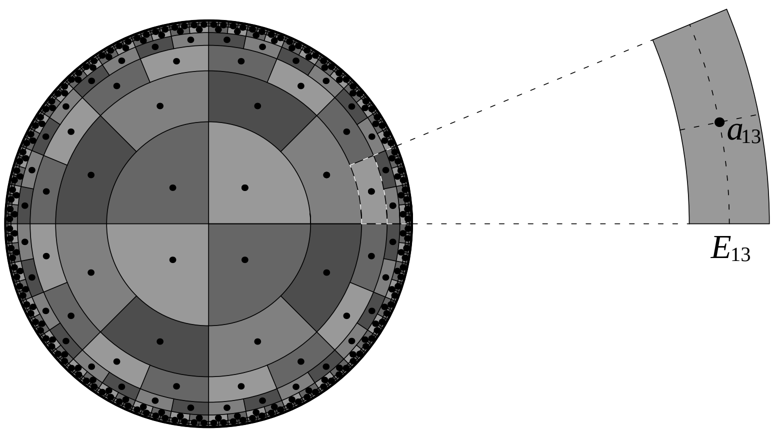

Figure 1. The disjoint sets and the center point a_{13} of the corresponding set E_{13}.

Remark 2.4. With some simple geometrical estimations, one can deduce that there exists a constant $\delta \in (0,1)$ such that

$$\sup_{w \in E_m} |\sigma_{a_m}(w)| \leq \delta \quad \text{for all } m \in \mathbf{N}.$$

In particular, using the stronger version of the triangle inequality associated with the pseudo-hyperbolic metric (see Lemma 1.4 in [Ga]), we then get that

$$\frac{1-\delta}{2(1+\delta)} \leq \frac{1-|\sigma_w(z)|^2}{1-|\sigma_{a_m}(z)|^2} \leq \frac{2(1+\delta)}{1-\delta}, \quad w \in E_m,\ z \in \mathbf{D}.$$

Remark 2.5. The $R(2,0,\beta)$-spaces, with $\beta \in (0,1)$, were introduced in [PX] as spaces which lie between the Hardy space H^1 and the Bergman space A^2. They showed that $R(2,0,\beta)$ is the dual of $Q_{\beta,0}$ as well as the predual of Q_β.

Remark 2.6. For $p > \max\{1, 1+\alpha+\beta\}$ it is easy to see that $\|\cdot\|_{R(p,\alpha,\beta)} \lesssim \|\cdot\|_{H^\infty}$. Indeed, let $f \in H^\infty$. Then the representation of f can be chosen to be f itself, since f satisfies

$$\begin{aligned}
&\left(\int_{\mathbf{D}} |f(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z)\right)^{\frac{p-1}{p}} \\
&\leq \|f\|_{H^\infty} \left((1-|a_m|^2)^{-\frac{\beta}{p-1}} \int_{\mathbf{D}} \frac{(1-|z|^2)^{-\frac{\alpha+\beta}{p-1}}}{|1-\overline{a}_m z|^{-\frac{2\beta}{p-1}}}\, dA(z)\right)^{\frac{p-1}{p}} < \infty,
\end{aligned}$$

for any center point a_m.

Proposition 2.7. *For $f \in R(p,\alpha,\beta)$ and for all $z \in \mathbf{D}$,*

$$|f(z)| \lesssim \frac{\|f\|_{R(p,\alpha,\beta)}}{(1-|z|)^{2-\frac{\alpha+2}{p}}}.$$

Proof. Fix $z \in \mathbf{D}$. Using the inequality on p. 39 in [HKZ], which states that for $f \in H(\mathbf{D})$, $s \in \mathbf{R}$ and $q \in (0,\infty)$,

$$(6) \qquad (1-|z|^2)^s |f(z)|^q \lesssim \int_{\mathbf{D}} (1-|w|^2)^{s-2} |f(w)|^q dA(w),$$

we obtain

$$|f_m(z)|^{\frac{p}{p-1}} \lesssim \frac{1}{(1-|z|)^{2-\frac{\alpha}{p-1}}} \int_{\mathbf{D}} |f_m(w)|^{\frac{p}{p-1}} (1-|w|^2)^{-\frac{\alpha}{p-1}}\, dA(w)$$

and hence,

$$(1-|z|)^{2-\frac{\alpha+2}{p}}|f(z)| \le (1-|z|)^{2-\frac{\alpha+2}{p}} \sum_{m=1}^{\infty} \left(|f_m(z)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\lesssim (1-|z|)^{2-\frac{\alpha+2}{p}} \sum_{m=1}^{\infty} \left(\frac{1}{(1-|z|)^{2-\frac{\alpha}{p-1}}} \int_{\mathbf{D}} |f_m(w)|^{\frac{p}{p-1}} (1-|w|^2)^{-\frac{\alpha}{p-1}}\, dA(w) \right)^{\frac{p-1}{p}}$$

$$\le \sum_{m=1}^{\infty} \left(\int_{\mathbf{D}} |f_m(w)|^{\frac{p}{p-1}} (1-|w|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(w)|^2)^{-\frac{\beta}{p-1}}\, dA(w) \right)^{\frac{p-1}{p}}.$$

Taking infimum over all the representations of f finishes the proof. □

Remark 2.8. In particular, the norm-topology of $R(p,\alpha,\beta)$ is finer than the compact-open topology. Moreover, using the completeness criterion (see for example Lemma I.1.8 in [W]), one can verify that the normed space $R(p,\alpha,\beta)$ is complete.

3. The $F_0(p,\alpha,\beta) - R(p,\alpha,\beta)$ duality

Some of the results in this section are highly inspired by the corresponding ones in [PX]. We begin by stating the main theorem of this section.

Theorem 3.1. *Let $p > \max\{1, 1+\alpha+\beta\}$. Then*

$$R(p,\alpha,\beta) \cong F_0(p,\alpha,\beta)^* \quad \text{under the pairing} \quad \langle f, g\rangle = \int_{\mathbf{D}} \overline{f(z)} g'(z)\, dA(z).$$

That is, every $f \in R(p,\alpha,\beta)$ induces a bounded linear functional $\langle f,\cdot\rangle \colon F_0(p,\alpha,\beta) \to \mathbf{C}$. Conversely, if $L \in F_0(p,\alpha,\beta)^*$, then there exists $f \in R(p,\alpha,\beta)$ such that $L(g) = \langle f, g\rangle$ for all $g \in F_0(p,\alpha,\beta)$. Moreover, for every $f \in R(p,\alpha,\beta)$,

$$\|f\|_{R(p,\alpha,\beta)} \approx \sup_{g \in B_{F_0(p,\alpha,\beta)}} |\langle f, g\rangle|.$$

Remark 3.2. Theorem 3.1 reduces to Theorem 1.2 in [PX] by choosing $p=2$, $\alpha=0$ and $\beta \in (0,1)$.

In order to prove Theorem 3.1 we will need the following two lemmas and Theorem 3.5.

Lemma 3.3. *Let $g \in H(\mathbf{D})$ be given by $g(z) = \sum_{k=1}^{\infty} b_k z^k$ and define the invertible linear operator $D^n \colon (H(\mathbf{D}), co) \to (H(\mathbf{D}), co)$ by*

$$D^n g(z) := \frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} b_{k+1} z^k, \quad \text{where } n \in \mathbf{N}.$$

Then $g \in F(p,\alpha,\beta)$ if and only if

$$\sup_{a\in\mathbf{D}} \left(\int_{\mathbf{D}} |D^n g(z)|^p (1-|z|^2)^{(n-1)p+\alpha} (1-|\sigma_a(z)|^2)^{\beta}\, dA(z) \right)^{\frac{1}{p}} < \infty.$$

Similarly, $g \in F_0(p,\alpha,\beta)$ if and only if

$$\lim_{|a|\to 1}\left(\int_{\mathbf{D}} |D^n g(z)|^p (1-|z|^2)^{(n-1)p+\alpha}(1-|\sigma_a(z)|^2)^\beta \, dA(z)\right)^{\frac{1}{p}} = 0.$$

Proof. A straightforward calculation shows that $D^1 g(z) = g'(z)$ and that for $n \geq 2$, we have that

$$(7) \qquad D^n g(z) = \frac{1}{(n-1)!}\left(z^{n-1} g^{(n)}(z) + \sum_{j=1}^{n-2} c_{n,j} z^{n-1-j} g^{(n-j)}(z) + n D^{n-1} g(z)\right),$$

where $c_{n,j} \in \mathbf{N}$. Theorem 4.2.1 and Theorem 4.2.3 in [R] state that $g \in F(p,\alpha,\beta)$ if and only if

$$\sup_{a\in\mathbf{D}}\left(\int_{\mathbf{D}} |g^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+\alpha}(1-|\sigma_a(z)|^2)^\beta \, dA(z)\right)^{\frac{1}{p}} < \infty$$

and $g \in F_0(p,\alpha,\beta)$ if and only if

$$\lim_{|a|\to 1}\left(\int_{\mathbf{D}} |g^{(n)}(z)|^p (1-|z|^2)^{(n-1)p+\alpha}(1-|\sigma_a(z)|^2)^\beta \, dA(z)\right)^{\frac{1}{p}} = 0,$$

respectively. Hence, by (7) the result follows by induction. □

Lemma 3.4. *For $p > \max\{1, 1+\alpha+\beta\}$ and*

$$n \in \mathbf{N}, \quad \textit{with} \quad n > \max\left\{\frac{2\beta}{p-1}\,,\, 1 + \frac{\beta-\alpha-1}{p}\right\},$$

we define the linear operator S on the set of Borel measurable functions H on $\mathbf{D}$ by

$$S(H)(w) := (1-|w|^2)^\gamma \int_{\mathbf{D}} H(z) \frac{(1-|z|^2)^{n-1-\gamma}}{(1-\overline{z}w)^{n+1}} \, dA(z),$$

where $w \in \mathbf{D}$ and $\gamma \in \big(\max\{0\,,\, \beta-\alpha-(n-1)(p-1)\}\,,\, \min\{n\,,\, p-1-\alpha-\beta\}\big)$. Then S maps $L^\infty(\mathbf{D}, d\mu_a)$ and $L^1(\mathbf{D}, d\mu_a)$ boundedly into $L^\infty(\mathbf{D}, d\mu_a)$ and $L^1(\mathbf{D}, d\mu_a)$, respectively, where

$$d\mu_a(z) := \frac{(1-|z|^2)^{-\frac{\alpha+\beta+\gamma p}{p-1}}}{|1-\overline{a}z|^{-\frac{2\beta}{p-1}}} \, dA(z), \quad a \in \mathbf{D}.$$

Proof. Note that the assumptions made on p and n guarantee that we can always find γ in the interval specified above. Since $\gamma > 0$ and $\gamma < n$, the integral formula (2) gives

$$\begin{aligned}\|S(H)\|_{L^\infty(\mathbf{D},d\mu_a)} &\leq \|H\|_{L^\infty(\mathbf{D},d\mu_a)} \sup_{w\in\mathbf{D}} (1-|w|^2)^\gamma \int_{\mathbf{D}} \frac{(1-|z|^2)^{n-1-\gamma}}{|1-\overline{w}z|^{n+1}} \, dA(z) \\ &\lesssim \|H\|_{L^\infty(\mathbf{D},d\mu_a)}.\end{aligned}$$

On the other hand, we have that

$$
\begin{aligned}
&\|S(H)\|_{L^1(\mathbf{D},d\mu_a)} \\
&\leq \int_{\mathbf{D}} (1-|w|^2)^{\gamma} \int_{\mathbf{D}} |H(z)| \frac{(1-|z|^2)^{n-1-\gamma}}{|1-\overline{z}w|^{n+1}}\, dA(z) \frac{(1-|w|^2)^{-\frac{\alpha+\beta+\gamma p}{p-1}}}{|1-\overline{a}w|^{-\frac{2\beta}{p-1}}}\, dA(w) \\
&= \int_{\mathbf{D}} |H(z)|(1-|z|^2)^{n-1-\gamma} \underbrace{\int_{\mathbf{D}} \frac{(1-|w|^2)^{-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{z}w|^{n+1}|1-\overline{a}w|^{-\frac{2\beta}{p-1}}}\, dA(w)}_{=:M(a)}\, dA(z).
\end{aligned}
$$

Thus, it suffices to show that

$$
M(a) \lesssim \frac{(1-|z|^2)^{-(n-1)-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{a}z|^{-\frac{2\beta}{p-1}}}.
$$

Since $\gamma < p-1-\alpha-\beta$, we have that the function $a \mapsto M(a)|1-\overline{a}z|^{-\frac{2\beta}{p-1}}$ is continuous on $\overline{\mathbf{D}}$ and subharmonic on $\mathbf{D}$. By the maximum principle we can (and will) assume that $|a| = 1$. A change of variable, $w \mapsto \sigma_z(w)$ gives

$$
\begin{aligned}
M(a) &= \int_{\mathbf{D}} \frac{(1-|\sigma_z(w)|^2)^{-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{z}\sigma_z(w)|^{n+1}|1-\overline{a}\sigma_z(w)|^{-\frac{2\beta}{p-1}}} |\sigma_z'(w)|^2\, dA(w) \\
&= \int_{\mathbf{D}} \frac{(1-|z|^2)^{-(n-1)-\frac{\alpha+\beta+\gamma}{p-1}}(1-|w|^2)^{-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{z}w|^{-(n-3)-\frac{2(\alpha+\gamma)}{p-1}}|1-\overline{a}z+w(\overline{a}-\overline{z})|^{-\frac{2\beta}{p-1}}}\, dA(w) \\
&\leq 2^{\frac{2\beta}{p-1}} \frac{(1-|z|^2)^{-(n-1)-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{a}z|^{-\frac{2\beta}{p-1}}} \int_{\mathbf{D}} \frac{(1-|w|^2)^{-\frac{\alpha+\beta+\gamma}{p-1}}}{|1-\overline{z}w|^{-(n-3)-\frac{2(\alpha+\gamma)}{p-1}}}\, dA(w),
\end{aligned}
$$

where we in the inequality have estimated $|1-\overline{a}z+w(\overline{a}-\overline{z})| \leq 2|1-\overline{a}z|$. Using the integral formula (2) we notice that the last integral is bounded in z due to the fact that $\gamma > \beta-\alpha-(n-1)(p-1)$. $\square$

Theorem 3.5. *For $p > \max\{1, 1+\alpha+\beta\}$ and*

$$
n \in \mathbf{N}, \quad \textit{with} \quad n > \max\left\{\frac{2\beta}{p-1}, 1+\frac{\beta-\alpha-1}{p}\right\},
$$

we define the Banach spaces $X_m \subseteq H(\mathbf{D})$ and $Y_m \subseteq H(\mathbf{D})$ by

$$
\|g\|_{X_m} := \left(\int_{\mathbf{D}} |D^n g(z)|^p (1-|z|^2)^{(n-1)p+\alpha}(1-|\sigma_{a_m}(z)|^2)^{\beta}\, dA(z)\right)^{\frac{1}{p}} < \infty,
$$

$$
\|f\|_{Y_m} := \left(\int_{\mathbf{D}} |f(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}}(1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z)\right)^{\frac{p-1}{p}} < \infty,
$$

where $D^n g$ is defined as in Lemma 3.3 and where we always assume that $g(0) = 0$. Then

$$(X_m)^* = Y_m \quad \textit{under the pairing} \quad \langle f, g\rangle = \int_{\mathbf{D}} \overline{f(z)} g'(z)\, dA(z).$$

Moreover, for every $f \in Y_m$,

$$\|f\|_{Y_m} \approx \sup_{g \in B_{X_m}} |\langle f, g\rangle|,$$

where the constants do not depend on m.

Remark 3.6. Using inequality (6) we get that for $g \in X_m$ and for all $z \in \mathbf{D}$,

$$|D^n g(z)| \lesssim (1 - |a_m|)^{-\frac{\beta}{p}} \frac{\|g\|_{X_m}}{(1 - |z|^2)^{n-1+\frac{\alpha+\beta+2}{p}}}.$$

Thus, using the invertibility of $D^n \colon (H(\mathbf{D}), co) \to (H(\mathbf{D}), co)$, we conclude that the norm-topology of X_m is finer than the compact-open topology. Hence, by Fatou's lemma X_m is complete. Since Y_m is a space of Bergman type, it is clearly a Banach space.

Proof of Theorem 3.5. For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in Y_m$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k \in X_m$, a straightforward calculation, using the Cauchy product, shows that

$$\int_{\mathbf{D}} \overline{f(z)} g'(z)\, dA(z) = \sum_{k=0}^{\infty} \overline{a_k} b_{k+1} = \int_{\mathbf{D}} \overline{f(z)} D^n g(z) (1 - |z|^2)^{n-1}\, dA(z).$$

Using this and Hölder's inequality we obtain

$$\begin{aligned} |\langle f, g\rangle| &\le \left(\int_{\mathbf{D}} |D^n g(z)|^p (1 - |z|^2)^{(n-1)p+\alpha} (1 - |\sigma_{a_m}(z)|^2)^{\beta}\, dA(z) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\mathbf{D}} |f(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{\alpha}{p-1}} (1 - |\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z) \right)^{\frac{p-1}{p}} \\ &= \|g\|_{X_m} \|f\|_{Y_m}. \end{aligned}$$

That is, $Y_m \subseteq (X_m)^*$.

Conversely, let $L \in (X_m)^*$ and consider $T \colon X_m \to L^p$ given by

$$T(g) := D^n g(z) (1 - |z|^2)^{n-1+\frac{\alpha}{p}} (1 - |\sigma_{a_m}(z)|^2)^{\frac{\beta}{p}}.$$

Let $G := T(X_m)$. Then, using the Hahn–Banach theorem, $L \circ T^{-1} \colon G \to \mathbf{C}$ can be extended (preserving the norm) to a bounded linear functional on L^p, denoted here by $\widetilde{L \circ T^{-1}}$. Thus, we can find $h_0 \in L^{\frac{p}{p-1}}$ such that

$$(\widetilde{L \circ T^{-1}})(f) = \int_{\mathbf{D}} f(z) \overline{h_0(z)}\, dA(z) \quad \text{for all } f \in L^p$$

and

$$\|L \circ T^{-1}\| = \left(\int_{\mathbf{D}} |h_0(z)|^{\frac{p}{p-1}}\, dA(z) \right)^{\frac{p-1}{p}}.$$

Especially we have that

$$L(g) = \int_{\mathbf{D}} D^n g(z) \underbrace{\overline{h_0(z)}(1-|z|^2)^{\frac{\alpha}{p}}(1-|\sigma_{a_m}(z)|^2)^{\frac{\beta}{p}}}_{=:\overline{h(z)}}(1-|z|^2)^{n-1}\, dA(z)$$

for all $g \in X_m$ and

$$\|L\| = \left(\int_{\mathbf{D}} |h(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{\alpha}{p-1}}(1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z)\right)^{\frac{p-1}{p}}.$$

For $s > -1$, every analytic f in $L^1(\mathbf{D}, (1-|z|^2)^s dA(z))$ has the following reproducing formula (see Corollary 1.5 in [HKZ]):

$$f(z) = (s+1)\int_{\mathbf{D}} f(w)\frac{(1-|w|^2)^s}{(1-z\overline{w})^{s+2}}\, dA(w).$$

We now claim that $D^n g \in L^1(\mathbf{D}, (1-|z|^2)^{n-1}\, dA(z))$ whenever $g \in X_m$. Indeed,

$$\begin{aligned}
&\int_{\mathbf{D}} |D^n g(z)|(1-|z|^2)^{n-1}\, dA(z)\\
&= \int_{\mathbf{D}} |D^n g(z)|(1-|z|^2)^{n-1+\frac{\alpha}{p}}(1-|\sigma_{a_m}(z)|^2)^{\frac{\beta}{p}}(1-|z|^2)^{-\frac{\alpha}{p}}(1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p}}\, dA(z)\\
&\le \|g\|_{X_m}\left((1-|a_m|^2)^{-\frac{\beta}{p-1}}\int_{\mathbf{D}} \frac{(1-|z|^2)^{-\frac{\alpha+\beta}{p-1}}}{|1-\overline{a}_m z|^{-\frac{2\beta}{p-1}}}\, dA(z)\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $p > 1+\alpha+\beta$, the last integral is bounded.

In other words, if $g \in X_m$, then

$$\begin{aligned}
L(g) &= \int_{\mathbf{D}} D^n g(z)\overline{h(z)}(1-|z|^2)^{n-1}\, dA(z)\\
&= \int_{\mathbf{D}} n\int_{\mathbf{D}} D^n g(w)\frac{(1-|w|^2)^{n-1}}{(1-z\overline{w})^{n+1}}\, dA(w)\,\overline{h(z)}(1-|z|^2)^{n-1}\, dA(z)\\
&= \int_{\mathbf{D}} D^n g(w)(1-|w|^2)^{n-1}\, \underbrace{n\int_{\mathbf{D}} \overline{h(z)}\frac{(1-|z|^2)^{n-1}}{(1-z\overline{w})^{n+1}}\, dA(z)}_{=:\overline{f_0(w)}}\, dA(w).
\end{aligned}$$

Using Theorem 1.10 in [HKZ] and the assumptions made on n, we notice that the function f_0 defined above is analytic. Thus, it remains to show that $\|f_0\|_{Y_m} \lesssim \|L\|$, where the constant does not depend on m. To do this, it suffices to show that

$$\|f_0\|_{Y_m}^{\frac{p}{p-1}} \lesssim \int_{\mathbf{D}} |h(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{\alpha}{p-1}}(1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z),$$

which reduces to

$$\int_{\mathbf{D}} |f_0(w)|^{\frac{p}{p-1}}\frac{(1-|w|^2)^{-\frac{\alpha+\beta}{p-1}}}{|1-\overline{a}_m w|^{-\frac{2\beta}{p-1}}}\, dA(w) \lesssim \int_{\mathbf{D}} |h(z)|^{\frac{p}{p-1}}\frac{(1-|z|^2)^{-\frac{\alpha+\beta}{p-1}}}{|1-\overline{a}_m z|^{-\frac{2\beta}{p-1}}}\, dA(z). \tag{8}$$

So fix $\gamma \in \big(\max\{0\,,\,\beta-\alpha-(n-1)(p-1)\}\,,\,\min\{n\,,\,p-1-\alpha-\beta\}\big)$. Again, the assumptions made on p and n guarantee that the interval is non-empty. Define the functions F and H by

$$F(w) := f_0(w)(1-|w|^2)^\gamma \quad \text{and} \quad H(z) := h(z)(1-|z|^2)^\gamma.$$

Then equation (8) reduces to

$$\int_{\mathbf{D}} |F(w)|^{\frac{p}{p-1}} \frac{(1-|w|^2)^{-\frac{\alpha+\beta+\gamma p}{p-1}}}{|1-\overline{a}_m w|^{-\frac{2\beta}{p-1}}}\, dA(w) \lesssim \int_{\mathbf{D}} |H(z)|^{\frac{p}{p-1}} \frac{(1-|z|^2)^{-\frac{\alpha+\beta+\gamma p}{p-1}}}{|1-\overline{a}_m z|^{-\frac{2\beta}{p-1}}}\, dA(z), \tag{9}$$

where

$$F(w) = n(1-|w|^2)^\gamma \int_{\mathbf{D}} H(z) \frac{(1-|z|^2)^{n-1-\gamma}}{(1-\overline{z}w)^{n+1}}\, dA(z).$$

To prove equation (9), we use the combination of Lemma 3.4 and the Riesz–Thorin interpolation theorem. Note that the constants obtained are independent of m. $\square$

Proof of Theorem 3.1. Let $f \in R(p,\alpha,\beta)$ and $g \in F_0(p,\alpha,\beta)$. Given $\varepsilon > 0$, there exists a representation of f such that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \int_{\mathbf{D}} |\overline{f_m(z)} g'(z)|\, dA(z) \\
&\leq \|g\|_{F(p,\alpha,\beta)} \sum_{m=1}^{\infty} \left(\int_{\mathbf{D}} |f_m(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z) \right)^{\frac{p-1}{p}} \\
&\leq \|g\|_{F(p,\alpha,\beta)} \left(\|f\|_{R(p,\alpha,\beta)} + \varepsilon \right).
\end{aligned}$$

Thus,

$$|\langle f, g\rangle| \leq \|g\|_{F(p,\alpha,\beta)} \|f\|_{R(p,\alpha,\beta)}. \tag{10}$$

Conversely, take $L \in F_0(p,\alpha,\beta)^*$. Let X_m and Y_m be the Banach spaces introduced in Theorem 3.5 and define

$$X := \left(\bigoplus_{m=1}^{\infty} X_m \right)_{c_0} \quad \text{and} \quad Y := \left(\bigoplus_{m=1}^{\infty} Y_m \right)_{\ell^1}.$$

Instead of taking the supremum over $\mathbf{D}$ in the definition of the $F(p,\alpha,\beta)$-norm it suffices to take it over the set $\{a_m : m \in \mathbf{N}\}$ (see Remark 2.4). This norm is equivalent to the complete norm $\sup_m \|\cdot\|_{X_m}$ on $F(p,\alpha,\beta)$ (see Lemma 3.3 and Remark 3.6). Moreover, by endowing $F_0(p,\alpha,\beta)$ with this new norm it becomes a normed subspace of X. Since X^* is isometrically isomorphic to $(\bigoplus_{m=1}^{\infty}(X_m)^*)_{\ell^1}$, Theorem 3.5 implies that $X^* \cong Y$ under the pairing

$$\sum_{m=1}^{\infty} \langle f_m, g_m\rangle, \quad f_m \in Y_m,\ g_m \in X_m.$$

By the Hahn–Banach theorem, L can now be extended (preserving the norm) to a functional $\widetilde{L} \in X^*$. Hence, there exist $f_m \in Y_m$, $m \in \mathbf{N}$, such that

$$L(g) = \sum_{m=1}^{\infty} \langle f_m, g \rangle, \quad g \in F_0(p,\alpha,\beta) \quad \text{and} \quad \sum_{m=1}^{\infty} \|f_m\|_{Y_m} \lesssim \|L\|.$$

By the proof of Proposition 2.7, we know that for all $z \in \mathbf{D}$,

$$(1-|z|^2)^{2-\frac{\alpha+2}{p}} |f_m(z)| \lesssim \|f_m\|_{Y_m}.$$

This implies that $f(z) := \sum_{m=1}^{\infty} f_m(z)$ converges uniformly on compact subsets of $\mathbf{D}$. Thus, $f \in H(\mathbf{D})$, $\|f\|_{R(p,\alpha,\beta)} \le \sum_{m=1}^{\infty} \|f_m\|_{Y_m}$ and $L(g) = \langle f, g \rangle$ for all $g \in F_0(p,\alpha,\beta)$. $\square$

Lemma 3.7. *For $p > \max\{1, 1+\alpha+\beta\}$, the polynomials are dense in $R(p,\alpha,\beta)$.*

Proof. Using Theorem 3.5 and the second part of the proof of Lemma 4.2 in [PX] the result follows directly. $\square$

Proposition 3.8. *Let $\alpha \ge 0$, $p > 1+\alpha+\beta$ and $f \in R(p,\alpha,\beta)$. Then $\|f - f_r\|_{R(p,\alpha,\beta)} \to 0$ as $r \to 1$.*

Proof. Clearly, $f_r \in R(p,\alpha,\beta)$ whenever $f \in R(p,\alpha,\beta)$. A direct computation with polynomials gives that $\langle f, g_r \rangle = r \langle f_r, g \rangle$. Hence, Theorem 3.1 and inequality (10) give that

$$\begin{aligned} \|f_r\|_{R(p,\alpha,\beta)} &\approx \sup_{g \in B_{F_0(p,\alpha,\beta)}} |\langle f_r, g \rangle| = \frac{1}{r} \sup_{g \in B_{F_0(p,\alpha,\beta)}} |\langle f, g_r \rangle| \\ &\le \frac{1}{r} \sup_{g \in B_{F_0(p,\alpha,\beta)}} \|f\|_{R(p,\alpha,\beta)} \|g_r\|_{F(p,\alpha,\beta)}. \end{aligned}$$

Since $\|g_r\|_{F(p,\alpha,\beta)} \le \|g\|_{F(p,\alpha,\beta)}$ we get that for given $\delta > 0$, $\|f_r\|_{R(p,\alpha,\beta)} \lesssim \|f\|_{R(p,\alpha,\beta)}$ for all $f \in R(p,\alpha,\beta)$ and for all $r \in (\delta, 1)$. Thus, let $f \in R(p,\alpha,\beta)$ and fix $\varepsilon > 0$. According to Lemma 3.7 we can find a polynomial p_0 such that $\|f - p_0\|_{R(p,\alpha,\beta)} < \varepsilon$. By the continuity of p_0 on $\overline{\mathbf{D}}$ we can choose r_0 close enough to 1 so that

$$\|p_0 - (p_0)_r\|_{R(p,\alpha,\beta)} \lesssim \|p_0 - (p_0)_r\|_{H^\infty} < \varepsilon \quad \text{whenever } r \ge r_0.$$

Hence, for $r \ge r_0$,

$$\begin{aligned} \|f - f_r\|_{R(p,\alpha,\beta)} &\le \|f - p_0\|_{R(p,\alpha,\beta)} + \|p_0 - (p_0)_r\|_{R(p,\alpha,\beta)} + \|(p_0)_r - f_r\|_{R(p,\alpha,\beta)} \\ &\lesssim \|f - p_0\|_{R(p,\alpha,\beta)} + \|p_0 - (p_0)_r\|_{R(p,\alpha,\beta)} \lesssim \varepsilon. \qquad \square \end{aligned}$$

4. The $E(p,\alpha,\beta) - F(p,\alpha,\beta)$ duality

Let

$$E(p,\alpha,\beta) := \{L \in F(p,\alpha,\beta)^* : L|_{(B_{F(p,\alpha,\beta)}, co)} \text{ is continuous}\}$$

be the closed subspace of $F(p,\alpha,\beta)^*$. Since $\big(B_{F(p,\alpha,\beta)}, co\big)$ is compact by Lemma 2.1, the Dixmier-Ng theorem [N] gives that

$$J\colon F(p,\alpha,\beta) \to E(p,\alpha,\beta)^*, \quad J\colon f \mapsto (L \mapsto L(f)), \quad g \in F(p,\alpha,\beta),\ L \in E(p,\alpha,\beta),$$

is an isometric isomorphism. Hence, we have

Theorem 4.1. *The $E(p,\alpha,\beta)$-space is a Banach space and $J\colon F(p,\alpha,\beta) \to E(p,\alpha,\beta)^*$ is an isometric isomorphism.*

For $z \in \mathbf{D}$, let $\delta_z : F(p,\alpha,\beta) \to \mathbf{C}$ be the point evaluation defined by $\delta_z(f) := f(z)$. Clearly, $\delta_z \in E(p,\alpha,\beta)$. We claim that the closed linear span of the set $\{\delta_z : z \in \mathbf{D}\}$ coincides with $E(p,\alpha,\beta)$. Indeed, if not, then by the Hahn–Banach theorem there is a $J(g) \not\equiv 0$, where $g \in F(p,\alpha,\beta)$, such that $J(g)\delta_z = g(z) = 0$ for all $z \in \mathbf{D}$, which is a contradiction. In particular, this shows that $E(p,\alpha,\beta)$ is separable.

For $L \in F_0(p,\alpha,\beta)^*$ and $r \in (0,1)$ we denote by L_r the functional given by

$$L_r(g) := L(g_r).$$

Since $\|g_r\|_{F(p,\alpha,\beta)} \le \|g\|_{F(p,\alpha,\beta)}$ for every $g \in F(p,\alpha,\beta)$, we have that $L_r \in F(p,\alpha,\beta)^*$ and that $\|L_r\| \le \|L\|$. Hence, by Alaoglu's theorem the net $\{L_r\}$ has a weak*-cluster point. Define $\tilde{L} \in F(p,\alpha,\beta)^*$ to be any such point. By Proposition 2.3, we know that $\lim_{r\to 1} L_r(g)$ exists for all $g \in F_0(p,\alpha,\beta)$. Therefore the restriction of $\tilde{L}$ to $F_0(p,\alpha,\beta)$ is equal to L and $\|\tilde{L}\| = \|L\|$.

Theorem 4.2. *For $\alpha \ge 0$ and $p > 1+\alpha+\beta$, the restriction map $E(p,\alpha,\beta) \to F_0(p,\alpha,\beta)^*$, $L \mapsto L|_{F_0(p,\alpha,\beta)}$, is an isometric isomorphism. In particular, every $L \in F_0(p,\alpha,\beta)^*$ has an extension $\tilde{L} \in F(p,\alpha,\beta)^*$ such that $\tilde{L}|_{(B_{F(p,\alpha,\beta)},co)}$ is continuous.*

Proof. The restriction map is clearly well-defined, linear and bounded. Next we show that it is surjective. For $L \in F_0(p,\alpha,\beta)^*$ we claim that

$$\|\tilde{L} - L_r\| \to 0 \quad \text{as} \quad r \to 1,$$

where $\tilde{L} \in F(p,\alpha,\beta)^*$ is the weak*-cluster point of the net $\{L_r\}$. This is proved by showing that $\{L_r\}$ is a Cauchy net. Using Theorem 3.1 and inequality (10) we obtain that for some $f \in R(p,\alpha,\beta)$,

$$\begin{aligned}|L_s(g) - L_r(g)| &= |L(g_s) - L(g_r)| = |\langle f, g_s\rangle - \langle f, g_r\rangle| \\ &= |\langle sf_s - rf_r, g\rangle| \le \|sf_s - rf_r\|_{R(p,\alpha,\beta)}\|g\|_{F(p,\alpha,\beta)}.\end{aligned}$$

In other words,

$$\sup_{g\in B_{F(p,\alpha,\beta)}} |L_s(g) - L_r(g)| \lesssim \|f - f_s\|_{R(p,\alpha,\beta)} + |s-r|\,\|f\|_{R(p,\alpha,\beta)} + \|f - f_r\|_{R(p,\alpha,\beta)},$$

which by Proposition 3.8 tends to zero as r and s tend to 1. Therefore $\tilde{L}$ must be the limit of the net $\{L_r\}$.

Note that $L_r \in E(p,\alpha,\beta)$ for every r with $r \in (0,1)$. Indeed, let $L_r \in F(p,\alpha,\beta)^*$ and take a sequence $\{g_n\} \in B_{F(p,\alpha,\beta)}$ which converges to g on compact subsets of

$\mathbf{D}$. Then

$$\begin{aligned}|L_r(g) - L_r(g_n)| &\le \|L\|\,\|g_r - (g_n)_r\|_{F(p,\alpha,\beta)} \\ &\le \|L\| \left(\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |g'(rz) - g_n'(rz)|^p (1-|z|^2)^\alpha (1-|\sigma_a(z)|^2)^\beta dA(z)\right)^{\frac{1}{p}} \\ &\lesssim \|L\| \sup_{z\in\mathbf{D}} |g'(rz) - g_n'(rz)|,\end{aligned}$$

which tends to zero by assumption.

Hence, we conclude that $\tilde{L} \in E(p,\alpha,\beta)$ and the surjectivity is thereby proved.

Finally, we show that the restriction map is an isometry. For any $g \in B_{F(p,\alpha,\beta)}$, the net $\{g_r\} \subset B_{F_0(p,\alpha,\beta)}$ and converges to g uniformly on compact subsets of $\mathbf{D}$. If $L \in E(p,\alpha,\beta)$, then $L(g_r) \to L(g)$ and therefore

$$\sup_{g\in B_{F(p,\alpha,\beta)}} |L(g)| = \sup_{g\in B_{F_0(p,\alpha,\beta)}} |L(g)|$$

and we are finished. □

Corollary 4.3. *Let $\alpha \ge 0$ and $p > 1+\alpha+\beta$. Then $F_0(p,\alpha,\beta)^{**}$ is isometrically isomorphic to $F(p,\alpha,\beta)$ and $R(p,\alpha,\beta)$ is isomorphic to $E(p,\alpha,\beta)$.*

Remark 4.4. Since $E(p,\alpha,\beta)$ is a separable dual space it has the Radon–Nikodym property. Hence, by p. 144 in [Go], we get that $E(p,\alpha,\beta)$ is the unique isometric predual of $F(p,\alpha,\beta)$.

Remark 4.5. Due to the restriction $p > \max\{1, 1+\alpha+\beta\}$, the results in sections 3 and 4 cannot be applied to all of the spaces mentioned in list (1). However, for the spaces that the results do not apply to, there are in fact easier ways to obtain similar results.

5. Multiplication operators on $R(p,\alpha,\beta)$

Multiplication operators on $F(p,\alpha,\beta)$-spaces and subspaces thereof have been studied in many papers. For example in [Z2], the author used the $F(p,\alpha,\beta)$-spaces to characterize the pointwise multipliers from weighted Bergman spaces and Hardy spaces to weighted Bergman spaces. It is still an open problem to give a complete characterization of the bounded multiplication operators on the Q_β-spaces for $\beta \in (0,1)$ (see [X]). In this section we characterize the bounded multiplication operators on $R(p,\alpha,\beta)$.

Theorem 5.1. *Let $p > \max\{1, 1+\alpha+\beta\}$. Then M_ψ is bounded on $R(p,\alpha,\beta)$ if and only if $\psi \in H^\infty$.*

Proof. Assume that $\psi \in H^\infty$ and let $f \in R(p,\alpha,\beta)$. Then

$$\begin{aligned}&\|M_\psi f\|_{R(p,\alpha,\beta)} \\ &\le \|\psi\|_{H^\infty} \sum_{m=1}^{\infty} \left(\int_{\mathbf{D}} |f_m(z)|^{\frac{p}{p-1}} (1-|z|^2)^{-\frac{\alpha}{p-1}} (1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\, dA(z)\right)^{\frac{p-1}{p}}.\end{aligned}$$

Taking infimum over all the representations of f on both sides gives that M_ψ is bounded on $R(p,\alpha,\beta)$.

Conversely, assume that M_ψ is bounded on $R(p,\alpha,\beta)$. Fix

$$\eta > \frac{\beta}{p} \quad \text{and define} \quad f_w(z) := \frac{(1-|w|^2)^\eta}{(1-\overline{w}z)^{\eta+2-\frac{\alpha+2}{p}}}, \quad w \in \mathbf{D}.$$

Trivially, $f_w \in H(\mathbf{D})$. We now claim that

$$\sup_{w\in\mathbf{D}} \|f_w\|_{R(p,\alpha,\beta)} \lesssim 1. \tag{11}$$

Fix $w \in \mathbf{D}$. Then $w \in E_m$ for some $m \in \mathbf{N}$. By Remark 2.4 there exists $\delta \in (0,1)$ (which does not depend on w and m) such that $|\sigma_{a_m}(w)| \le \delta$. Therefore we obtain (with a change of variable $z \mapsto \sigma_w(z)$) that

$$\begin{aligned}
&\|f_w\|_{R(p,\alpha,\beta)}\\
&\le \left(\int_{\mathbf{D}} |f_w(z)|^{\frac{p}{p-1}}(1-|z|^2)^{-\frac{\alpha}{p-1}}(1-|\sigma_{a_m}(z)|^2)^{-\frac{\beta}{p-1}}\,dA(z)\right)^{\frac{p-1}{p}}\\
&= \left(\int_{\mathbf{D}} \frac{(1-|w|^2)^{\frac{\eta p}{p-1}+2}}{|1-\overline{w}\sigma_w(z)|^{\frac{(\eta+2)p-\alpha-2}{p-1}}}\frac{(1-|\sigma_w(z)|^2)^{-\frac{\alpha}{p-1}}}{|1-\overline{w}z|^4}(1-|\sigma_{a_m}(\sigma_w(z))|^2)^{-\frac{\beta}{p-1}}\,dA(z)\right)^{\frac{p-1}{p}}\\
&= \left(\int_{\mathbf{D}} \frac{(1-|z|^2)^{-\frac{\alpha}{p-1}}}{|1-\overline{w}z|^{2-\frac{\alpha+\eta p}{p-1}}}(1-|\sigma_{\sigma_w(a_m)}(z))|^2)^{-\frac{\beta}{p-1}}\,dA(z)\right)^{\frac{p-1}{p}}\\
&\le 2^{\frac{2\beta}{p}}\underbrace{(1-|\sigma_w(a_m)|^2)^{-\frac{\beta}{p}}}_{\le(1-\delta^2)^{-\frac{\beta}{p}}}\left(\int_{\mathbf{D}} \frac{(1-|z|^2)^{-\frac{\alpha+\beta}{p-1}}}{|1-\overline{w}z|^{2-\frac{\alpha+\eta p}{p-1}}}\,dA(z)\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $\eta > \beta/p$, the integral formula (2) shows that the last integral is bounded in w. Hence, (11) is valid.

Since $f_w \in R(p,\alpha,\beta)$, we have that $\psi f_w \in R(p,\alpha,\beta)$ by assumption. Thus, using Proposition 2.7 and (11), we obtain

$$|\psi(z)f_w(z)| \lesssim \frac{\|\psi f_w\|_{R(p,\alpha,\beta)}}{(1-|z|)^{2-\frac{\alpha+2}{p}}} \lesssim \frac{\|f_w\|_{R(p,\alpha,\beta)}}{(1-|z|)^{2-\frac{\alpha+2}{p}}} \lesssim \frac{1}{(1-|z|)^{2-\frac{\alpha+2}{p}}} \quad \text{for all } z \in \mathbf{D}.$$

Especially for $z = w$ we get that $|\psi(w)| \lesssim 1$ for all $w \in \mathbf{D}$. □

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