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MATCHING UNIVALENT FUNCTIONS AND CONFORMAL WELDING

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Abstract. Given a conformal mapping f of the unit disk $\mathbf{D}$ onto a simply connected domain D in the complex plane bounded by a closed Jordan curve, we consider the problem of constructing a matching conformal mapping, i.e., the mapping of the exterior of the unit disk $\mathbf{D}^*$ onto the exterior domain D^* regarding to D. The answer is expressed in terms of a linear differential equation with a driving term given as the kernel of an operator dependent on the original mapping f. Examples are provided. This study is related to the problem of conformal welding and to representation of the Virasoro algebra in the space of univalent functions.

Introduction

One of the classical problems of complex analysis resides in finding the conformal mapping between a given simply connected hyperbolic domain D on the Riemann sphere $\overline{\mathbf{C}}$ and some canonical domain, e. g., the unit disk $\mathbf{D} := \{z : |z| < 1\}$ or its exterior $\mathbf{D}^* := \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ means the closure of $\mathbf{D}$. Despite the fact that the existence and essential uniqueness of the mapping is guaranteed by the Riemann mapping theorem, only in some particular cases it can be found analytically in a more or less explicit form. In the present paper we consider a special formulation of this problem, when the domain D is bounded by a closed Jordan curve and represented by means of the conformal mapping of $\mathbf{D}^*$ onto the exterior D^* of the domain D, $\infty \in D^*$.

If the boundary ∂D is C^∞ smooth, then this formulation is closely connected to Kirillov's representation of the Lie–Fréchet group $\mathrm{Diff}^+(S^1)$ of all orientation preserving C^∞-diffeomorphisms of the unit circle S^1, and to representation of the Virasoro algebra, which is a central extension by $\mathbf{C}$ of the complexified Lie algebra of vector fields on S^1. Virasoro algebra is known to play an important role in non-linear equations, where the Virasoro algebra is intrinsically related to the KdV canonical structure (see, e.g., [6, 8]), and in Conformal Field Theory, where the Virasoro–Bott group appears as the space of reparametrization of a closed string (see, e.g., [20]) .

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Let f be a conformal mapping of $\mathbf{D}$ onto a Jordan domain D and let φ be a conformal mapping of $\mathbf{D}^*$ onto a Jordan domain D^*. The functions f and φ are said to be *matching* if D and D^* are complementary domains, i.e., $D \cap D^* = \emptyset$ and $\partial D = \partial D^*$.

A pair of matching functions (f, φ), being continuously extended to S^1, defines a homeomorphism of S^1 given by the formula

$$\gamma = f^{-1} \circ \varphi. \tag{1}$$

Such a representation of homeomorphisms of S^1 is called the *conformal welding*.

Using Möbius transformations we can always assume that

(i) $0 \in D$ and $\infty \in D^*$;
(ii) $f(0) = f'(1) - 1 = 0$;
(iii) $\varphi(\infty) = \infty$.

Conformal weldings have close connection to theory of quasiconformal (q.c.) mappings. Denote by $\mathscr{S}$ the class of all univalent analytic functions f in $\mathbf{D}$ subject to condition (ii), and let $\mathscr{S}^{\mathrm{qc}}$ be the subclass of $\mathscr{S}$ consisting of functions which can be extended to a quasiconformal homeomorphism of $\overline{\mathbf{C}}$. If $f \in \mathscr{S}^{\mathrm{qc}}$, then φ also admits q.c. extention to $\overline{\mathbf{C}}$ and therefore $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$, where $\mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ stands for the group of all orientation preserving quasisymmetric (q.s.) homeomorphisms of S^1, i.e., γ satisfies

$$\sup\left\{\left|\frac{\gamma\big(e^{i(t+h)}\big) - \gamma\big(e^{it}\big)}{\gamma\big(e^{i(t-h)}\big) - \gamma\big(e^{it}\big)}\right| : t, h \in \mathbf{R},\ 0 < |h| < \pi\right\} < +\infty. \tag{2}$$

Moreover, it is known that for any $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ there exists a unique conformal welding (1) under conditions (i)–(iii). Given $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$, the construction of the pair (f, φ) of matching functions involves solution of the Beltrami equation

$$\bar{\partial} f = \mu\, \partial f,$$

where ∂ and $\bar{\partial}$ stand for $\big(\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial y}\big)/2$ respectively, with the coefficient $\mu = \mu(z)$ depending on γ. See Section 1 for details.

Without any attempt to make a complete list we mention papers [3, 4, 9, 11, 12, 14, 17], where further study of the existence and uniqueness of conformal welding and its generalizations can be found.

In this paper we establish a more explicit connection between f, φ and γ. We will use the notation Lip_α, $\alpha \in (0,1)$ for the class of Hölder continuous functions of exponent α, and $C^{n,\alpha}$ for the class of n-times differentiable functions with the n-th derivative from the class Lip_α. In order to indicate the domain of definition and admissible values of functions we will add them in the parenthesis, e.g., $\mathrm{Lip}_\alpha(S^1, \mathbf{R})$ will stand for the set of all real-valued functions which are from the class Lip_α on S^1. By $\mathscr{S}^{n,\alpha}$, $n \geqslant 1$, we denote the class of all functions $f \in \mathscr{S}$ that map $\mathbf{D}$ onto domains bounded by $C^{n,\alpha}$-smooth Jordan curves. According to the Kellog–Warschawski theorem (see, e.g., [21, p. 49]), $f \in \mathscr{S}^{n,\alpha}$ if and only if it can be continuously extended to S^1, with $f|_{S^1} \in C^{n,\alpha}$, and $f'|_{S^1}$ does not vanish. The class of all $f \in \mathscr{S}$ that map $\mathbf{D}$ onto domains bounded by C^∞-smooth Jordan curves will be denoted by $\mathscr{S}^\infty$.

Let $f \in \mathscr{S}^{1,\alpha}$. Consider the linear operator I_f from $\mathrm{Lip}_\alpha(S^1, \mathbf{R})$ to the space $\mathrm{Hol}(\mathbf{D})$ of all holomorphic functions in $\mathbf{D}$, defined by the formula

$$I_f[v](z) := -\frac{1}{2\pi i}\int_{S^1}\left(\frac{sf'(s)}{f(s)}\right)^2 \frac{v(s)}{f(s)-f(z)}\frac{ds}{s}, \quad z \in \mathbf{D}. \tag{3}$$

The following statement is our main result.

Theorem 1. *Suppose $f \in \mathscr{S}^{1,\alpha}$ and φ, $\varphi(\infty) = \infty$, are matching univalent functions. Then the kernel of the operator $I_f\colon \mathrm{Lip}_\alpha(S^1, \mathbf{R}) \to \mathrm{Hol}(\mathbf{D})$ is the one-dimensional manifold $\ker I_f = \mathrm{span}\{v_0\}$, where*

$$v_0(z) := \frac{1}{z}\,\frac{(\psi\circ f)(z)}{f'(z)(\psi'\circ f)(z)}, \quad \psi := \varphi^{-1}, \quad z \in S^1. \tag{4}$$

Moreover, the function v_0 is positive on S^1 and satisfies the condition

$$\int_0^{2\pi}\frac{dt}{v_0(e^{it})} = 2\pi. \tag{5}$$

Remark 1. Let $f \in \mathscr{S}^{1,\alpha}$ be given. Consider the problem of finding the conformal mapping ψ of $D^* := \overline{\mathbf{C}} \setminus \overline{f(\mathbf{D})}$ onto $\mathbf{D}^*$, $\psi(\infty) = \infty$, (subject to an additional condition ensuring the uniqueness). Theorem 1 reduces this problem to solution of the equation $I_f[v] = 0$. Indeed, given f and v_0, one can calculate ψ on the boundary of D^* by solving the following differential equation

$$\psi'(u) = H(u)\psi(u), \quad u \in \partial D^*,$$

where $H := \tilde{H} \circ f^{-1}$ and $\tilde{H}(z) := 1/\left[zf'(z)v_0(z)\right]$, $z \in S^1$.

Theorem 1 describes the real-valued solutions to the equation $I_f[v] = 0$. The set of complex solutions to this equation is much more extensive. Denote by $\mathrm{Hol}_C(\mathbf{D}^*)$ the class of all continuous functions $h\colon \mathbf{D}^* \cup S^1 \to \mathbf{C}$ which are analytic in $\mathbf{D}^*$.

Theorem 2. *Suppose $f \in \mathscr{S}^{1,\alpha}$ and φ, $\varphi(\infty) = \infty$, are matching univalent functions, and $\gamma := f^{-1} \circ \varphi$ is the induced homeomorphism of S^1. Then the kernel of the operator $I_f\colon \mathrm{Lip}_\alpha(S^1, \mathbf{C}) \to \mathrm{Hol}(\mathbf{D})$ coincides with the set of all functions v of the form*

$$v(z) = v_0(z)\cdot(h\circ\gamma^{-1})(z), \quad z \in S^1, \tag{6}$$

where h is an arbitrary function belonging to $\mathrm{Hol}_C(\mathbf{D}^*) \cap \mathrm{Lip}_\alpha(S^1, \mathbf{C})$ and v_0 is defined by (4).

In Section 2 we show how the operator I_f appears in a natural way within the identification of the Kirillov's homogeneous manifold $\mathscr{M} := \mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$ with $\mathscr{S}^\infty$ and deduce an analogue of Theorem 1 for the C^∞-smooth case.

Section 4 is devoted to the proof of Theorems 1 and 2. Examples of univalent matching functions and conformal weldings are given in Sections 5 and 6.

1. Conformal welding for quasisymmetric homeomorphisms of S^1

It is known that *conformal welding establishes a bijective correspondence between $\mathscr{S}^{\mathrm{qc}}$ and* $\mathrm{Homeo}^+_{\mathrm{qs}}(S^1)/\mathrm{Rot}(S^1)$, where $\mathrm{Rot}(S^1)$ stands for the group of rotations of S^1. For the history of the question, see, e.g., [10]. Here we briefly give a sketch of the proof, see also [22].

Let u, $u(\infty) = \infty$, be any q.c. automorphism of $\mathbf{D}^*$. Let us construct the quasiconformal homeomorphism $\tilde{f}$ of the Riemann sphere $\overline{\mathbf{C}}$, such that the functions $f := \tilde{f}|_{\mathbf{D}}$ and $\varphi := \big(\tilde{f}|_{\mathbf{D}^*}\big) \circ u$ are analytic in $\mathbf{D}$ and $\mathbf{D}^*$ respectively. It is easy to see that $\tilde{f}$ should satisfy the Beltrami equation

$$\bar{\partial}\tilde{f}(z) = \mu(z)\,\partial\tilde{f}(z), \quad \mu(z) := \begin{cases} \bar{\partial}\big(u^{-1}(z)\big)/\partial\big(u^{-1}(z)\big), & \text{if } z \in \mathbf{D}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

In order to have a unique solution we impose the following normalization

$$\tilde{f}(0) = \tilde{f}'(0) - 1 = 0, \;\; \tilde{f}(\infty) = \infty. \tag{8}$$

Then $f \in \mathscr{S}^{\mathrm{qc}}$ and φ are matching functions and the homeomorphism of the unit circle $\gamma := f^{-1} \circ \varphi$ coincides with the continuous extension of u to S^1.

It is known [2] that an orientation preserving homeomorphism $\gamma\colon S^1 \to S^1$ can be extended to a q.c. automorphism u of $\mathbf{D}^*$ if and only if it is quasisymmetric, i.e., satisfies (2). Moreover, by superposing u and a suitable q.c. automorphism of $\mathbf{D}^*$, identical on S^1, one can always assume that $u(\infty) = \infty$. It follows that for any $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ there exists a conformal welding with $f \in \mathscr{S}^{\mathrm{qc}}$.

Fix any q.c. extension $u\colon \mathbf{D}^* \to \mathbf{D}^*$; $\infty \mapsto \infty$, of $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ and let

$$\tilde{f}(z) := \begin{cases} f(z), & \text{if } z \in \mathbf{D}, \\ (\varphi \circ u^{-1})(z), & \text{otherwise,} \end{cases}$$

where $f \in \mathscr{S}$ and φ are matching univalent functions such that $\gamma = f^{-1} \circ \gamma$. Then $\tilde{f}$ satisfies (7)–(8). This defines $\tilde{f}$ uniquely (see, e.g., [18, p. 194]). It follows that for any $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ the conformal welding is unique.

On the hand, if $f \in \mathscr{S}^{\mathrm{qc}}$, then φ and consequently $\gamma = f^{-1} \circ \varphi$, can be extended to a quasiconformal homeomorphism of $\overline{\mathbf{C}}$. It follows that $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$. Since the condition $\phi(\infty) = \infty$ defines a conformal mapping onto $D^* := \overline{\mathbf{C}} \setminus \overline{f(\mathbf{D})}$ only up to rotations, f corresponds to the equivalence class $[\gamma] \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)/\mathrm{Rot}(S^1)$, rather than to an element of $\mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$.

Remark 2. If $\gamma\colon S^1 \to S^1$ is a diffeomorphism, then one of its q.c. extensions to $\mathbf{D}^*$ is given by the formula $u(re^{it}) := r\gamma(e^{it})$, and the Beltrami coefficient μ in (7) equals

$$\mu(re^{it}) = e^{2it}\,\frac{1 - \big(\gamma^{-1}\big)^{\#}(e^{it})}{1 + \big(\gamma^{-1}\big)^{\#}(e^{it})},$$

where we introduce the operator '#' by $\beta^{\#} := \big(\pi^{-1} \circ \beta \circ \pi\big)'$, and $\pi\colon \mathbf{R} \to S^1$ is the universal covering, $\pi(x) = e^{ix}$.

In Section 6 we consider a certain class of analytic diffeomorphisms γ for which Theorem 1 can be used to find the conformal welding without solving the Beltrami equation.

2. Kirillov's representation of $\mathrm{Diff}^+(S^1)$ via univalent functions

The group $\mathrm{Diff}^+(S^1)$ of all orientation preserving C^∞-diffeomorphisms of the unit circle S^1 is one of the simplest, and by this reason important, example of an infinite-dimensional Lie group. Denote by $\mathscr{F}$ the Fréchet space of all C^∞-smooth functions $h\colon S^1 \to \mathbf{R}$ endowed with the countable family of seminorms $\|h\|_n := \max_{x\in\mathbf{R}} \left|(d^n/dx^n)h(e^{ix})\right|$, $n \geqslant 0$. It is known (see, e.g., [5]) that $\mathrm{Diff}^+(S^1)$ becomes a Lie–Fréchet group if we define the structure of a C^∞-smooth manifold on $\mathrm{Diff}^+(S^1)$ by means of the covering mapping $h \mapsto \gamma[h]$, $\gamma[h](\zeta) := \zeta e^{ih(\zeta)}$, of the open set $\{h \in \mathscr{F} : dh(e^{ix})/dx > -1\}$ onto $\mathrm{Diff}^+(S^1)$. All the tangent spaces $\mathrm{T}_\gamma\mathrm{Diff}^+(S^1)$ are identified then in a natural way with $\mathscr{F}$.

Kirillov [15] suggested to use the correspondence between $\mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ and $\mathscr{S}^{\mathrm{qc}}$ established by means of conformal welding, in order to represent the homogeneous manifold $\mathscr{M} := \mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$, usually referred to as Kirillov's manifold, via univalent functions.

Consider the class $\mathscr{S}^\infty$ of all functions $f \in \mathscr{S}$ having C^∞-smooth extension to $\partial\mathbf{D}$ with non-vanishing derivative. By the Kellog–Warschawski theorem (see, e.g., [21, p. 49]), $f \in \mathscr{S}^\infty$ if and only if f has a C^∞-smooth extension to S^1 and the derivative $f'\big|_{S^1}$ does not vanish. It follows that $\mathscr{S}^\infty$ corresponds via conformal welding to a subset of $\mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$. According to the result of Kirillov [15], it actually coincides with $\mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$, and consequently one can identify $\mathscr{M}$ with $\mathscr{S}^\infty$.

Denote by $K\colon \mathscr{S}^\infty \to \mathscr{M}$ the mapping that takes each $f \in \mathscr{S}^\infty$ to the corresponding equivalence class of diffeomorphisms $[\gamma]$. The infinitesimal version of the inverse mapping is as follows.

Fix any $v \in \mathscr{F} \cong \mathrm{T}_{\mathrm{id}}\mathrm{Diff}^+(S^1)$ and consider the right-invariant vector field over $\mathrm{Diff}^+(S^1)$, $V\colon \gamma \mapsto v \circ \gamma \in \mathscr{F} \cong \mathrm{T}_\gamma\mathrm{Diff}^+(S^1)$ generated by v. This gives us the identification $\mathrm{T}_\gamma\mathrm{Diff}^+(S^1) \cong \mathrm{T}_{\mathrm{id}}\mathrm{Diff}^+(S^1) \cong \mathscr{F}$, which we adhere further on, and which is obviously different from the identification of $\mathrm{T}_\gamma\mathrm{Diff}^+(S^1)$ with $\mathscr{F}$ described above.

Thus, to each $v \in \mathscr{F}$ and each $\gamma \in \mathrm{Diff}^+(S^1)$ one associates the variation $\gamma_\varepsilon(\zeta) := \gamma(\zeta)\exp[i\varepsilon(v\circ\gamma)(\zeta)]$ of γ. According to [16], the corresponding variation of the function f equals to $f_\varepsilon := K^{-1}([\gamma_\varepsilon]) = f + \delta f + o(\varepsilon)$, where

$$\delta f(z) = \frac{\varepsilon}{2\pi}\int_{S^1}\left(\frac{sf'(s)}{f(s)}\right)^2 \frac{f^2(z)\,v(s)}{f(z)-f(s)}\frac{ds}{s} = i\varepsilon f^2(z) I_f[v](z), \quad z \in \mathbf{D}. \tag{9}$$

A natural consequence is that $I_f[v](z) = 0$ for all $z \in \mathbf{D}$ if and only if the variation of γ produces no variation of $[\gamma] \in \mathscr{M}$ (up to higher order terms). It can be reformulated as follows: *the element of* $\mathrm{T}_\gamma\mathrm{Diff}^+(S^1)$ *represented by* $v\circ\gamma$ *is tangent to the one-dimensional manifold*

$$\gamma \circ \mathrm{Rot}(S^1) = [\gamma] \subset \mathrm{Diff}^+(S^1).$$

The latter is equivalent to

$$v \in \mathrm{Ad}_\gamma\Big(\mathrm{T}_{\mathrm{id}}\mathrm{Rot}(S^1)\Big) = \mathrm{Ad}_\gamma\big\{\text{constant functions on } S^1\big\}.$$

Elementary calculations show that

$$\mathrm{Ad}_\gamma u = \frac{u\circ\gamma^{-1}}{\left(\gamma^{-1}\right)^{\#}}.$$

As a conclusion we get

Proposition 1. *The kernel of* $I_f\colon \mathscr{F} \to \mathrm{Hol}(\mathbf{D})$ *is one-dimensional and coincides with* $\mathrm{span}\{1/(\gamma^{-1})^{\#}\}$.

Remark 3. Proposition 1 reveals a version of Theorem 1 for C^∞-smooth case. It reduces the problem of calculating $K^{-1}(f)$ to solution of the equation $I_f[v]=0$. The nontrivial solution v_0 subject to the normalization

$$\int_0^{2\pi} \frac{dt}{v_0(e^{it})} = 2\pi$$

allows us to determine $[\gamma]$ by means of the equality

$$\gamma^{-1}(e^{ix}) = \exp\left(\int_0^x \frac{i\,dt}{v_0(e^{it})} + iC\right), \tag{10}$$

with the arbitrary constant C being responsible for the fact that (10) defines γ only up to the right action of $\mathrm{Rot}(S^1)$.

3. Virasoro algebra and complex structure on Kirillov's manifold

The Lie algebra of $\mathrm{Diff}^+(S^1)$ is the Fréchet space $\mathscr{F}$ endowed with the Lie bracket

$$\{v_1,v_2\}(e^{ix}) = v_2(e^{ix})\,\frac{dv_1(e^{ix})}{dx} - v_1(e^{ix})\,\frac{dv_2(e^{ix})}{dx}. \tag{11}$$

Remark 4. The expression (11) differs in sign from the commutator $[V_1,V_2]$ of the vector fields $V_j\colon \gamma \to v_j\circ\gamma$ generated by v_j, because V_j are right-invariant vector fields rather than left-invariant, which are usually considered in this context.

The simplest basis for the complexification $\mathscr{F}_{\mathbf{C}} := \{v_1+iv_2 : v_1,v_2\in\mathscr{F}\}$ of $\mathscr{F}$ is given by powers of z:

$$L_k(z) := iz^k, \quad k\in\mathbf{Z}.$$

Continuation of the Lie bracket $\{\cdot,\cdot\}\colon \mathscr{F}\times\mathscr{F}\to\mathscr{F}$ by complex bilinearity to $\mathscr{F}_{\mathbf{C}}$ gives the commutation relations $\{L_k,L_j\}=(j-k)L_{k+j}$.

The (complex) Virasoro algebra can defined now as the central extension of $\mathscr{F}_{\mathbf{C}}$ by $\mathbf{C}$ which is the Lie algebra over $\mathscr{F}_{\mathbf{C}}\oplus\mathbf{C}$ with the commutation relations

$$\big\{(L_k,a),(L_j,b)\big\} = \big(\{L_k,L_j\}, \tfrac{c}{12}k(k^2-1)\delta_{k,-j}\big).$$

Here c is a constant parameter referred to as the *central charge* in mathematical physics.

Unfortunately, it is not known whether the Lie–Fréchet algebra $\mathscr{F}_{\mathbf{C}}$ is the Lie algebra of any Lie–Fréchet group, which, if exists, can serve as complexification for $\mathrm{Diff}^+(S^1)$. There are strong reasons to believe that such a group does not exist [19]. Nevertheless, the infinitesimal action $\mathscr{F}\times\mathscr{M}\to \mathrm{T}\mathscr{M}$ induced by the left action of $\mathrm{Diff}^+(S^1)$, can be extended from $\mathscr{F}$ to $\mathscr{F}_{\mathbf{C}}$, due to the fact that the linear space spanned by the variations (9) has a natural complex structure, the operation of multiplication by i. This induces complex structure J_γ on $\mathscr{F}/\ker I_f \cong \mathrm{T}_{[\gamma]}\mathscr{M}$. We use Theorem 2 to obtain the explicit form of it. Instead of looking for the operator on $\mathscr{F}/\ker I_f$ we define J_γ as an operator on $\mathscr{F}$ with the property that $J_\gamma[v_0]=0$. For $v\in\mathscr{F}$ we have

$$iI_f[v] = I_f[iv] = I_f[J_\gamma v].$$

It follows that $J_\gamma v = iv-\tilde v$, where $\tilde v\in\mathscr{F}_{\mathbf{C}}$ is a solution of $I_f[\tilde v]=0$ satisfying the condition $\operatorname{Im}\tilde v = v$. Using the representation (6) for $\tilde v$ we obtain the formula

$$J_\gamma[v]\circ\gamma = (v_0\circ\gamma)\cdot J_0\left[\frac{v\circ\gamma}{v_0\circ\gamma}\right], \tag{12}$$

where $J_0\colon \mathscr{F} \to \mathscr{F}$ is the so-called conjugation,

$$J_0\left[\sum_{k\in\mathbf{Z}} a_k z^k\right] = i\sum_{k\in\mathbf{Z}} \operatorname{sgn}(k) a_k z^k.$$

Elementary calculations lead us to the following

Proposition 2. *The complex structure on* $\mathrm{T}\mathscr{M}$ *induced by the standard complex structure on* $\mathscr{F}_{\mathbf{C}}$ *via* I_f *is given by* $J_\gamma = \mathrm{Ad}_\gamma J_0 \,(\mathrm{Ad}_\gamma)^{-1}$, *where* Ad_γ *stands for the differential of* $A_\gamma\beta := \gamma\circ\beta\circ\gamma^{-1}$ *at the origin* $\beta = \mathrm{id}$.

Remark 5. The complex structure J_γ coincides with that introduced in [1] only for the case $\gamma = \mathrm{id}$ and thus it is not invariant under the right action of $\mathrm{Diff}^+(S^1)$ on $\mathscr{M}$. However, J_γ is left-invariant, which is proved by Kirillov [15] and easily follows from the fact that the differential of the left action of $\mathrm{Diff}^+(S^1)$ is given by $v \mapsto \mathrm{Ad}_\gamma v$, where $v \in \mathscr{F} \cong \mathrm{T}_\gamma\mathrm{Diff}^+(S^1)$.

Remark 6. In [7] Gardiner and Sullivan considered the group $\mathrm{Homeo}^+_{\mathrm{sym}}(\mathbf{R})$ of all symmetric orientation preserving homeomorphisms of $\mathbf{R}$. In particular, they showed that the tangent space to $\mathrm{Homeo}^+_{\mathrm{sym}}(\mathbf{R})$ at the identity can be identified with the class of all functions $F\colon \mathbf{R}\to\mathbf{R}$, "smooth" in the sense of Zygmund.

4. Proof of Theorems 1 and 2

Here we give a proof of Theorems 1 and 2 stated in the Introduction, which is based purely on complex analysis.

Proof of Theorem 1. Denote $D := f(\mathbf{D})$, $\Gamma := \partial D$,

$$H(u) := \frac{g(u)v(g(u))}{u^2 g'(u)}, \quad F(w) := -\frac{1}{2\pi i}\int_\Gamma \frac{H(u)}{u-w}\,du, \quad w \in \overline{\mathbf{C}}\setminus\Gamma,$$

where g stands for the inverse of the function f.

The equation $I_f[v](z) = 0$, $z \in \mathbf{D}$, is equivalent to

$$F(w) = 0, \quad w \in D. \tag{13}$$

Using the Sokhotsky–Plemelj formulas we conclude that if v is a solution to (13), then $H(u)$ is the boundary values of an analytic function in $D^* := \overline{\mathbf{C}}\setminus\overline{D}$ vanishing at $w=\infty$. The converse is also true due to the Cauchy integral formula for unbounded domains. It follows that v_0 is a solution to (13). Indeed, for $v = v_0$ we have

$$H(u) = \frac{\psi(u)}{u^2\psi'(u)}.$$

The function v_0 can be expressed as $v_0(z) = \zeta\varphi'(\zeta)/\big(zf'(z)\big)$, where $\zeta := \psi(f(z))$. Both vectors $\zeta\varphi'(\zeta)$ and $zf'(z)$ are the outer normal vectors of Γ at the point $w = f(z) = \varphi(\zeta)$. It follows that $v_0(z) > 0$. The continuous function $\tau(t)$ defined by $e^{i\tau(t)} = \psi\big(f(e^{it})\big)$, $t\in\mathbf{R}$, satisfies the conditions $\tau'(t) = 1/v_0(e^{it})$ and $\tau(t+2\pi) = \tau(t)+2\pi$. It follows that (5) holds.

It remains to prove that any real-valued solution $v \in \mathrm{Lip}_\alpha(S^1,\mathbf{R})$ to equation (13) is of the form $v = \lambda v_0$, $\lambda\in\mathbf{R}$. Assume $v_1 \in \mathrm{Lip}_\alpha(S^1,\mathbf{R})$ is a solution. And consider the one-parameter family of solutions defined by $v := v_0 + \varepsilon v_1$, where $\varepsilon \in \mathbf{R}$ is sufficiently small for v to be positive on S^1. By the above argument, the function

$G(u) := u\psi'(u)H(u)$, $u \in \Gamma$, has an analytic continuation to D^*, which will be denoted by $G(w)$.

The function G does not vanish in $D^* \cup \Gamma$ provided ε is small enough. Indeed, $G(w) \to \psi(w)/w$ as $\varepsilon \to 0$ uniformly in $D^* \cup \Gamma$, with the limit function $\psi(w)/w$ continuous and non-vanishing. It follows that $\tilde{G}(w) := \log G(w)$ is analytic in D^* and continuous on $D^* \cup \Gamma$. The inequality $v > 0$ implies that $\operatorname{Im} \tilde{G}(u) = \operatorname{Im} \log J(u)$, $u \in \Gamma$, where $J(u) := g(u)\psi'(u)/\big(ug'(u)\big)$. This equality determines $\tilde{G}$ up to a real constant term. Therefore, $v(z)$ is unique up to a positive constant coefficient. This completes the proof. □

By the same techniques one can prove Theorem 2.

Proof of Theorem 2. Let us look for solutions to $I_f[v] = 0$ in the form (6) *without any a priori assumptions* on h, except for that $h \in \operatorname{Lip}_\alpha(S^1, \mathbf{C})$. Any solution can be represented in this form because v_0 is positive. Now we use the change of variable $s = \gamma(t)$ in integral (3). Taking into account that

$$v_0(s) = 1/(\gamma^{-1})^{\#}(s) = (t/s) \cdot (ds/dt) \quad \text{and} \quad f'\big(\gamma(t)\big) \cdot (ds/dt) = \varphi'(t),$$

we conclude that

$$I_f[v](z) = -\frac{1}{2\pi i} \int_{S^1} \left(\frac{t\varphi'(t)}{\varphi(t)} \right)^2 \frac{h(t)}{\varphi(t) - w} \frac{dt}{t}, \quad w := f(z), \quad z \in \mathbf{D}.$$

Applying another change of variable $u = \varphi(t)$, we obtain the following expression for the above quantity

$$-\frac{1}{2\pi i} \int_\Gamma \frac{\psi(u)}{u\psi'(u)} \frac{h(\psi(u))/u}{u - w} \, du,$$

Due to the Sokhotsky–Plemelj formulas and to the Cauchy integral formula for unbounded domains, the above quantity equals zero for all $w \in D := f(\mathbf{D})$ if and only if h represents the boundary values of an analytic function in $\mathbf{D}^*$. This fact proves the theorem. □

5. Examples of matching univalent functions

Here we consider a class of examples, for which both matching functions f and φ are expressed by means of ordinary differential equations.

Given an integer $n > 1$, let us consider the following quadratic differentials

$$\Xi(\zeta)\, d\zeta^2 := -\frac{d\zeta^2}{\zeta^2};$$

$$W(w)\, dw^2 := -\frac{w^{n-2}\, dw^2}{P(w)}, \quad P(w) := \prod_{k=0}^{n-1} (w - w_k), \quad w_k := e^{2\pi i k/n};$$

$$Z(z)\, dz^2 := -\frac{z^{n-2}\, dz^2}{Q(z)}; \quad Q(z) := \varkappa \prod_{k=0}^{n-1} \frac{|z_k|}{z_k} (z_k - z)(z - 1/\overline{z_k}), \quad z_k := re^{2\pi i k/n},$$

where $r \in (0,1)$, and $\varkappa > 0$ is such that $\int_{S^1} \sqrt{Z(z)}\, dz = 2\pi$ for the appropriately chosen branch of the square root.

These quadratic differentials have the following structure of trajectories (see, e.g., [13, 23]). All the trajectories of $\Xi(\zeta)\, d\zeta^2$ are circles centered on the origin, with 0 and

∞ as critical points. Critical trajectories of $W(w)\,dw^2$ are line intervals joining $w=0$ with w_k. Denote the union of their closures by E_w. All the remaining trajectories are closed Jordan curves separating E_w and the critical point at infinity. The structure of trajectories of the quadratic differential $Z(z)\,dz^2$ is symmetric with respect to the unit circle, which is also a trajectory. Similarly to $W(w)\,dw^2$, singular trajectories of $Z(z)\,dz^2$ that lie in $\mathbf{D}$ are line intervals joining the origin with z_k. They form a continuum, which we denote by E_z. The singular trajectories lying outside $\mathbf{D}$ form a symmetric continuum E_z^*. All the remaining trajectories are Jordan curves separating E_z and E_z^*.

Let us choose any non-singular trajectory Γ of the quadratic differential $W(w)\,dw^2$ and construct the bijective conformal mappings $f\colon \mathbf{D}\to D$, $f(0)=0$, $f'(0)>0$, and $\varphi\colon \mathbf{D}^*\to D^*$, $\varphi(\infty)=\infty$, $\varphi'(\infty)>0$, where D and D^* are the interior and exterior of Γ, respectively.

The mapping f can be constructed as follows. Let us define the parameter r in $Z(z)\,dz^2$ by requiring that the moduli of the annular domains $\mathbf{D}\setminus E_z$ and $D\setminus E_w$ are equal. Consider the conformal mapping f of $\mathbf{D}\setminus E_z$ onto $D\setminus E_w$ normalized by $f(z_0)=w_0$. This mapping satisfies the following differential equation

$$W(w)\,dw^2 = Z(z)\,dz^2. \tag{14}$$

Indeed, the conformal mapping $\zeta=\varrho(z)$ of the ring domain $\overline{\mathbf{C}}\setminus(E_z\cup E_z^*)$ onto the domain of the form $G:=\{\zeta : \rho<|\zeta|<1/\rho\}$ normalized by $\varrho(z_0)=\rho$ satisfies the equation (see, e.g., [23, pp. 43–46])

$$Z(z)\,dz^2 = \Xi(\zeta)\,d\zeta^2. \tag{15}$$

Analogously, the conformal mapping $\zeta=\psi(w)$ of the circular domain $\overline{\mathbf{C}}\setminus E_w$ of the quadratic differential $W(w)\,dw^2$ onto the domain $\{z : |z|>\rho\}$ normalized by $\psi(\infty)=\infty$ and $\psi(w_0)=\rho$ satisfies the equation

$$W(w)\,dw^2 = \Xi(\zeta)\,d\zeta^2.$$

Since the moduli of the annular domains $\mathbf{D}\setminus E_z$ and $D\setminus E_w$ are equal, $\psi(D\setminus E_w)=G'$, $G':=\{\zeta : \rho<|\zeta|<1\}$, and consequently $f=\psi^{-1}\circ\varrho$. It follows that (14) holds.

Now using the symmetry of E_w and E_z one can prove that f extends analytically to E_z, i.e., f is the desired conformal mapping of $\mathbf{D}$ onto D.

It follows from the above consideration, that the exterior mapping is $\varphi=\psi^{-1}|_{\mathbf{D}^*}$.

By rescaling w-plane we can assure that $f\in\mathscr{S}$. Now we can easily calculate the function v_0 spanning the kernel of the operator $I_f[v_0]$, formula (3). According to Theorem 1 and equality (15),

$$v_0(z) = \left(-z^2 Z(z)\right)^{-1/2} = \sqrt{\frac{\varkappa}{r^n}}\prod_{k=0}^{n-1}|z-re^{ikt/n}|, \quad z\in S^1.$$

Remark 7. The choice of the coefficient $\varkappa$ in the construction of quadratic differential $Z(z)dz^2$ garantees that v_0 satisfies normalization (5).

Remark 8. The circle diffeomorphism γ coincides on S^1 with ϱ^{-1}. Consequently, it can be extended analytically from S^1 to the ring G.

Remark 9. For the case $n = 2$ the curve Γ is an ellipse with foci $w = \pm 1$ and the mapping f is

$$f(z) = \sin\left(\frac{\pi \mathbf{F}(\frac{z}{r}, r^2)}{2\mathbf{K}(r^2)}\right),$$

where $\mathbf{F}(z,k)$ is the first elliptic integral,

$$\mathbf{F}(z,k) = \int_0^z \frac{dq}{\sqrt{(1-q^2)(1-k^2q^2)}},$$

and $\mathbf{K}(k) = \mathbf{F}(1,k)$. The eccentricity of the ellipse Γ equals $\lambda = 1/f(1)$. The exterior mapping is just the Joukowski mapping

$$\varphi(\zeta) = \frac{1}{2}\left(c_\lambda \zeta + \frac{1}{c_\lambda \zeta}\right), \quad c_\lambda := \frac{1+\sqrt{1-\lambda^2}}{\lambda},$$

and

$$v_0(z) = \frac{1}{(\varphi^{-1}\circ f)^\#(z)} = \frac{2r\mathbf{K}(r^2)\sqrt{(r^2-z^2)(z^2-r^{-2})}}{\pi z} = \frac{2\mathbf{K}(r^2)|r^2-z^2|}{\pi}.$$

6. Conformal welding for a class of circle diffeomorphisms

Consider a diffeomorphism $\gamma\colon S^1 \to S^1$ such that the function $v_0 := 1/(\gamma^{-1})^\#$ has the form $v_0(z) = \sum_{k=-n}^{n} a_k z^k$, in which case, since v_0 is positive, $a_{-k} = \overline{a_k}$, and so we have two equivalent representations:

$$v_0(z) = a_0 + \sum_{k=1}^{n} a_k z^k + \frac{\overline{a_k}}{z^k} = \varkappa \prod_{k=1}^{n} \frac{e^{-it_k}}{z}(r_k e^{it_k} - z)(z - e^{it_k}/r_k), \tag{16}$$

where $r_k \in (0,1)$, $t_k \in \mathbf{R}$, $k = 1, \ldots, n$, and the coefficients $\varkappa$ and a_k's are subject to the conditions $v_0 > 0$ and $\int_0^{2\pi} dt/v_0(e^{it}) = 2\pi$.

The set of all diffeomorphisms γ satisfying the above condition is dense in many important spaces of circle homeomorphisms. Let us consider the problem of finding the function $f \in \mathscr{S}^\infty$ corresponding to v_0 given by (16). In general, for a diffeomorphism $\gamma \in C^{1,\alpha}$, $\alpha \in (0,1)$, the conformal welding is given by a unique solution to the equation

$$I_f[v_0](z) := -\frac{1}{2\pi i}\int_{S^1}\left(\frac{sf'(s)}{f(s)}\right)^2 \frac{v_0(s)}{f(s)-f(z)}\frac{ds}{s} = 0, \quad z \in \mathbf{D}, \tag{17}$$

regarded as an equation with respect to $f \in \mathscr{S}^{1,\alpha}$. The existence and uniqueness of the solution to (17) is implied by Theorem 1 and the fact that for any $\gamma \in \mathrm{Homeo}^+_{\mathrm{qs}}(S^1)$ there exists a unique conformal welding with $f \in \mathscr{S}$.

If v_0 is of the form (16), then (17) can be substantially simplified by means of calculus of residues. The residue of the expression under the integral at $s = z$ equals $zf'(z)v_0(z)/(f(z))^2$ and the residue at the origin is of the form $P_0\big(1/f(z)\big)/f(z)$, where P_0 is a polynomial of degree n with coefficients depending on a_k's and the first Taylor coefficients of f. It follows that the function $w = f(z)$ satisfies the differential equation

$$\frac{w^{n-1}\,dw}{P(w)} = \frac{z^{n-1}\,dz}{Q(z)}, \tag{18}$$

where

$$P(w) := b_0 \prod_{k=1}^{n} (w - w_k),$$

b_0 and w_k's are unknown parameters and

$$Q(z) := z^n v_0(z) = \varkappa \prod_{k=1}^{n} \frac{|z_k|}{z_k} (z_k - z)(z - 1/\overline{z_k}), \quad z_k := r_k e^{it_k}.$$

Since f is univalent and analytic in $\mathbf{D}$, w_k's are exactly the images of z_k's and we can suppose that they are numbered so that $w_k = f(z_k)$.

For simplicity we suppose that all the roots of Q are simple. Then $w_k \neq w_j$ for $k \neq j$ and comparing residues of $z^{n-1}/Q(z)$ and $f'(z)\big(f(z)\big)^{n-1}/P(f(z))$ we obtain the following system of algebraic equations:

$$\frac{w_k^{n-1}}{P_k(w_k)} = A_k, \quad k = 1, \ldots, n, \tag{19}$$

where

$$P_k(w) := \frac{P(w)}{w - w_k}, \quad A_k := \operatorname*{Res}_{z=z_k} \frac{z^{n-1}}{Q(z)}.$$

Using the residue theorem we further conclude that

$$\frac{1}{b_0} = \sum_{k=1}^{n} A_k = \int_0^{2\pi} \frac{dt}{v_0(e^{it})} = 1.$$

In view of (18) the condition $f'(0) = 1$ results in the equality

$$\prod_{k=1}^{n} w_k = (-1)^n Q(0) = \varkappa \prod_{k=1}^{n} \frac{z_k}{|z_k|}. \tag{20}$$

Now we can summarize the above consideration as following

Proposition 3. *Suppose* $\gamma \in \operatorname{Diff}^+(S^1)$ *is such that* $v_0 := 1/(\gamma^{-1})^{\#}$ *is of the form* (16). *Then the function* $f \in \mathscr{S}^\infty$ *that corresponds to* γ *via conformal welding, is a solution to differential equation* (18) *with* $b_0 := 1$ *and* $w_k := f(z_k)$. *Moreover, the vector* $(w_1, \ldots, w_n)$ *satisfies system* (19), (20), *provided all the roots* z_k *of* Q *are simple.*

Remark 10. Given any non-vanishing values of the parameters w_k, $k = 1, \ldots, n$, differential equation (18) with $b_0 := 1$ has a unique analytic solution $w = w(z)$ in a neighborhood of $z = 0$ that satisfies the condition $w(0) = w'(0) - 1 = 0$. At the same time the number of solutions of system (19), (20) grows drastically as n increases.

The simplest case $n = 1$ corresponds to the subgroup $\text{Möb}(S^1) \subset \operatorname{Diff}^+(S^1)$ consisting of Möbius transformations of the unit disk restricted to S^1 (excluding rotations, which correspond to $n = 0$) and f has the form $z/(1 - c_1 z)$, $|c_1| \in (0, 1)$. But even for $n = 2$ the expressions turn out to be quite complicated.

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