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ON A PROBLEM OF NEVANLINNA

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Abstract. If f is a meromorphic function on the plane, let

$$K(f) = \limsup_{r\to\infty} \frac{N(r,0,f)+N(r,\infty,f)}{T(r,f)},$$

where we use standard functionals from Nevanlinna theory. It has long been conjectured for all meromorphic functions of finite nonintegral order ρ that $K(f) \geq K(L_\rho)$, where the entire function L_ρ is the canonical product with positive zeros satisfying $n(r,0,L_\rho) = [r^\rho]$. This conjecture has been established only for $\rho < 1$. We show the existence of $\rho_0 > 1$ such that if $1 < \rho < \rho_0$ then $K(f) \geq K(L_\rho)$ for all meromorphic f of order ρ satisfying $N(r,0,f)+N(r,\infty,f) \sim cr^\rho$ for some $c > 0$.

1. Introduction

If f is a nonconstant meromorphic function on the complex plane, let

$$K(f) = \limsup_{r\to\infty} \frac{N(r,0,f)+N(r,\infty,f)}{T(r,f)}.$$

(We assume familiarity with the basic concepts and notation of value distribution theory.) For $0 \leq \rho < +\infty$, let

$$k(\rho) = \inf K(f),$$

where f varies over all meromorphic functions of order ρ. The example $\exp(z^\rho)$ shows that $k(\rho) = 0$ if ρ is a positive integer. In [14], Nevanlinna showed that $k(\rho) > 0$ for positive nonintegral ρ and posed the problem of finding the exact value of $k(\rho)$.

For nonintegral $\rho > 0$, let L_ρ denote the Lindelöf function of order ρ, i.e., the canonical product with positive zeros and $n(r,0,L_\rho) = [r^\rho]$. Nevanlinna [14] observed with $q = [\rho]$ that

$$K(L_\rho) = k_L(\rho) := \begin{cases} \dfrac{|\sin \pi\rho|}{q + |\sin\pi\rho|}, & q \leq \rho \leq q + \frac{1}{2}, \\ \dfrac{|\sin\pi\rho|}{q+1}, & q + \frac{1}{2} \leq \rho \leq q+1. \end{cases}$$

In fact it is known [8] for any nonintegral ρ that if f is entire with positive zeros and $N(r,0,f) \sim r^\rho\sigma(r)$, where $\sigma(r)$ is slowly varying in the sense that $\sigma(br)/\sigma(r) \to 1$ as $r \to \infty$ for all $b \in (0,+\infty)$, then

$$K(f) = k_L(\rho). \tag{1.1}$$

Since L_ρ has order ρ, it is evident that $k(\rho) \leq k_L(\rho)$ for $\rho > 0$. It is generally presumed that $k(\rho) = k_L(p)$.

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The quantity $k(\rho)$ has attracted the attention of many authors. It is known [2] that

$$k(\rho) = k_L(\rho), \quad 0 \le \rho \le 1; \tag{1.2}$$

that [9]

$$K(f) \ge k_L(\rho), \quad 0 \le \rho < \infty,$$

for entire f of order ρ with positive zeros; and that [11]

$$k(\rho) \ge \frac{9\,|\sin \pi\rho|}{10(\rho+1)}, \quad \rho > 1.$$

For other results concerning $k(\rho)$, see [1], [3], [4], [5], [6], [7], [8], [10], [12], [15], and [16].

Denote $N(r,0,f) + N(r,\infty,f)$ by $N(r)$ and $n(r,0,f) + n(r,\infty,f)$ by $n(r)$. For ρ slightly greater than 1 we show that $K(f) \ge k_L(\rho)$ for meromorphic f of order ρ for which the growth of $N(r)$ is suitably regular.

Theorem. *There exists $\rho_0 > 1$ such that if $1 < \rho < \rho_0$ and f is meromorphic of order ρ with $N(r) \sim cr^\rho$ for some $c > 0$, then $K(f) \ge k_L(\rho)$.*

An important property of meromorphic functions of order less than one is that if

$$f(z) = \frac{\prod_\nu \left(1 - \dfrac{z}{z_\nu}\right)}{\prod_\nu \left(1 - \dfrac{z}{w_\nu}\right)}$$

and

$$\hat{f}(z) = \prod_\nu \left(1 - \frac{z}{|z_\nu|}\right) \cdot \prod_\nu \left(1 - \frac{z}{|w_\nu|}\right),$$

then ([5], [9])

$$T(r,f) \le T(r,\hat{f}), \quad r > 0, \tag{1.3}$$

and hence $K(f) \ge K(\hat{f})$. Thus to establish (1.2) it is sufficient to restrict attention to entire functions with positive zeros.

The greater difficulty in determining $k(\rho)$ for $\rho > 1$ is in part a result of the fact that the analogue of (1.3) is not in general valid for $\rho > 1$; thus the arguments of the zeros and poles of f play an essential role in the analysis for $\rho > 1$.

To construct an example where the analogue of (1.3) fails at least for some $r > 0$, we consider ρ slightly greater than 1 and for large r_ν replace a single factor $E_1(z/r_\nu)$ of $L_\rho(z)$ by $E_1(z/r_\nu e^{2\pi i/3})$, obtaining a function $\widetilde{L}_\rho(z)$. (Here and throughout $E_1(z) = (1-z)e^z$ is the Weierstrass factor of genus 1.) It is known ([8], [10], [14]) that, for $1 < \rho < 3/2$ and large r, $\left\{\theta : \log\left|L_\rho\left(re^{i\theta}\right)\right| > 0\right\}$ is essentially the interval $\{\theta : |\theta| < (\pi/2)(1 + (\rho-1)/\rho)\}$. It is elementary for all $r > 0$ that $\log\left|L_\rho(re^{i\theta})\right|$ is an even function of θ, and it is also elementary that $\log\left|L_\rho(re^{i\theta})\right|$ is decreasing for $\pi/2 < \theta < \pi$ (a consequence, for example, of (2.20) below). From these observations and known properties of $L_\rho(re^{i\theta})$ (see [8, last three lines of page 489]), we conclude that while there may exist a small set of values of θ near $\theta = 0$ for which

$\log\left|L_\rho(re^{i\theta})\right| < 0$, for all large r there exists a small $\delta = \delta_r > 0$ such that

$$T(r,f) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|L_\rho(re^{i\theta})\right| d\theta. \tag{1.4}$$

We may use (2.12) below with $\gamma = 0$ and $\delta = \delta_{r_\nu/2}$ to compare the integrals over $\left[-\frac{\pi}{2}-\delta,\ \frac{\pi}{2}+\delta\right]$ of $\log\left|L_\rho\left(\frac{r_\nu}{2}e^{i\theta}\right)\right|$ and $\log\left|\widetilde{L}_\rho\left(\frac{r_\nu}{2}e^{i\theta}\right)\right|$. Since $\delta > 0$ is small it follows that the terms in (2.12) of the sum on the right corresponding to small even values of m are negligible and in fact that

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|\widetilde{L}_\rho\left(\frac{r_\nu}{2}e^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|L_\rho\left(\frac{r_\nu}{2}e^{i\theta}\right)\right| d\theta > \frac{1}{4}\left(\frac{1}{5^2}\right)\left(\frac{1}{2^5}\right), \tag{1.5}$$

where the dominant contribution to the difference of the integrals arises from the fifth term of the Fourier series in (2.12). (Note that there is no contribution to the left side of (1.5) from the third term of the Fourier series (2.12) since the argument of the newly-introduced zero of $\widetilde{L}_\rho$ is $2\pi/3$.) We conclude from (1.4) and (1.5) that

$$T\left(\frac{r_\nu}{2}, \widetilde{L}_\rho\right) \geq \frac{1}{2\pi}\int_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|\widetilde{L}_\rho\left(\frac{r_\nu}{2}e^{i\theta}\right)\right| d\theta > T\left(\frac{r_\nu}{2}, L_\rho\right),$$

as desired.

It is worth noting that $T(r_\nu/2, L_\rho)$ is increased by changing the argument of one of the zeros of L_ρ by a substantial amount, namely $2\pi/3$. We shall refer to zeros and poles of a meromorphic function as outliers if their arguments differ by a substantial amount (specified in (3.24) and (3.38)) from the arguments of the majority of zeros and poles of comparable modulus. In the proof of the Theorem, we first modify f to obtain a new meromorphic function F by replacing outlying zeros and poles (such as $r_\nu e^{2\pi i/3}$ in the case of $\widetilde{L}_\rho$) by zeros and poles of the same modulus that are not outliers. The above example shows for the resulting modified function F that we do not in general have $T(r,f) \leq T(r,F)$ for all $r > 0$; however, we show (see (3.47)) that $T(r,f)$ is dominated to within an acceptably small error by $T(r,F)$ in a suitable average sense.

For the function F, which has no outlying zeros or poles, we show (see (3.90)) that the characteristic function is increased for all large r to within a very small error by replacing all zeros and poles of F by zeros of the same modulus on a single ray through the origin, completing the proof. This latter argument is based on certain monotonicity properties of $\log\left|E_1(re^{i\theta})\right|$, $-\pi \leq \theta \leq \pi$. (See Lemmas 4 through 9.)

We establish the Theorem for $\rho_0 = 1 + 10^{-9}$. A refinement of our argument could well yield a larger ρ_0; we make no attempt to determine the best ρ_0 our method can produce. It is clear, however, that our approach cannot succeed unless $\rho - 1 > 0$ is small. Our argument depends heavily on the fact that if f were a possible counterexample to the Theorem, then the Fourier series of $\log\left|f\left(re^{i\theta}\right)\right|$ would be dominated by the two terms $c_1(r,f)e^{i\theta}$ and $c_{-1}(r,f)e^{-i\theta}$, and a continuous argument of $c_1(r,f)$ would vary quite slowly with r. (See Lemma 11.) These considerations apply only when $\rho - 1 > 0$ is small.

Our proof can be extended to obtain the same conclusion under the more general hypothesis that $N(r) \sim r^\rho \sigma(r)$ for any slowly varying function $\sigma(r)$. To avoid excessive complications in the exposition, we provide complete details only under the simpler assumption that $N(r) \sim cr^\rho$.

2. Preliminaries

We suppose $1 < \rho < \rho_0 = 1 + 10^{-9}$ and assume, as we may, that $c = 1/\rho$ in the statement of the Theorem. Routine arguments [3], based only on the monotonicity of $n(r)$, then show that $n(r) \sim r^\rho$. For $\tilde{\varepsilon} > 0$, let $R_0 = R_0(\tilde{\varepsilon})$ be such that for $r > R_0$,

$$\left|\rho \frac{N(r)}{r^\rho} - 1\right| < \tilde{\varepsilon} \tag{2.1}$$

and

$$\left|\frac{n(r)}{r^\rho} - 1\right| < \tilde{\varepsilon}. \tag{2.2}$$

In addition, we assume for later convenience that

$$R_0 > \left(\frac{2}{\tilde{\varepsilon}}\right)^{\left(\frac{1}{\rho-1}\right)}. \tag{2.3}$$

Suppose

$$g(z) = \frac{\prod_\nu E_1\left(\dfrac{z}{z_\nu}\right)}{\prod_\nu E_1\left(\dfrac{z}{w_\nu}\right)} \tag{2.4}$$

is a quotient of convergent Weierstrass products of genus 1. In the course of our proof we apply results established below for g to two different functions of the above form, namely the function f of the Theorem and a modified function F obtained from f by altering the arguments (but not the moduli) of some of the zeros and poles of f. Because the counting functions $N(r)$ and $n(r)$ are the same for F as for f, we may presume the counting functions of g satisfy (2.1) and (2.2), with the same value of R_0 in the two cases $g = f$ and $g = F$. For $r > 0$ with $|z_\nu| \neq r$ and $|w_\nu| \neq r$ for all ν, we have

$$\log\left|g\left(re^{i\theta}\right)\right| = \sum_{m=-\infty}^{\infty} c_m(r,g)e^{im\theta}, \tag{2.5}$$

where [11], since g has genus 1,

$$c_0(r,g) = N(r,0,g) - N(r,\infty,g),$$
$$c_1(r,g) = \frac{1}{2}\sum_{|z_\nu|<r}\left(\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right) - \frac{1}{2}\sum_{|w_\nu|<r}\left(\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right),$$

$$c_m(r,g) = -\frac{1}{2m}\sum_{|z_\nu|\le r}\left(\frac{\overline{z_\nu}}{r}\right)^m - \frac{1}{2m}\sum_{|z_\nu|>r}\left(\frac{r}{z_\nu}\right)^m + \frac{1}{2m}\sum_{|w_\nu|\le r}\left(\frac{\overline{w_\nu}}{r}\right)^m + \frac{1}{2m}\sum_{|w_\nu|>r}\left(\frac{r}{w_\nu}\right)^m,\ m\ge 2, \tag{2.6}$$

and

$$c_{-m}(r,g) = \overline{c_m(r,g)},\ m \ge 1.$$

We note for each m that $c_m(r,g)$ is a continuous function of r. In view of the fact that the set of values of r for which the Fourier series (2.5) does not converge uniformly is countable and hence negligible for our purposes, for ease of exposition we treat (2.5) as valid for all $r > 0$. Analysis of the logarithm of the modulus of a meromorphic function via its Fourier series originated with F. Nevanlinna [13].

Lemma 1. *Suppose the meromorphic function g is given by (2.4). Suppose for small $\tilde\varepsilon > 0$ there is an associated $R_0 = R_0(\tilde\varepsilon)$ such that (2.1), (2.2), and (2.3) are satisfied where $N(r)$ and $n(r)$ are the counting functions of g. Let*

$$R_1 = \frac{R_0}{(\tilde\varepsilon)^{\frac{1}{2+\rho}}}.$$

If $r > R_1$, we have

$$\log\left|g\left(re^{i\theta}\right)\right| = \sum_{m=-\infty}^{\infty} c_m(r,g)e^{im\theta} = g_a\left(re^{i\theta}\right) + g_b\left(re^{i\theta}\right),$$

where

$$g_a\left(re^{i\theta}\right) = N(r,0,g) - N(r,\infty,g) + c_1(r,g)e^{i\theta} + c_{-1}(r,g)e^{-i\theta} + \sum_{|m|\ge 2} a_m(r)e^{im\theta}$$

with $a_{-m}(r) = \overline{a_m(r)}$ and

$$|a_m(r)| \le (1+3\tilde\varepsilon)\,\frac{\rho^2 N(r)}{m^2-\rho^2},\quad m\ge 2, \tag{2.7}$$

and

$$g_b\left(re^{i\theta}\right) = \sum_{|m|\ge 2} b_m(r)e^{im\theta}$$

with $b_{-m}(r) = \overline{b_m(r)}$ and

$$|b_m(r)| \le \frac{6\tilde\varepsilon\rho N(r)}{m},\quad m \ge 2.$$

In particular,

$$\|g_b\|_2 \le 12\,\tilde\varepsilon N(r). \tag{2.8}$$

Proof. We make the following elementary observations:

(i) if $t > r$, then

$$n(t) < (1+\tilde\varepsilon)\,t^\rho < \frac{1+\tilde\varepsilon}{1-\tilde\varepsilon}\,n(r)\left(\frac{t}{r}\right)^\rho < (1+3\tilde\varepsilon)\;n(r)\left(\frac{t}{r}\right)^\rho; \tag{2.9}$$

(ii) if $R_0 < t < r$, then

$$n(t) > (1-\tilde{\varepsilon})t^\rho > \frac{1-\tilde{\varepsilon}}{1+\tilde{\varepsilon}}\,n(r)\left(\frac{t}{r}\right)^\rho > (1-3\tilde{\varepsilon})\,n(r)\left(\frac{t}{r}\right)^\rho, \tag{2.10}$$

and

(iii)

$$n(r) < (1+\tilde{\varepsilon})r^\rho < \frac{1+\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\rho\,N(r). \tag{2.11}$$

Upon applying integration by parts as well as (2.6), (2.9), (2.10), and (2.11), we have for all $m \ge 2$ and all $r > R_1$ that

$$\begin{aligned}
|c_m(r,g)| &\le \frac{1}{2m}\int_0^r \left(\frac{t}{r}\right)^m dn(t) + \frac{1}{2m}\int_r^\infty \left(\frac{r}{t}\right)^m dn(t)\\
&= -\frac{1}{2}\int_0^{R_0}\left(\frac{t}{r}\right)^m \frac{n(t)}{t}\,dt - \frac{1}{2}\int_{R_0}^r \left(\frac{t}{r}\right)^m \frac{n(t)}{t}\,dt + \frac{1}{2}\int_r^\infty \left(\frac{r}{t}\right)^m \frac{n(t)}{t}\,dt\\
&< \frac{-(1-3\tilde{\varepsilon})\,n(r)}{2}\int_{R_0}^r \left(\frac{t}{r}\right)^{m+\rho}\frac{dt}{t} + \frac{(1+3\tilde{\varepsilon})\,n(r)}{2}\int_r^\infty \left(\frac{r}{t}\right)^{m-\rho}\frac{dt}{t}\\
&\le \frac{(1-3\tilde{\varepsilon})\,n(r)}{2(m+\rho)}\left(\frac{R_0}{r}\right)^{m+\rho} + \frac{n(r)}{2}\left(\frac{1}{m-\rho} - \frac{1}{m+\rho}\right)\\
&\quad + \frac{3\tilde{\varepsilon}}{2}\,n(r)\left(\frac{1}{m+\rho} + \frac{1}{m-\rho}\right)\\
&< \frac{\tilde{\varepsilon}n(r)}{2(m+\rho)} + \frac{\rho n(r)}{m^2-\rho^2} + \frac{3m\tilde{\varepsilon}n(r)}{m^2-\rho^2} < \frac{\rho n(r)}{m^2-\rho^2} + \frac{5\tilde{\varepsilon}n(r)}{m}\\
&< \frac{1+\tilde{\varepsilon}}{1-\tilde{\varepsilon}}\,N(r)\left(\frac{\rho^2}{m^2-\rho^2} + \frac{5\tilde{\varepsilon}\rho}{m}\right) < (1+3\tilde{\varepsilon})\frac{\rho^2 N(r)}{m^2-\rho^2} + \frac{6\tilde{\varepsilon}\rho N(r)}{m}.
\end{aligned}$$

Thus for $|m| \ge 2$ we may write $c_m(r,g) = a_m(r) + b_m(r)$ where $a_m(r)$ and $b_m(r)$ satisfy the required inequalities. We note that g_a and g_b are real. □

We observe that near a zero or pole of g, $\log\left|g(re^{i\theta})\right|$ is unbounded and, in view of (2.7), evidently $g_b(re^{i\theta})$ makes a dominant contribution to $\log\left|g(re^{i\theta})\right|$; nevertheless, $\left|g_b(re^{i\theta})\right|$ is small in the sense of (2.8).

Lemma 2. *For $\gamma \in [-\pi,\pi]$, $\alpha \in [-\pi,\pi]$, $0 \le \delta < \frac{\pi}{2}$, $r > 0$, and $\tilde{r} > 0$, let*

$$I(\gamma,\alpha,\delta,r,\tilde{r}) = \frac{1}{2\pi}\int\limits_{\gamma-\frac{\pi}{2}-\delta}^{\gamma+\frac{\pi}{2}+\delta} \log\left|E_1\!\left(\frac{re^{i\theta}}{\tilde{r}e^{i\alpha}}\right)\right| d\theta.$$

Then for $r < \tilde{r}$

$$I(\gamma,\alpha,\delta,r,\tilde{r}) = -\frac{1}{\pi}\sum_{m=2}^\infty \frac{1}{m^2}\left(\frac{r}{\tilde{r}}\right)^m \cos m(\gamma-\alpha)\left(\sin\frac{m\pi}{2}\cos m\delta + \cos\frac{m\pi}{2}\sin m\delta\right)$$

$$= \frac{1}{4\pi}\left(\frac{r}{\tilde r}\right)^2 \cos 2(\gamma-\alpha)\sin 2\delta + \frac{1}{9\pi}\left(\frac{r}{\tilde r}\right)^3 \cos 3(\gamma-\alpha)\cos\, 3\delta$$

$$- \frac{1}{\pi}\sum_{m=4}^{\infty}\frac{1}{m^2}\left(\frac{r}{\tilde r}\right)^m \cos m(\gamma-\alpha)\left(\sin\frac{m\pi}{2}\cos m\delta + \cos\frac{m\pi}{2}\sin m\delta\right), \tag{2.12}$$

and for $r > \tilde r$

$$I(\gamma,\alpha,\delta,r,\tilde r) = \left(\frac{1}{2}+\frac{\delta}{\pi}\right)\log\frac{r}{\tilde r} + \frac{1}{\pi}\left(\frac{r}{\tilde r}\right)\cos(\gamma-\alpha)\cos\delta$$

$$- \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{\tilde r}{r}\right)^m \cos m(\gamma-\alpha)\left(\sin\frac{m\pi}{2}\cos m\delta + \cos\frac{m\pi}{2}\sin m\delta\right). \tag{2.13}$$

Proof. We apply (2.6) to $E_1\Big(\frac{z}{\tilde r e^{i\alpha}}\Big)$. If $r < \tilde r$, then

$$\log\left|E_1\left(\frac{re^{i\theta}}{\tilde r e^{i\alpha}}\right)\right| = -\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{r}{\tilde r}\right)^m \cos m(\theta-\alpha).$$

If $r > \tilde r$, then

$$\log\left|E_1\left(\frac{re^{i\theta}}{\tilde r e^{i\alpha}}\right)\right| = \log\frac{r}{\tilde r} + \frac{r}{\tilde r}\cos(\theta-\alpha) - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\tilde r}{r}\right)^m \cos m(\theta-\alpha).$$

Term-by-term integration now establishes the lemma. □

Lemma 3. *Suppose g, $\tilde\varepsilon > 0$, and R_0 are as in Lemma 1. Suppose $R_2 \geq R_0$. For $r > R_2$, set*

$$Q_r = \frac{1}{2}\sum_{R_2<|z_\nu|<r}\left|\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right| + \frac{1}{2}\sum_{R_2<|w_\nu|<r}\left|\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right|$$

and

$$P_r = \frac{1}{2}\int_0^r \left(\frac{r}{t} - \frac{t}{r}\right)\rho t^{\rho-1}\,dt = \frac{\rho r^\rho}{\rho^2-1}. \tag{2.14}$$

If $r > R_2$, we have

$$\text{(i)} \qquad |Q_r| < (1+\tilde\varepsilon)P_r\left(1 - \left(\frac{R_2}{r}\right)^{\rho-1}\right)$$

and

$$\text{(ii)} \qquad c_1(r, L_\rho) > \frac{P_r}{1+\tilde\varepsilon},$$

where L_ρ is the Lindelöf function of order ρ.

Proof. We observe for $r > R_2$ that

$$\begin{aligned} J &:= \frac{1}{2}\int_{R_2}^{r}\left(\frac{r}{t}-\frac{t}{r}\right)\rho t^{\rho-1}\,dt \\ &= \frac{\rho}{2}\left(\frac{r^\rho}{\rho-1}-\frac{r^\rho}{\rho+1}-\frac{r^\rho}{\rho-1}\left(\frac{R_2}{r}\right)^{\rho-1}+\frac{r^\rho}{\rho+1}\left(\frac{R_2}{r}\right)^{\rho+1}\right) \\ &\le \frac{\rho}{2}\left(\frac{r^\rho}{\rho-1}-\frac{r^\rho}{\rho+1}-\frac{r^\rho}{\rho-1}\left(\frac{R_2}{r}\right)^{\rho-1}+\frac{r^\rho}{\rho+1}\left(\frac{R_2}{r}\right)^{\rho-1}\right) \\ &= P_r\left(1-\left(\frac{R_2}{r}\right)^{\rho-1}\right). \end{aligned} \tag{2.15}$$

We further observe from integration by parts that

$$J = \frac{1}{2}\left(\frac{R_2}{r}-\frac{r}{R_2}\right)R_2^\rho + \frac{1}{2}\int_{R_2}^{r}\left(\frac{r}{t}+\frac{t}{r}\right)\frac{t^\rho}{t}\,dt. \tag{2.16}$$

From (2.2), (2.15), and (2.16) we have

$$\begin{aligned} |Q_r| &= \frac{1}{2}\int_{R_2}^{r}\left(\frac{r}{t}-\frac{t}{r}\right)dn(t) \\ &= \frac{1}{2}\left(\frac{R_2}{r}-\frac{r}{R_2}\right)n(R_2)+\frac{1}{2}\int_{R_2}^{r}\left(\frac{r}{t}+\frac{t}{r}\right)\frac{n(t)}{t}\,dt \\ &\le \frac{1}{2}\left(\frac{R_2}{r}-\frac{r}{R_2}\right)\left(n(R_2)-(1+\tilde{\varepsilon})\,R_2^\rho\right) \\ &\quad+\frac{1}{2}\left(\frac{R_2}{r}-\frac{r}{R_2}\right)(1+\tilde{\varepsilon})R_2^\rho+\frac{(1+\tilde{\varepsilon})}{2}\int_{R_2}^{r}\left(\frac{r}{t}+\frac{t}{r}\right)\frac{t^\rho}{t}\,dt \\ &< (1+\tilde{\varepsilon})J < (1+\tilde{\varepsilon})P_r\left(1-\left(\frac{R_2}{r}\right)^{\rho-1}\right), \end{aligned}$$

establishing (i) for all $r > R_2$.

Using (2.6), integration by parts, and the fact that $0 \le t^\rho - [t^\rho] < 1$ for all t, for $r > 1$ we have

$$\begin{aligned} P_r - c_1(r, L_\rho) &= \frac{1}{2}\int_{0}^{r}\left(\frac{r}{t}-\frac{t}{r}\right)d(t^\rho-[t^\rho]) \\ &= \frac{1}{2}\int_{0}^{1}\left(\frac{r}{t}-\frac{t}{r}\right)\rho t^{\rho-1}\,dt+\frac{1}{2}\int_{1}^{r}\left(\frac{r}{t}-\frac{t}{r}\right)d(t^\rho-[t^\rho]) \\ &\le \frac{\rho r}{2(\rho-1)}+\frac{1}{2}\int_{1}^{r}\left(\frac{r}{t}+\frac{t}{r}\right)\frac{dt}{t} \\ &= \frac{\rho r}{2(\rho-1)}+\frac{1}{2}\left(r-\frac{1}{r}\right) < \frac{3\tilde{\varepsilon}P_r}{4}, \end{aligned}$$

where the last inequality follows from (2.3) and (2.14). This establishes (ii). □

We next prove a sequence of lemmas leading to Lemma 9, a result critical to our analysis of the characteristic of F, the function obtained from the original f by replacing the outlying zeros and poles of f by zeros and poles that are not outliers.

Using Lemma 9, we show that because F has no outlying zeros or poles, its characteristic is effectively dominated for all large r by the characteristic of a canonical product with zeros on a ray through the origin at the moduli of the zeros and poles of F. (See (3.90).)

Suppose $a = |a|\, e^{i\alpha} \neq 0$. Let

$$\varphi(z) = E_1\left(\frac{z}{a}\right)$$

and

$$\psi(z) = 1/E_1\left(\frac{z}{-a}\right).$$

Note for all $r > 0$ and all θ that

$$\psi(re^{i(\theta+\pi)}) = 1/\varphi(re^{i\theta}),$$

and hence

$$\log\left|\psi(re^{i(\theta+\pi)})\right| = -\log\left|\varphi(re^{i\theta})\right|. \tag{2.17}$$

Lemma 4. *If $\frac{\pi}{4} < |\theta - \alpha| < 3\pi/4$, then for all $r > 0$*

$$\log\left|\varphi(re^{i\theta})\right| + \log\left|\varphi(re^{i(\theta+\pi)})\right| > 0.$$

Proof. Since $|\cos(\theta - \alpha)| < \frac{1}{\sqrt{2}}$, we have

$$\begin{aligned}
2\log\left|\varphi(re^{i\theta})\right| + 2\log\left|\varphi(re^{i(\theta+\pi)})\right| &= \log\left|1 - \frac{r}{|a|}e^{i(\theta-\alpha)}\right|^2 + \log\left|1 + \frac{r}{|a|}e^{i(\theta-\alpha)}\right|^2 \\
&= \log\left(\left(1 + \frac{r^2}{|a|^2}\right)^2 - \frac{4r^2}{|a|^2}\cos^2(\theta - \alpha)\right) \\
&\geq \log\left(1 + \frac{r^4}{|a|^4}\right) > 0. \qquad \square
\end{aligned}$$

Lemma 5. *For all $r > 0$,*

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\varphi(re^{i\theta})\right| d\theta = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\psi(re^{i\theta})\right| d\theta + \log^+ \frac{r}{|a|}.$$

Proof. Applying (2.17) and Jensen's Theorem, we have

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\varphi(re^{i\theta})\right| d\theta = -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\psi(re^{i(\theta+\pi)})\right| d\theta$$

$$= -\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\left|\psi(re^{i\theta})\right| d\theta = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\psi(re^{i\theta})\right| d\theta + \log^+ \frac{r}{|a|}. \qquad \square$$

Lemma 6. *Suppose $|\alpha| < \frac{\pi}{6}$ and $0 \le \delta < \frac{\pi}{12}$. Then for all $r > 0$*

$$\frac{1}{2\pi}\int\limits_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|\varphi(re^{i\theta})\right| d\theta \ge \frac{1}{2\pi}\int\limits_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|\psi(re^{i\theta})\right| d\theta + \log^+ \frac{r}{|a|}.$$

Proof. Noting that the $\delta = 0$ case is treated in Lemma 5, we suppose $0 < \delta < \frac{\pi}{12}$. If $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \delta$, then

$$\frac{\pi}{4} < \frac{\pi}{2} - \alpha < \theta - \alpha < \frac{\pi}{2} + \delta + \frac{\pi}{6} < \frac{3\pi}{4}.$$

From (2.17) and Lemma 4 we conclude for such θ that

$$\log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| = \log\left|\varphi(re^{i\theta})\right| + \log\left|\varphi(re^{i(\theta+\pi)})\right| > 0. \tag{2.18}$$

If $-\frac{\pi}{2} - \delta < \theta < -\frac{\pi}{2}$, then

$$-\frac{3\pi}{4} < -\frac{\pi}{2} - \delta - \frac{\pi}{6} < \theta - \alpha < -\frac{\pi}{2} + \frac{\pi}{6} < -\frac{\pi}{4}.$$

From (2.17) and Lemma 4 we conclude that (2.18) holds for such θ as well. Lemma 5 combined with (2.18) now yields Lemma 6. □

Lemma 7. *Suppose $|\gamma - \alpha| < \frac{\pi}{6}$ and $0 \le \delta < \frac{\pi}{12}$. Let $A = [\gamma - \frac{\pi}{2} - \delta,\ \gamma + \frac{\pi}{2} + \delta]$ and $B = [\gamma - \frac{\pi}{2} - \delta,\ \gamma + \frac{3\pi}{2} - \delta] - A$. Then for all $r > 0$*

$$\frac{1}{2\pi}\int\limits_A \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta \ge \frac{1}{2\pi}\int\limits_B \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta.$$

Proof. Noting that the zero of $\varphi(e^{i\gamma}z)$ has argument $\alpha - \gamma$ and that $\psi(e^{i\gamma}z) = 1/\varphi(-e^{i\gamma}z)$, we apply Lemma 6 to conclude

$$\begin{aligned}\frac{1}{2\pi}\int\limits_A \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta &= \frac{1}{2\pi}\int\limits_{-\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \log\left|\varphi(re^{i(\theta+\gamma)})\right| - \log\left|\psi(re^{i(\theta+\gamma)})\right| d\theta \\ &\ge \log^+ \frac{r}{|a|}.\end{aligned}$$

From Jensen's Theorem we have

$$\frac{1}{2\pi}\int\limits_{\gamma-\frac{\pi}{2}-\delta}^{\gamma+\frac{3\pi}{2}-\delta} \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta = 2\log^+ \frac{r}{|a|}.$$

Subtraction yields

$$\frac{1}{2\pi}\int\limits_B \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta \le \log^+ \frac{r}{|a|} \le \frac{1}{2\pi}\int\limits_A \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta,$$

establishing Lemma 7. □

For $r > 0$, let

$$u_r(\theta) = \log\left|E_1(re^{i\theta})\right| = \frac{1}{2}\left(\log(1 - 2r\cos\theta + r^2) + 2r\cos\theta\right).$$

As we are concerned with the set where the logarithm of the modulus of a quotient of convergent Weierstrass products of genus 1 is positive, we analyze the behavior of $u_r(\theta)$ in considerable detail.

Since

$$u_r'(\theta) = r^2 \sin\theta \left(\frac{2\cos\theta - r}{1 - 2r\cos\theta + r^2} \right), \tag{2.19}$$

we observe for all $r > 0$ that

$$\begin{aligned} &u_r'(\theta) < 0, \ \frac{\pi}{2} < \theta < \pi, \text{ and} \\ &u_r'(\theta) > 0, \ -\pi < \theta < -\frac{\pi}{2}. \end{aligned} \tag{2.20}$$

From (2.19) it is clear that $u_r(\theta)$ is a decreasing function of θ for $0 < \theta < \pi$ if $r > 2$. Since $u_r(\theta)$ is even for all $r > 0$ we conclude that if $r > 2$, I is an interval of length less than 2π, and α is any real number, then

$$\int_I u_r(\theta - \alpha)\, d\theta \le \int_I u_r(\theta - \gamma)\, d\theta$$

where γ is the midpoint of I. Equivalently, with I as above, $s > 0$, and $2|z_\nu| < s$, for all real α we have

$$\int_I \log \left| E_1\!\left(\frac{se^{i\theta}}{|z_\nu| e^{i\alpha}} \right) \right| d\theta \le \int_I \log \left| E_1\!\left(\frac{se^{i\theta}}{|z_\nu| e^{i\gamma}} \right) \right| d\theta \tag{2.21}$$

if γ is the midpoint of I.

Set $U_r(\theta) = u_r(\theta) - u_r(\theta + \pi)$. We note that $U_r\left(-\frac{\pi}{2}\right) = 0$ since u_r is even. We have

$$U_r'(\theta) = u_r'(\theta) - u_r'(\theta + \pi) = \frac{2r^3 \sin\theta \,(4\cos^2\theta - (1 + r^2))}{(1 - 2r\cos\theta + r^2)(1 + 2r\cos\theta + r^2)}.$$

Thus $U_r'(\theta) = 0$ if $\theta = 0$ or if $\cos^2\theta = \frac{1+r^2}{4}$. Since $U_r'\left(-\frac{\pi}{2}\right) > 0$ and $U_r'(\theta)$ is of constant sign on $\left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right)$, we conclude that U_r is increasing on $\left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right)$ and thus

$$\begin{aligned} &U_r(\theta) < 0, \ -\frac{2\pi}{3} < \theta < -\frac{\pi}{2}, \text{ and} \\ &U_r(\theta) > 0, \ -\frac{\pi}{2} < \theta < -\frac{\pi}{3}. \end{aligned} \tag{2.22}$$

Lemma 8. *Suppose $0 \le \delta < \frac{\pi}{6}$. For $r > 0$, the only solution of*

$$u_r(\theta + \pi + 2\delta) = u_r(\theta) \tag{2.23}$$

in $\left(-\frac{2\pi}{3} - 2\delta, -\frac{\pi}{3}\right)$ occurs at $\theta = -\frac{\pi}{2} - \delta$. In fact we have

$$u_r(\theta + \pi + 2\delta) - u_r(\theta) > 0, \ -\frac{2\pi}{3} - 2\delta < \theta < -\frac{\pi}{2} - \delta,$$

and

$$u_r(\theta + \pi + 2\delta) - u_r(\theta) < 0, \ -\frac{\pi}{2} - \delta < \theta < -\frac{\pi}{3}.$$

Proof. Noting that the case $\delta = 0$ is a restatement of (2.22), we suppose $0 < \delta < \frac{\pi}{6}$. We first observe that $\theta = -\frac{\pi}{2} - \delta$ is a solution of (2.23) since u_r is an even function.

We consider the interval $-\frac{\pi}{2} - \delta < \theta \le -\frac{\pi}{2}$. We have

$$\frac{\pi}{2} < \frac{\pi}{2} + \delta < \theta + \pi + 2\delta \le \frac{\pi}{2} + 2\delta < \pi,$$

and conclude from (2.20) that on this interval $u_r(\theta)$ is increasing and $u_r(\theta + \pi + 2\delta)$ is decreasing, implying

$$u_r(\theta + \pi + 2\delta) - u_r(\theta) < 0, \quad -\frac{\pi}{2} - \delta < \theta \le -\frac{\pi}{2}. \tag{2.24}$$

Now suppose $-\frac{\pi}{2} < \theta < -\frac{\pi}{3}$. From (2.22) we have

$$u_r(\theta) - u_r(\theta + \pi) = U_r(\theta) > 0.$$

We also have

$$\frac{\pi}{2} < \theta + \pi < \theta + \pi + 2\delta < \frac{2\pi}{3} + 2\delta < \pi,$$

which in conjunction with (2.20) implies

$$u_r(\theta + \pi) > u_r(\theta + \pi + 2\delta).$$

Combining the last two observations with (2.24), we conclude

$$u_r(\theta + \pi + 2\delta) - u_r(\theta) < 0, \quad -\frac{\pi}{2} - \delta < \theta < -\frac{\pi}{3}. \tag{2.25}$$

We next consider θ in $\left(-\frac{2\pi}{3} - 2\delta, -\frac{\pi}{2} - \delta\right)$. We have

$$-\frac{\pi}{2} - \delta < -\theta - \pi - 2\delta < -\frac{\pi}{3}.$$

From (2.25) we conclude

$$u_r(-\theta) < u_r(-\theta - \pi - 2\delta).$$

Since u_r is even we have

$$u_r(\theta) < u_r(\theta + \pi + 2\delta), \quad -\frac{2\pi}{3} - 2\delta < \theta < -\frac{\pi}{2} - \delta,$$

which combined with (2.25) yields Lemma 8. $\square$

The following lemma plays a central role in establishing inequality (3.90). Suppose I is an interval of length somewhat greater than π and $z_\nu = |z_\nu|e^{i\alpha}$ where α is close to the midpoint of I. Lemma 9 asserts for any $r > 0$ that the integral over I of the logarithm of the modulus on $|z| = r$ of the Weierstrass factor of genus 1 with zero at z_ν is maximized by choosing α to be the midpoint of I. Note that the example in the Introduction shows that this conclusion does not hold in general if α is permitted to stray as far as $2\pi/3$ from the midpoint of I. Note also that (2.21) asserts that if $|z_\nu| < r/2$, then choosing α to be the midpoint of I is extremal for all α, not just for those α close to the midpoint of I.

Lemma 9. *Suppose* $0 \le \delta < \frac{\pi}{6}$, $\gamma \in [-\pi, \pi]$, $|\alpha - \gamma| < \frac{\pi}{6} + \delta$, $z_\nu = |z_\nu|e^{i\alpha}$, *and* $r > 0$. *Let*

$$A(\alpha) = \int_{\gamma - \frac{\pi}{2} - \delta}^{\gamma + \frac{\pi}{2} + \delta} u_r(\theta - \alpha)\, d\theta = \int_{\gamma - \frac{\pi}{2} - \delta}^{\gamma + \frac{\pi}{2} + \delta} \log \left| E_1\!\left(\frac{r|z_\nu|e^{i\theta}}{z_\nu}\right) \right| d\theta.$$

Then $A(\alpha) \le A(\gamma)$.

Proof. We have

$$A'(\alpha) = u_r\Big(\gamma - \frac{\pi}{2} - \delta - \alpha\Big) - u_r\Big(\gamma + \frac{\pi}{2} + \delta - \alpha\Big).$$

If $0 < \alpha - \gamma < \frac{\pi}{6} + \delta$, then

$$-\frac{2\pi}{3} - 2\delta < \gamma - \frac{\pi}{2} - \delta - \alpha < -\frac{\pi}{2} - \delta,$$

and we deduce from the first conclusion of Lemma 8 that $A'(\alpha) < 0$. If $-\frac{\pi}{6} - \delta < \alpha - \gamma < 0$, then

$$-\frac{\pi}{2} - \delta < \gamma - \frac{\pi}{2} - \delta - \alpha < -\frac{\pi}{3},$$

and the second conclusion of Lemma 8 implies $A'(\alpha) > 0$. The first derivative test now establishes Lemma 9. □

We conclude this section with the following elementary fact.

Lemma 10. *Suppose V and W are integrable real-valued functions on an interval I,*

$$\{\theta \in I : V(\theta) > 0\} \subset A \subset \{\theta \in I : V(\theta) \geq 0\}, \quad B = I - A,$$

and

$$\int_A W(\theta)\, d\theta \geq \int_B W(\theta)\, d\theta.$$

Then $\|V\|_1 \leq \|V + W\|_1$.

Proof. We have

$$\begin{aligned}\int_I V^+(\theta)\, d\theta = \int_A V(\theta)\, d\theta &\leq \int_A V(\theta) + W(\theta)\, d\theta - \int_B W(\theta)\, d\theta \\ &\leq \int_I (V(\theta) + W(\theta))^+\, d\theta - \int_B W(\theta)\, d\theta,\end{aligned}$$

as well as

$$\begin{aligned}\int_I V^-(\theta)\, d\theta = -\int_B V(\theta)\, d\theta &= -\int_B V(\theta) + W(\theta)\, d\theta + \int_B W(\theta)\, d\theta \\ &\leq \int_I (V(\theta) + W(\theta))^-\, d\theta + \int_B W(\theta)\, d\theta.\end{aligned}$$

Adding these inequalities establishes Lemma 10. □

3. Proof of the Theorem

By the Hadamard Factorization Theorem we may suppose f has the form of g in (2.4). We suppose there exists a small $\varepsilon > 0$ and large r_0 such that

$$\frac{N(r)}{T(r,f)} < \frac{k_L(\rho)}{1+\varepsilon}, \quad r > r_0, \tag{3.1}$$

and seek a contradiction. We suppose, as we may, that

$$\frac{N(r,0,L)}{T(r,L)} > \frac{k_L(\rho)}{1+\frac{\varepsilon}{2}}, \quad r > r_0, \tag{3.2}$$

where L is either the Lindelöf function L_ρ of order ρ, or, in view of (1.1),

$$L(z) = \prod_\nu E_1\left(\frac{z}{|z_\nu|\, e^{i\gamma}}\right) \cdot \prod_\nu E_1\left(\frac{z}{|w_\nu|\, e^{i\gamma}}\right)$$

for some real γ. (Here of course z_ν and w_ν are the zeros and poles of f.)

Without exception we consider only $\tilde\varepsilon > 0$ satisfying

$$\text{(3.3)}\qquad \begin{array}{ll} \text{(i)} & 2\tilde\varepsilon < k_L(\rho) < |\sin \pi\rho| < \pi(\rho-1) < 10^{-9}\pi, \text{ and} \\ \text{(ii)} & (1+2\tilde\varepsilon)^2\left(1+\dfrac{9\varepsilon\rho}{10} + 13\tilde\varepsilon(1+\tilde\varepsilon)\rho\right) < 1+\varepsilon. \end{array}$$

To such $\tilde\varepsilon > 0$ we associate a quantity $R_0(\tilde\varepsilon)$ with (2.1), (2.2), and (2.3) satisfied where $N(r)$ and $n(r)$ are the counting functions of f, as well as a quantity $R_1(\tilde\varepsilon) > R_0(\tilde\varepsilon)$ as in Lemma 1. In the proof we frequently use without comment the fact that $k_L(\rho)$ is small.

It is critical to our argument for the values of ρ under consideration that the dominant terms of the Fourier series of $\log\left|f(re^{i\theta})\right|$ are $c_1(r,f)e^{i\theta}$ and $c_{-1}(r,f)e^{-i\theta}$, and that a continuous argument of $c_1(r,f)$ varies quite slowly. To this end we prove

Lemma 11. *Suppose $\tilde\varepsilon > 0$. Let $\beta(r,f)$ be a continuous argument of $c_1(r,f)$. If $R_3 > \max(R_1, r_0)$ is such that*

$$\frac{1}{2}\sum_{|z_\nu|\le R_0}\frac{1}{|z_\nu|} + \frac{1}{2}\sum_{|w_\nu|\le R_0}\frac{1}{|w_\nu|} < \tilde\varepsilon R_3^{\rho-1},$$

then

$$|\beta(r,f) - \beta(R_a,f)| < \frac{\pi}{150}$$

if

$$R_3 < R_a \le r < R_a\left(\frac{10^{-4}}{3k_L(\rho)}\right)^{\left(\frac{1}{\rho-1}\right)}.$$

Proof. For $1 \le p \le \infty$, let $m_p(r,f)$ denote the L^p norm of $\log\left|f(re^{i\theta})\right|$. For $r > R_3$ we have by (3.1) that

$$\text{(3.4)}\qquad \frac{N(r)}{m_1(r,f)+N(r)} = \frac{N(r)}{2T(r,f)} < \frac{k_L(\rho)}{2(1+\varepsilon)},$$

or

$$m_1(r,f) > N(r)\left(\frac{2(1+\varepsilon)}{k_L(\rho)} - 1\right).$$

We write

$$\text{(3.5)}\qquad \log\left|f(re^{i\theta})\right| = c_1(r,f)e^{i\theta} + c_{-1}(r,f)e^{-i\theta} + q(re^{i\theta}),$$

and conclude from Lemma 1 applied with $g = f$ that

$$\text{(3.6)}\quad \|q\|_1 \le \|q\|_2 \le N(r)\left\{1 + 2(1+3\tilde\varepsilon)^2\sum_{m=2}^{\infty}\left(\frac{\rho^2}{m^2-\rho^2}\right)^2\right\}^{\frac{1}{2}} + 12\tilde\varepsilon N(r) \le 2N(r).$$

It follows that

$$\|2c_1(r,f)\cos(\theta+\beta(r,f))\|_1 \ge \left\|\log\left|f(re^{i\theta})\right|\right\|_1 - \left\|q(re^{i\theta})\right\|_1 > N(r)\left(\frac{2(1+\varepsilon)}{k_L(\rho)} - 3\right),$$

implying

$$(3.7) \qquad |c_1(r,f)| > \frac{\pi N(r)}{4}\left(\frac{2(1+\varepsilon)}{k_L(\rho)} - 3\right).$$

By (3.2) the analogue of (3.4) for L_ρ is

$$\frac{N(r,0,L_\rho)}{m_1(r,L_\rho)+N(r,0,L_\rho)} = \frac{N(r,0,L_\rho)}{2T(r,L_\rho)} > \frac{k_L(\rho)}{2\left(1+\frac{\varepsilon}{2}\right)},$$

and the above argument leads to

$$c_1(r,L_\rho) < \frac{\pi N(r,0,L_\rho)}{4}\left(\frac{2\left(1+\frac{\varepsilon}{2}\right)}{k_L(\rho)} + 1\right).$$

Using (2.1) we conclude for $r > R_3$ that

$$\frac{|c_1(r,f)|}{c_1(r,L_\rho)} > \frac{N(r)}{N(r,0,L_\rho)}\left(\frac{\dfrac{2(1+\varepsilon)}{k_L(\rho)} - 3}{\dfrac{2\left(1+\frac{\varepsilon}{2}\right)}{k_L(\rho)} + 1}\right) \geq (1-\tilde{\varepsilon})\left(1-2k_L(\rho)\right) > 1 - 3k_L(\rho),$$

where in the last inequality we have used (3.3i).

Let

$$M = \frac{1}{3k_L(\rho)} > \frac{1}{3\pi(\rho-1)} > 10^8.$$

We note that

$$(3.8) \qquad \frac{10^4}{M} = (10^4)\,3k_L(\rho) < 3\pi(10^4)(\rho-1) < 3\pi(10^{-5}) < 10^{-4}.$$

From above we have

$$(3.9) \qquad \frac{|c_1(r,f)|}{c_1(r,L_\rho)} > 1 - 3k_L(\rho) = 1 - \frac{1}{M}, \quad r > R_3,$$

and from (3.3i)

$$(3.10) \qquad \tilde{\varepsilon} < \frac{1}{6M}.$$

For r and R_a under consideration we further note from (3.8) that

$$(3.11) \qquad \left(\frac{R_a}{r}\right)^{\rho-1} > \frac{10^4}{M}.$$

We let

$$A_r = \frac{1}{2}\sum_{|z_\nu|\leq R_a}\left(\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right) - \frac{1}{2}\sum_{|w_\nu|\leq R_a}\left(\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right)$$

and

$$B_r = \frac{1}{2}\sum_{R_a<|z_\nu|\leq r}\left(\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right) - \frac{1}{2}\sum_{R_a<|w_\nu|\leq r}\left(\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right).$$

Since

$$A_r = \frac{1}{2}\sum_{|z_\nu|\le R_0}\left(\frac{r}{z_\nu}-\frac{\overline{z_\nu}}{r}\right)-\frac{1}{2}\sum_{|w_\nu|\le R_0}\left(\frac{r}{w_\nu}-\frac{\overline{w_\nu}}{r}\right)$$
$$+\frac{1}{2}\sum_{R_0<|z_\nu|\le R_a}\left(\frac{r}{z_\nu}-\frac{\overline{z_\nu}}{r}\right)-\frac{1}{2}\sum_{R_0<|w_\nu|\le R_a}\left(\frac{r}{w_\nu}-\frac{\overline{w_\nu}}{r}\right),$$

we have by the choice of R_3 and conclusion (i) of Lemma 3 applied to f with $R_2 = R_0$ that

$$|A_r + B_r| \le |A_r| + |B_r| < \tilde{\varepsilon} r\, R_3^{\rho-1} + \frac{1}{2}\sum_{R_0<|z_\nu|\le r}\left|\frac{r}{z_\nu}-\frac{\overline{z_\nu}}{r}\right| + \frac{1}{2}\sum_{R_0<|w_\nu|\le r}\left|\frac{r}{w_\nu}-\frac{\overline{w_\nu}}{r}\right|$$

$$< \tilde{\varepsilon} r\, R_3^{\rho-1} + (1+\tilde{\varepsilon})P_r\left(1-\left(\frac{R_0}{r}\right)^{\rho-1}\right) \tag{3.12}$$
$$< \tilde{\varepsilon} r^{\rho} + (1+\tilde{\varepsilon})P_r < (1+2\tilde{\varepsilon})P_r.$$

By (3.9), (3.10), and conclusion (ii) of Lemma 3 we have

$$\begin{aligned}|A_r + B_r| = |c_1(r,f)| &> \left(1-\frac{1}{M}\right)c_1(r,L_\rho)\\ &> \left(1-\frac{1}{M}\right)\frac{P_r}{1+\tilde{\varepsilon}} > \left(1-\frac{3}{2M}\right)P_r.\end{aligned} \tag{3.13}$$

An application of conclusion (i) of Lemma 3 with $R_2 = R_a$ yields

$$|B_r| < (1+\tilde{\varepsilon})P_r\left(1-\left(\frac{R_a}{r}\right)^{\rho-1}\right). \tag{3.14}$$

From (3.10), (3.11), (3.13), and (3.14) we conclude

$$\begin{aligned}|A_r| &\ge |A_r + B_r| - |B_r|\\ &> P_r\left(1-\frac{3}{2M}\right) - (1+\tilde{\varepsilon})P_r\left(1-\left(\frac{R_a}{r}\right)^{\rho-1}\right)\\ &> P_r\left(1-\frac{3}{2M}-(1+\tilde{\varepsilon})\left(1-\frac{10^4}{M}\right)\right)\\ &> P_r\left(\frac{9998}{M}\right).\end{aligned} \tag{3.15}$$

Let

$$\alpha(r) = \arg\left(A_r + B_r\right) - \arg A_r = \beta(r,f) - \arg A_r$$

for some continuous $\arg A_r$. We have

$$|B_r|^2 = |A_r + B_r - A_r|^2 = \left(|A_r + B_r| - |A_r|\right)^2 + 2\,|A_r|\;|A_r + B_r|\;\left(1-\cos\alpha(r)\right),$$

or

$$\begin{aligned}&(|B_r| + |A_r| - |A_r + B_r|)(|B_r| + |A_r + B_r| - |A_r|)\\ &= 2\,|A_r|\;|A_r + B_r|\,(1-\cos\alpha(r)).\end{aligned} \tag{3.16}$$

From (3.10), (3.12), and (3.13) we have

$$|B_r| + |A_r| - |A_r + B_r| < (1 + 2\tilde{\varepsilon})P_r - \left(1 - \frac{3}{2M}\right) P_r < \frac{2P_r}{M}. \tag{3.17}$$

By (3.14) we have

$$|B_r| + |A_r + B_r| - |A_r| \le 2\,|B_r| < 2(1 + \tilde{\varepsilon})P_r. \tag{3.18}$$

Inserting (3.13), (3.15), (3.17), and (3.18) into (3.16) yields

$$\frac{4(1+\tilde{\varepsilon})}{M} > 2\left(1 - \frac{3}{2M}\right)\left(\frac{9998}{M}\right)(1 - \cos\alpha(r)) > 2\left(\frac{9997}{M}\right)(1 - \cos\alpha(r)).$$

For an appropriate branch of $\alpha(r)$ we have

$$\frac{(\alpha(r))^2}{2} - \frac{(\alpha(r))^4}{4!} < \frac{2(1+\tilde{\varepsilon})}{9997},$$

implying

$$|\alpha(r)| < \frac{2}{99}. \tag{3.19}$$

We rewrite A_r as

$$\begin{aligned} A_r &= \frac{r}{R_a}\left(\frac{1}{2}\sum_{|z_\nu| \le R_a}\left(\frac{R_a}{z_\nu} - \frac{\overline{z_\nu}}{R_a}\right) - \frac{1}{2}\sum_{|w_\nu| \le R_a}\left(\frac{R_a}{w_\nu} - \frac{\overline{w_\nu}}{R_a}\right)\right) \\ &\quad + \frac{r}{R_a}\left(\frac{1}{2}\sum_{|z_\nu| \le R_a}\left(\frac{\overline{z_\nu}}{R_a} - \frac{\overline{z_\nu}R_a}{r^2}\right) - \frac{1}{2}\sum_{|w_\nu| \le R_a}\left(\frac{\overline{w_\nu}}{R_a} - \frac{\overline{w_\nu}R_a}{r^2}\right)\right) \\ &= \frac{r}{R_a}\left(c_1(R_a, f) + X\right), \end{aligned}$$

where by (2.1), (2.2), (3.3i), and (3.7)

$$|X| \le \frac{n(R_a)}{2} < \frac{(1+\tilde{\varepsilon})R_a^\rho}{2} < \frac{(1+\tilde{\varepsilon})\rho N(R_a)}{2(1-\tilde{\varepsilon})} < 2(\rho - 1)\,|c_1(R_a, f)|\,.$$

From the continuity of $\beta(\cdot, f)$ at R_a and (3.19), we first conclude for r slightly greater than R_a that

$$|\arg A_r - \beta(R_a, f)| < 2/98,$$

and in turn conclude from the continuity of $\arg A_r$ and the above estimate on $|X|$ that

$$|\arg A_r - \beta(R_a, f)| \le \sin^{-1}(2(\rho - 1)) < 3(\rho - 1)$$

for all r and R_a under consideration. From (3.19) we conclude

$$|\beta(r, f) - \beta(R_a, f)| \le \frac{2}{99} + 3(10^{-9}) < \frac{\pi}{150},$$

establishing Lemma 11. □

For $r > R_3$ we define the following sets:

$$I_r = \Big[-\beta(r,f) - \frac{\pi}{2},\ -\beta(r,f) + \frac{\pi}{2}\Big),$$

$$L_r = \Big[-\beta(r,f) + \frac{\pi}{2},\ -\beta(r,f) + \frac{3\pi}{2}\Big),$$

$$X_r = \Big\{\theta \in \Big[-\beta(r,f) - \frac{\pi}{2},\ -\beta(r,f) + \frac{3\pi}{2}\Big) : \log\left|f(re^{i\theta})\right| \geq 0\Big\},$$

$$Y_r = \Big\{\theta \in \Big[-\beta(r,f) - \frac{\pi}{2},\ -\beta(r,f) + \frac{3\pi}{2}\Big) : \log\left|f(re^{i\theta})\right| < 0\Big\},$$

$$J_r = \Big[-\beta(r,f) - \frac{\pi}{2} + (k_L(\rho))^{2/3}, -\beta(r,f) + \frac{\pi}{2} - (k_L(\rho))^{2/3}\Big] \cap Y_r,$$

$$K_r = \Big[-\beta(r,f) + \frac{\pi}{2} + (k_L(\rho))^{2/3}, -\beta(r,f) + \frac{3\pi}{2} - (k_L(\rho))^{2/3}\Big] \cap X_r,$$

and

$$\begin{aligned} C_r = {} & \Big[-\beta(r,f) - \frac{\pi}{2},\ -\beta(r,f) - \frac{\pi}{2} + (k_L(\rho))^{2/3}\Big) \\ & \cup \Big(-\beta(r,f) + \frac{\pi}{2} - (k_L(\rho))^{2/3},\ -\beta(r,f) + \frac{\pi}{2} + (k_L(\rho))^{2/3}\Big) \\ & \cup \Big(-\beta(r,f) + \frac{3\pi}{2} - (k_L(\rho))^{2/3},\ -\beta(r,f) + \frac{3\pi}{2}\Big). \end{aligned}$$

Because $\rho - 1$ is small and consequently the Fourier series of $\log\left|f(re^{i\theta})\right|$ is dominated (see Lemma 1 and (3.7)) by $c_1(r,f)e^{i\theta} + c_{-1}(r,f)e^{-i\theta} = 2\left|c_1(r,f)\right|\cos(\theta + \beta(r,f))$, to a first approximation we have $X_r = I_r$ and $Y_r = L_r$. In order to make precise statements, we study these sets in some detail and in particular obtain upper bounds on the measure of the symmetric differences $X_r \triangle I_r$ and $Y_r \triangle L_r$.

We note that

$$X_r - I_r \subset K_r \cup C_r$$

and

$$I_r - X_r \subset J_r \cup C_r,$$

and thus

$$X_r \triangle I_r \subset K_r \cup J_r \cup C_r. \tag{3.20}$$

We also note that

$$Y_r \triangle L_r = X_r \triangle I_r. \tag{3.21}$$

Suppose $\theta \in J_r$. By (3.7)

$$2\left|c_1(r,f)\right|\cos\left(\theta + \beta(r,f)\right) > \frac{3N(r)}{k_L(\rho)}\left(k_L(\rho)\right)^{2/3} = \frac{3N(r)}{(k_L(\rho))^{1/3}}.$$

Since $\theta \,\varepsilon\, J_r$, from (3.5) we have

$$q(re^{i\theta}) = \log\left|f(re^{i\theta})\right| - 2\left|c_1(r,f)\right|\cos(\theta + \beta(r,f)) < -\frac{3N(r)}{\left(k_L(\rho)\right)^{1/3}}.$$

Applying (3.6) we conclude that

$$\frac{m(J_r)}{2\pi}\left(\frac{3N(r)}{(k_L(\rho))^{1/3}}\right) \leq \frac{1}{2\pi}\int\limits_{J_r}\left|q(re^{i\theta})\right| d\theta \leq \|q\|_2\left(\frac{m(J_r)}{2\pi}\right)^{1/2} \leq 2N(r)\left(\frac{m(J_r)}{2\pi}\right)^{1/2},$$

implying

$$m(J_r) < \frac{8\pi}{9}\left(k_L(\rho)\right)^{2/3}.$$

A similar argument yields

$$m(K_r) < \frac{8\pi}{9}(k_L(\rho))^{2/3}.$$

From (3.8), (3.20), and (3.21) we conclude

$$\begin{aligned} m(X_r \bigtriangleup I_r) = m(Y_r \bigtriangleup L_r) &< \left(\frac{16\pi}{9} + 4\right)(k_L(\rho))^{2/3} \\ &< \left(\frac{16\pi}{9} + 4\right)\left(\pi(10^{-9})\right)^{2/3} < (2.06)10^{-5}. \end{aligned} \tag{3.22}$$

Clearly the integral of $\cos\theta$ over an interval of length π is largest when the interval is centered at 0. For later use we now express this fact in more quantitative terms. Suppose $|\alpha_1| < \frac{\pi}{150}$ and $\frac{\pi}{36} - \frac{\pi}{150} < |\alpha_2| < \frac{4\pi}{3}$. Elementary calculations yield

$$\frac{1}{2\pi}\int_{\alpha_1-\frac{\pi}{2}}^{\alpha_1+\frac{\pi}{2}} \cos\theta\, d\theta = \frac{\cos\alpha_1}{\pi} > \frac{\cos\left(\frac{\pi}{150}\right)}{\pi} > \frac{.99978}{\pi},$$

and

$$\frac{1}{2\pi}\int_{\alpha_2-\frac{\pi}{2}}^{\alpha_2+\frac{\pi}{2}} \cos\theta\, d\theta = \frac{\cos\alpha_2}{\pi} < \frac{\cos\left(\frac{\pi}{36} - \frac{\pi}{150}\right)}{\pi} < \frac{.99781}{\pi}.$$

Thus

$$\frac{1}{2\pi}\int_{\alpha_1-\frac{\pi}{2}}^{\alpha_1+\frac{\pi}{2}} \cos\theta\, d\theta - \frac{1}{2\pi}\int_{\alpha_2-\frac{\pi}{2}}^{\alpha_2+\frac{\pi}{2}} \cos\theta\, d\theta > \frac{.00197}{\pi}. \tag{3.23}$$

Suppose that z_ν is a zero of f such that for a choice of $\arg z_\nu$ satisfying $|\arg z_\nu + \beta(|z_\nu|, f)| \le \pi$, we in fact have

$$|\arg z_\nu + \beta(|z_\nu|, f)| \ge \frac{\pi}{36}. \tag{3.24}$$

For such z_ν we set

$$z_\nu^* = |z_\nu|\, e^{-i\beta(|z_\nu|,f)}.$$

We regard such zeros as outliers and replace them with z_ν^*. After a similar replacement of outlying poles (see (3.38)), we obtain a new meromorphic function F whose characteristic function effectively dominates that of f in a suitable average sense. (See (3.47).) We now begin a detailed analysis of this replacement process.

Let

$$G_1(z) = \log\left|E_1\left(\frac{z}{z_\nu^*}\right)\right|$$

and

$$H_1(z) = \log\left|E_1\left(\frac{z}{z_\nu}\right)\right|.$$

Suppose $r < s = |z_\nu|$. If $K(z)$ denotes either $G_1(z)$ or $H_1(z)$, from (2.6) we have

$$K(re^{i\theta}) = \sum_{|m|\geq 2} d_m e^{im\theta}$$

where

$$|d_m| = \frac{1}{2m}\left(\frac{r}{s}\right)^m.$$

Thus

$$\|K\|_2^2 \leq \frac{2}{4}\left(\frac{r}{s}\right)^4\left(\frac{\pi^2}{6}-1\right) < \left(\frac{r}{s}\right)^4,$$

or

$$\|K\|_2 \leq \left(\frac{r}{s}\right)^2.$$

We conclude

$$\max\left(\left\|G_1(re^{i\theta})\right\|_2, \|H_1(re^{i\theta})\|_2\right) \leq \left(\frac{r}{s}\right)^2. \tag{3.25}$$

Now suppose that $r > s$. We have from (2.6) that

$$G_1(re^{i\theta}) = \log\frac{r}{s} + \frac{r}{s}\cos\left(\theta + \beta(s,f)\right) - \mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\overline{z_\nu^*}}{r}\right)^m e^{im\theta}.$$

Set

$$g_1(re^{i\theta}) = \frac{r}{s}\cos\left(\theta + \beta(s,f)\right)$$

and

$$g_2(re^{i\theta}) = -\mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\overline{z_\nu^*}}{r}\right)^m e^{im\theta}.$$

Similarly we write

$$\begin{aligned} H_1(re^{i\theta}) &= \log\frac{r}{s} + \frac{r}{s}\cos\left(\theta - \arg z_\nu\right) - \mathrm{Re}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\overline{z}_\nu}{r}\right)^m e^{im\theta} \\ &= \log\frac{r}{s} + h_1(re^{i\theta}) + h_2(re^{i\theta}). \end{aligned}$$

Trivially we have

$$\|g_1 - h_1\|_1 \leq \|g_1 - h_1\|_\infty \leq \frac{2r}{s} \tag{3.26}$$

as well as

$$\max\left(\|g_2\|_2, \|h_2\|_2\right) \leq \frac{\pi}{\sqrt{6}}. \tag{3.27}$$

Thus

$$\|G_1 - H_1\|_2 \leq \|g_1 - h_1\|_2 + \|g_2\|_2 + \|h_2\|_2 \leq \frac{2r}{s} + \pi. \tag{3.28}$$

For $t > 1$, define $t'' < t < t'$ by

$$\left(\frac{t}{t''}\right)^{\rho-1} = \left(\frac{t'}{t}\right)^{\rho-1} = \frac{10^{-4}}{3k_L(\rho)} = \frac{M}{10^4} > \frac{10^5}{3\pi} > 10^4, \tag{3.29}$$

where we have used (3.8). Throughout the rest of the proof, for any $t > 1$ the quantities t'' and t' are specified by (3.29). We note that with this notation the conclusion of Lemma 11 asserts that if $R_3 < R_a = s \le r < s'$, then

$$|\beta(r,f) - \beta(s,f)| < \frac{\pi}{150}. \tag{3.30}$$

We now suppose $R_A > R_3$ and consider z_ν satisfying (3.24) with $|z_\nu| = s$ where $R_A < s < s' \le R_B$, with $R_B > R'_A$ yet to be specified. We set

$$\begin{aligned}
\mathrm{I} &= \int_{R_A}^{R_B} \left\{ \frac{1}{2\pi} \int_{X_r} G_1(re^{i\theta}) - H_1(re^{i\theta})\, d\theta - \frac{1}{2\pi} \int_{Y_r} G_1(re^{i\theta}) - H_1(re^{i\theta})\, d\theta \right\} \frac{dr}{r^{\rho+1}} \\
&= \int_{R_A}^{s} + \int_{s}^{s'} + \int_{s'}^{R_B} = \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Our first goal is to show that $\mathrm{I} > 0$. To this end we show that $|\mathrm{II}|$ and $|\mathrm{IV}|$ are relatively small and that III is positive and relatively large because for $s < 2r$ the dominant term $g_1(re^{i\theta}) - h_1(re^{i\theta})$ of $G_1(re^{i\theta}) - H_1(re^{i\theta})$ makes a significant positive contribution to III in view of (3.22), (3.23), and (3.30).

Clearly

$$|\mathrm{II}| \le \int_{R_A}^{s} \left\| G_1(re^{i\theta}) - H_1(re^{i\theta}) \right\|_1 \frac{dr}{r^{\rho+1}} \le \frac{2}{(2-\rho)s^\rho} \tag{3.31}$$

by (3.25). Likewise by (3.8) and (3.28)

$$\begin{aligned}
|\mathrm{IV}| &\le \int_{s'}^{R_B} \left(\frac{2r}{s} + \pi \right) \frac{dr}{r^{\rho+1}} \le \frac{10^4}{Ms^\rho} \left(\frac{2}{\rho - 1} + \frac{\pi}{\rho} \right) \\
&< \frac{3\pi}{(10^5)s^\rho} \left(\frac{2}{\rho - 1} + \frac{\pi}{\rho} \right) < \frac{2(10^{-4})}{(\rho-1)s^\rho}.
\end{aligned} \tag{3.32}$$

We write

$$\begin{aligned}
\mathrm{III} &= \int_{s}^{s'} \left\{ \frac{1}{2\pi} \int_{X_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta - \frac{1}{2\pi} \int_{Y_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta \right\} \frac{dr}{r^{\rho+1}} \\
&\quad + \int_{s}^{s'} \left\{ \frac{1}{2\pi} \int_{X_r} g_2(re^{i\theta}) - h_2(re^{i\theta})\, d\theta - \frac{1}{2\pi} \int_{Y_r} g_2(re^{i\theta}) - h_2(re^{i\theta})\, d\theta \right\} \frac{dr}{r^{\rho+1}} \\
&= \mathrm{III}_A + \mathrm{III}_B.
\end{aligned}$$

By (3.27) we have

$$|\mathrm{III}_B| \le \int_{s}^{s'} \left\| g_2(re^{i\theta}) - h_2(re^{i\theta}) \right\|_1 \frac{dr}{r^{\rho+1}} \le \frac{\pi}{\rho s^\rho}. \tag{3.33}$$

Let

$$\mathrm{III}_C = \int_s^{s'} \left\{ \frac{1}{2\pi} \int_{I_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta - \frac{1}{2\pi} \int_{L_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta \right\} \frac{dr}{r^{\rho+1}}.$$

From (3.22), (3.26), and (3.29) we conclude

$$\begin{aligned} |\mathrm{III}_A - \mathrm{III}_C| &\leq \frac{1}{2\pi} \int_s^{s'} \left\{ \int_{X_r \triangle I_r} \left| g_1(re^{i\theta}) - h_1(re^{i\theta}) \right| d\theta \right. \\ &\qquad \left. + \int_{Y_r \triangle L_r} \left| g_1(re^{i\theta}) - h_1(re^{i\theta}) \right| d\theta \right\} \frac{dr}{r^{\rho+1}} \\ &\leq \left(\frac{4.12}{\pi} \right) (10^{-5}) \int_s^{s'} \left(\frac{r}{s} \right) \frac{dr}{r^{\rho+1}} \\ &= \left(\frac{4.12}{\pi} \right) (10^{-5}) \left(\frac{1}{\rho - 1} \right) \left(\frac{1}{s^\rho} \right) \left(1 - \frac{10^4}{M} \right). \end{aligned} \tag{3.34}$$

We note that

$$\int_{I_r} g_1(re^{i\theta})\, d\theta = \left(\frac{r}{s} \right) \int_{-\beta(r,f)-\frac{\pi}{2}}^{-\beta(r,f)+\frac{\pi}{2}} \cos(\theta + \beta(s,f))\, d\theta = \left(\frac{r}{s} \right) \int_{\beta(s,f)-\beta(r,f)-\frac{\pi}{2}}^{\beta(s,f)-\beta(r,f)-\frac{\pi}{2}} \cos\theta\, d\theta$$

and

$$\int_{I_r} h_1(re^{i\theta})\, d\theta = \left(\frac{r}{s} \right) \int_{-\beta(r,f)-\frac{\pi}{2}}^{-\beta(r,f)+\frac{\pi}{2}} \cos(\theta - \arg z_\nu)\, d\theta = \left(\frac{r}{s} \right) \int_{-\arg z_\nu - \beta(r,f)-\frac{\pi}{2}}^{-\arg z_\nu - \beta(r,f)+\frac{\pi}{2}} \cos\theta\, d\theta.$$

For $s < r < s'$ and a choice of $\arg z_\nu$ satisfying $|\arg\, z_\nu + \beta\,(|z_\nu|, f)| \leq \pi$, we have from (3.24) and (3.30)

$$\begin{aligned} \frac{4\pi}{3} &> \pi + \frac{\pi}{150} \geq |-\arg\, z_\nu - \beta(s,f)| + |\beta(s,f) - \beta(r,f)| \\ &\geq |-\arg\, z_\nu - \beta(r,f)| \geq |-\arg\, z_\nu - \beta(s,f)| - |\beta(s,f) - \beta(r,f)| \\ &> \frac{\pi}{36} - \frac{\pi}{150}. \end{aligned}$$

Combining this inequality with (3.23) and (3.30) we obtain

$$\frac{1}{2\pi} \int_{I_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta > \frac{.00197}{\pi} \left(\frac{r}{s} \right).$$

Since

$$-\frac{1}{2\pi} \int_{L_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{I_r} g_1(re^{i\theta}) - h_1(re^{i\theta})\, d\theta,$$

we conclude using (3.29) that

$$\mathrm{III}_C > \frac{2(.00197)}{\pi}\int_s^{s'}\left(\frac{r}{s}\right)\frac{dr}{r^{\rho+1}} = \frac{2(.00197)}{\pi s^\rho}\left(\frac{1}{\rho-1}\right)\left(1-\frac{10^4}{M}\right). \tag{3.35}$$

Combining (3.33), (3.34), and (3.35) we have

$$\begin{aligned}\mathrm{III} &= \mathrm{III}_C + \mathrm{III}_A - \mathrm{III}_C + \mathrm{III}_B\\ &\geq \mathrm{III}_C - |\mathrm{III}_A - \mathrm{III}_C| - |\mathrm{III}_B|\\ &> \left(\frac{1}{\rho-1}\right)\left(\frac{1}{s^\rho}\right)\left(1-\frac{10^4}{M}\right)(.00125-.00002) - \frac{\pi}{\rho s^\rho}\\ &> \frac{.00122}{(\rho-1)s^\rho}.\end{aligned} \tag{3.36}$$

The combination of (3.31), (3.32), and (3.36) yields

$$\mathrm{I} > \mathrm{III} - |\mathrm{II}| - |\mathrm{IV}| > 0. \tag{3.37}$$

Now suppose w_ν is a pole of f such that for a choice of $\arg w_\nu$ satisfying

$$|\arg w_\nu + \beta(|w_\nu|, f) + \pi| \leq \pi,$$

we in fact have

$$|\arg w_\nu + \beta(|w_\nu|, f) + \pi| \geq \frac{\pi}{36}. \tag{3.38}$$

For such w_ν, set

$$w_\nu^* = |w_\nu|\, e^{-i(\beta(|w_\nu|,f)+\pi)}.$$

Such poles w_ν are considered outliers and are replaced by poles w_ν^*. If $R_A < |w_\nu| < |w_\nu|' \leq R_B$, write

$$G_2(z) = -\log\left|E_1\left(\frac{z}{w_\nu^*}\right)\right|$$

and

$$H_2(z) = -\log\left|E_1\left(\frac{z}{w_\nu}\right)\right|.$$

If

$$\widetilde{\mathrm{I}} = \int_{R_A}^{R_B}\left\{\frac{1}{2\pi}\int_{X_r} G_2(re^{i\theta}) - H_2(re^{i\theta})\,d\theta - \frac{1}{2\pi}\int_{Y_r} G_2(re^{i\theta}) - H_2(re^{i\theta})\,d\theta\right\}\frac{dr}{r^{\rho+1}},$$

minor modifications of the above argument show that

$$\widetilde{\mathrm{I}} > 0. \tag{3.39}$$

(The modifications involve the minus signs in the definitions of G_2 and H_2, and the fact that the intervals of integration in the analogue of (3.23) are each translated by π. These effects cancel one another, and we apply the modified (3.23) to obtain the analogue of (3.35) with the sense of the inequality preserved.)

For z_ν not satisfying (3.24) and w_ν not satisfying (3.38), we now define $z_\nu^* = z_\nu$ and $w_\nu^* = w_\nu$. Let $\widetilde{F}$ be the meromorphic function obtained from f by replacing the Weierstrass factor $E_1(z/z_\nu)$ by $E_1(z/z_\nu^*)$ for each zero z_ν of f satisfying $R_A < |z_\nu| < |z_\nu|' \leq R_B$ and replacing each Weierstrass factor $E_1(z/w_\nu)$ of f by $E_1(z/w_\nu^*)$ for each pole w_ν of f such that $R_A < |w_\nu| < |w_\nu|' \le R_B$. The combination of (3.37) and (3.39) yields

$$\begin{aligned}\int_{R_A}^{R_B} \frac{\big\|\log|f(re^{i\theta})|\big\|_1}{r^{\rho+1}}\,dr &= \int_{R_A}^{R_B}\left\{\frac{1}{2\pi}\int_{X_r}\log|f(re^{i\theta})|d\theta - \frac{1}{2\pi}\int_{Y_r}\log|f(re^{i\theta})|d\theta\right\}\frac{dr}{r^{\rho+1}}\\ &\le \int_{R_A}^{R_B}\left\{\frac{1}{2\pi}\int_{X_r}\log|\widetilde{F}(re^{i\theta})|d\theta - \frac{1}{2\pi}\int_{Y_r}\log|\widetilde{F}(re^{i\theta})|d\theta\right\}\frac{dr}{r^{\rho+1}}\\ &\le \int_{R_A}^{R_B}\frac{\big\|\log|\widetilde{F}(re^{i\theta})|\big\|_1}{r^{\rho+1}}\,dr.\end{aligned}$$

Since f and $\widetilde{F}$ have zeros and poles of the same modulus, we conclude from the first fundamental theorem that

$$\int_{R_A}^{R_B}\frac{T(r,f)}{r^{\rho+1}}\,dr \le \int_{R_A}^{R_B}\frac{T(r,\widetilde{F})}{r^{\rho+1}}\,dr. \tag{3.40}$$

Our next task is to show that if all outlying zeros z_ν and outlying poles w_ν are replaced by z_ν^* and w_ν^* as above (and not just those of modulus s where $R_A < s < s' \le R_B$) to obtain a new function F, then a version of (3.40) holds for F with a small error term (see (3.47)). We begin by letting

$$f_A(z) = \frac{\prod_{|z_\nu|\le R_A} E_1\left(\frac{z}{z_\nu}\right)}{\prod_{|w_\nu|\le R_A} E_1\left(\frac{z}{w_\nu}\right)},$$

$$\tilde{f}_A(z) = \frac{\prod_{|z_\nu|\le R_A} E_1\left(\frac{z}{z_\nu^*}\right)}{\prod_{|w_\nu|\le R_A} E_1\left(\frac{z}{w_\nu^*}\right)},$$

and

$$\hat{f}_A(z) = \prod_{|z_\nu|\le R_A} E_1\left(\frac{z}{|z_\nu|}\right)\cdot\prod_{|w_\nu|\le R_A} E_1\left(\frac{z}{|w_\nu|}\right).$$

It is elementary from (2.6) or from [11, page 380] that $|c_m(r,f_A)| \le |c_m(r,\hat{f}_A)|$ for all m and all $r>0$ and consequently, by Parseval's identity, that $m_2(r,f_A) \le m_2(r,\hat{f}_A)$ as well. For $r > e\,R_A$ we have

$$\begin{aligned}N(r,\infty,f_A) &= N(R_A,\infty,f_A) + n(R_A,\infty,f_A)\log\frac{r}{R_A}\\ &\le N(eR_A,\infty,f)\left(1+\log\frac{r}{R_A}\right) < \frac{r}{R_A}N(eR_A,\infty,f_A).\end{aligned}$$

For such r and all m we in fact have

$$\left|c_m(r,\hat{f}_A)\right| \le \frac{r}{R_A}\left|c_m(eR_A,\hat{f}_A)\right|. \tag{3.41}$$

For $m=0$, (3.41) is established as above with ∞ replaced by 0 and f_A by $\hat{f}_A$. For $|m|\ge 2$, (3.41) is a routine consequence of (2.6). To establish (3.41) for $|m|=1$, for $|z_\nu|\le R_A$ set

$$\kappa(r)=\frac{r}{|z_\nu|}-\frac{|z_\nu|}{r}$$

and

$$\eta(r)=\frac{r}{R_A}\left(\frac{eR_A}{|z_\nu|}-\frac{|z_\nu|}{eR_A}\right).$$

Straightforward calculations yield $\kappa(eR_A)<\eta(eR_A)$ and $\kappa'(r)<\eta'(r)$ for $r>eR_A$, implying $\kappa(r)<\eta(r)$ for $r>eR_A$ and establishing (3.41) for $|m|=1$. Combining the above observations, we have

$$\begin{aligned}
\int_{R_A}^{R_B}\frac{T(r,f_A)}{r^{\rho+1}}\,dr &\le \int_{R_A}^{R_B}\frac{N(r,\infty,f_A)+m_2(r,f_A)}{r^{\rho+1}}\,dr\\
&\le \int_{R_A}^{R_B}\frac{N(r,\infty,f_A)+m_2(r,\hat{f}_A)}{r^{\rho+1}}\,dr\\
&\le \int_{R_A}^{eR_A}\frac{N(r,\infty,f_A)+m_2(r,\hat{f}_A)}{r^{\rho+1}}\,dr\\
&\quad+\int_{eR_A}^{\infty}\frac{r}{R_A}\Big(N(eR_A,\infty,f_A)+m_2(eR_A,\hat{f}_A)\Big)\frac{dr}{r^{\rho+1}}\\
&<\frac{\varepsilon}{10k_L(\rho)}\log\frac{R_B}{R_A}
\end{aligned}$$

for a sufficiently large choice of R_B since the last integral above is convergent. Identical reasoning shows that this inequality holds with f_A replaced by $\tilde{f}_A$. We conclude that

$$\int_{R_A}^{R_B}\frac{T(r,f_A/\tilde{f}_A)}{r^{\rho+1}}\,dr \le \int_{R_A}^{R_B}\frac{T(r,f_A)+T(r,\tilde{f}_A)}{r^{\rho+1}}\,dr < \frac{\varepsilon}{5k_L(\rho)}\log\frac{R_B}{R_A} \tag{3.42}$$

if R_B is sufficiently large.

Let

$$f_B(z)=\frac{\prod_{|z_\nu|>R_B''}E_1\!\left(\dfrac{z}{z_\nu}\right)}{\prod_{|w_\nu|>R_B''}E_1\!\left(\dfrac{z}{w_\nu}\right)},$$

$$\tilde{f}_B(z) = \frac{\prod\limits_{|z_\nu|>R''_B} E_1\left(\dfrac{z}{z^*_\nu}\right)}{\prod\limits_{|w_\nu|>R''_B} E_1\left(\dfrac{z}{w^*_\nu}\right)},$$

and

$$\hat{f}_B(z) = \prod_{|z_\nu|>R''_B} E_1\left(\frac{z}{|z_\nu|}\right) \cdot \prod_{|w_\nu|>R''_B} E_1\left(\frac{z}{|w_\nu|}\right).$$

It follows from [11, second paragraph of page 381] (or alternatively from (2.6), (2.7), (2.8), and part (i) of Lemma 3 applied with $R_2 = R''_B$) that $m_2(r, \hat{f}_B) \leq 3r^\rho/k_L(\rho)$ for $R''_B \leq r \leq R_B$. Thus we have from (2.6) and (3.29)

$$\begin{aligned}\int_{R''_B}^{R_B} \frac{T(r, f_B)}{r^{\rho+1}}\,dr &\leq \int_{R''_B}^{R_B} \frac{N(r, \infty, f_B) + m_2(r, f_B)}{r^{\rho+1}}\,dr \\ &\leq 2\int_{R''_B}^{R_B} \frac{m_2(r, \hat{f}_B)}{r^{\rho+1}}\,dr \leq \frac{6}{k_L(\rho)} \log\frac{R_B}{R''_B} = \frac{6(\log M - \log 10^4)}{(\rho-1)k_L(\rho)}.\end{aligned} \tag{3.43}$$

For $r < R''_B$ and $m \geq 2$, we have from (2.2) and (2.6)

$$|c_m(r, f_B)| \leq \left|c_m(r, \hat{f}_B)\right| = \frac{1}{2m}\int_{R''_B}^{\infty} \left(\frac{r}{s}\right)^m dn(s) < \frac{(1+\tilde{\varepsilon})r^m}{2(m-\rho)(R''_B)^{m-\rho}}.$$

Thus

$$\begin{aligned}\int_{R_A}^{R''_B} \frac{T(r, f_B)}{r^{\rho+1}}\,dr &\leq \int_0^{R''_B} \frac{m_\infty(r, f_B)}{r^{\rho+1}}\,dr \\ &< (1+\tilde{\varepsilon})\sum_{m=2}^{\infty}\left(\frac{1}{m-\rho}\right)\left(\frac{1}{R''_B}\right)^{m-\rho}\int_0^{R''_B} r^{m-\rho-1}\,dr \\ &= (1+\tilde{\varepsilon})\sum_{m=2}^{\infty}\left(\frac{1}{m-\rho}\right)^2 \leq 2.\end{aligned} \tag{3.44}$$

The estimates establishing (3.43) and (3.44) do not depend on the arguments of the zeros and poles of f_B, and thus hold for $\tilde{f}_B$ as well. We choose R_B so large that

$$2 + \frac{6\,(\log M - \log 10^4)}{(\rho-1)k_L(\rho)} < \frac{\varepsilon}{10k_L(\rho)}\log\frac{R_B}{R_A}, \tag{3.45}$$

and conclude from (3.43) and (3.44) that

$$\int_{R_A}^{R_B} \frac{T(r, f_B/\tilde{f}_B)}{r^{\rho+1}}\,dr \leq \int_{R_A}^{R_B} \frac{T(r, f_B) + T(r, \tilde{f}_B)}{r^{\rho+1}}\,dr < \frac{\varepsilon}{5k_L(\rho)}\log\frac{R_B}{R_A}. \tag{3.46}$$

We let F be the meromorphic function obtained from f by replacing every factor $E_1(z/z_\nu)$ of f by $E_1(z/z_\nu^*)$ and every factor $E_1(z/w_\nu)$ of f by $E_1(z/w_\nu^*)$. We note that $\widetilde{F} = f_A f_B F/\tilde{f}_A\tilde{f}_B$. We further note that the quantities R_0, R_1, R_3, and r_0 depend only on $\tilde{\varepsilon} > 0$ and the moduli of the zeros and poles of f, and hence can be taken to have the same values for the function F as for f. Combining (3.40), (3.42), and (3.46) we conclude

$$\int_{R_A}^{R_B} \frac{T(r,f)}{r^{\rho+1}}\,dr \le \int_{R_A}^{R_B} \frac{T(r,F)}{r^{\rho+1}}\,dr + \frac{2\varepsilon}{5k_L(\rho)}\log\frac{R_B}{R_A} \tag{3.47}$$

for $R_A > R_3$ provided that R_B/R_A is large enough that (3.42) and (3.45) are satisfied.

Having removed the outlying zeros and poles of f from further consideration by replacing f by F, we simplify the notation by henceforth denoting the zeros of F by z_ν and the poles of F by w_ν. We next show for all large r that $\{\theta : \log\left|F(re^{i\theta})\right| > 0\}$ is effectively an interval.

Lemma 12. *Suppose $\tilde{\varepsilon} > 0$ and R_3 is associated with $\tilde{\varepsilon}$ and F as in Lemma 11. Let $b > 0$ be the modulus of the zero or pole of F of smallest modulus. For*

$$r > R_4 = \max\left(R_3', \left(\frac{R_0^\rho}{b\,\tilde{\varepsilon}}\right)^{(\frac{1}{\rho-1})}\right) \tag{3.48}$$

there exist

$$\theta_1 \in \left(-\beta(r,f) - \frac{\pi}{2} - \left(\frac{\pi}{36}+\frac{\pi}{120}\right),\, -\beta(r,f) - \frac{\pi}{2} + \left(\frac{\pi}{36}+\frac{\pi}{120}\right)\right)$$

and

$$\theta_2 \in \left(-\beta(r,f) + \frac{\pi}{2} - \left(\frac{\pi}{36}+\frac{\pi}{120}\right),\, -\beta(r,f) + \frac{\pi}{2} + \left(\frac{\pi}{36}+\frac{\pi}{120}\right)\right)$$

such that

(i) $\log\left|F(re^{i\theta_j})\right| = 0$, $\;j = 1, 2$, *and*

(ii) *there are no other solutions of* $\log\left|F(re^{i\theta})\right| = 0$ *in*

$$\left(-\beta(r,f) - \frac{2\pi}{3}, -\beta(r,f) - \frac{\pi}{3}\right) \cup \left(-\beta(r,f) + \frac{\pi}{3}, -\beta(r,f) + \frac{2\pi}{3}\right).$$

Furthermore, if

$$Y(re^{i\theta}) = \begin{cases} \log^+\left|F(re^{i\theta})\right|, & \theta_1 \le \theta \le \theta_2, \\ -\log^-\left|F(re^{i\theta})\right|, & \theta_2 \le \theta < \theta_1 + 2\pi, \end{cases}$$

then

$$\left\|\log\left|F(re^{i\theta})\right| - Y(re^{i\theta})\right\|_1 \le 12\,\tilde{\varepsilon}N(r).$$

Proof. From Lemma 11, (3.24), and (3.38) we conclude that if $r'' < |z_\nu| < r'$ or $r'' < |w_\nu| < r'$, then for appropriate choices of the arguments we have

$$|\arg z_\nu + \beta(r,f)| < \frac{\pi}{36} + \frac{\pi}{150} \tag{3.49}$$

and

$$|\arg w_\nu + \pi + \beta(r,f)| < \frac{\pi}{36} + \frac{\pi}{150}. \tag{3.50}$$

We write $F = GH$ where

$$G(z) = \frac{\prod\limits_{\frac{r}{2}<|z_\nu|\le 2r} E_1\left(\dfrac{z}{z_\nu}\right)}{\prod\limits_{\frac{r}{2}<|w_\nu|\le 2r} E_1\left(\dfrac{z}{w_\nu}\right)}.$$

For $m \ge 2$, using (2.2) and integration by parts we have with obvious notation

$$\frac{1}{2m}\int\limits_{2r}^{\infty}\left(\frac{r}{t}\right)^m dn_H(t) \le -\frac{1}{2m}\left(\frac{1}{2}\right)^m n_H(2r) + \frac{(1+\tilde{\varepsilon})}{2(m-\rho)}2^{\rho-m}r^\rho.$$

For $m \ge 1$ we have

$$\frac{1}{2m}\int\limits_{0}^{r/2}\left(\frac{t}{r}\right)^m dn_H(t) = \frac{1}{2m}\left(\frac{1}{2}\right)^m n_H\left(\frac{r}{2}\right) - \frac{1}{2}\int\limits_{0}^{r/2}\left(\frac{t}{r}\right)^m \frac{n_H(t)}{t}\,dt. \tag{3.51}$$

Since $n_H\left(\frac{r}{2}\right) = n_H(2r)$, from (2.6) we conclude for $m \ge 2$ that

$$|c_m(r,H)| \le \frac{1}{2m}\int\limits_{0}^{r/2}\left(\frac{t}{r}\right)^m dn_H(t) + \frac{1}{2m}\int\limits_{2r}^{\infty}\left(\frac{r}{t}\right)^m dn_H(t) \le \frac{1+\tilde{\varepsilon}}{2(m-\rho)}2^{\rho-m}r^\rho. \tag{3.52}$$

We have from (2.1) and (3.29)

$$\begin{aligned}\frac{1}{2}\int\limits_{0}^{r''}\left(\frac{r}{t}\right)dn_H(t) &= \frac{1}{2}\left(\frac{r}{r''}\right)n_H(r'') + \frac{1}{2}\int\limits_{0}^{R_0}\left(\frac{r}{t}\right)\frac{n_H(t)}{t}\,dt + \frac{1}{2}\int\limits_{R_0}^{r''}\left(\frac{r}{t}\right)\frac{n_H(t)}{t}\,dt\\ &< \frac{(1+\tilde{\varepsilon})}{2}r(r'')^{\rho-1} + \frac{n(R_0)r}{2b} + \frac{(1+\tilde{\varepsilon})}{2(\rho-1)}r(r'')^{\rho-1}\\ &< \frac{(1+\tilde{\varepsilon})}{2}r^\rho\left(\left(\frac{r''}{r}\right)^{\rho-1} + \frac{R_0^\rho}{br^{\rho-1}} + \left(\frac{1}{\rho-1}\right)\left(\frac{r''}{r}\right)^{\rho-1}\right)\\ &< \frac{(1+\tilde{\varepsilon})}{2}r^\rho\left(\frac{10^4}{M}\left(\frac{\rho}{\rho-1}\right)+\tilde{\varepsilon}\right),\end{aligned} \tag{3.53}$$

where we have also used (3.48).

We note from (2.1) and (3.29) that

$$\begin{aligned}\frac{1}{2}\int\limits_{r''}^{r/2}\left(\frac{r}{t}\right)dn_H(t) &> \left(\frac{-r}{2r''}\right)n_H(r'') + \frac{(1-\tilde{\varepsilon})}{2(\rho-1)}r^\rho\left(\left(\frac{1}{2}\right)^{\rho-1} - \left(\frac{r''}{r}\right)^{\rho-1}\right)\\ &> r^\rho\left(-\frac{(1+\tilde{\varepsilon})}{2}\left(\frac{10^4}{M}\right) + \frac{(1-\tilde{\varepsilon})}{2(\rho-1)}\left(\left(\frac{1}{2}\right)^{\rho-1} - \frac{10^4}{M}\right)\right).\end{aligned} \tag{3.54}$$

Combining (3.49), (3.50), (3.51), (3.53), and (3.54), we have

$$\begin{aligned}
|c_1(r,H)| &\geq \left| \frac{1}{2} \sum_{r''<|z_\nu|\leq \frac{r}{2}} \frac{r}{z_\nu} - \frac{1}{2} \sum_{r''<|w_\nu|\leq \frac{r}{2}} \frac{r}{w_\nu} \right| \\
&\quad - \left| \frac{1}{2} \sum_{|z_\nu|\leq r''} \frac{r}{z_\nu} - \frac{1}{2} \sum_{|w_\nu|\leq r''} \frac{r}{w_\nu} \right| - \left| \frac{1}{2} \sum_{|z_\nu|\leq \frac{r}{2}} \frac{\overline{z_\nu}}{r} - \frac{1}{2} \sum_{|w_\nu|\leq \frac{r}{2}} \frac{\overline{w_\nu}}{r} \right| \\
&\geq \frac{\cos\left(\frac{\pi}{36} + \frac{\pi}{150}\right)}{(2.02)(\rho-1)} r^\rho - \frac{(10^{-4})r^\rho}{2(\rho-1)} - \frac{(1+\tilde{\varepsilon})}{4} \left(\frac{r}{2}\right)^\rho \\
&> \left(\frac{.49}{\rho-1}\right) r^\rho.
\end{aligned} \tag{3.55}$$

Trivially we have

$$|c_1(r,G)| \leq \frac{1}{2}\left(2 - \frac{1}{2}\right) n(r) < \frac{3}{4}(1+\tilde{\varepsilon}) r^\rho, \tag{3.56}$$

and thus

$$|c_1(r,F)| \geq |c_1(r,H)| - |c_1(r,G)| > \frac{.48}{\rho-1} r^\rho. \tag{3.57}$$

We write $c_1(r,F) = c_1^A(r,F) + c_1^B(r,F)$, where

$$c_1^A(r,F) = \frac{1}{2} \sum_{|z_\nu|\leq r''} \left(\frac{r}{z_\nu} - \frac{\overline{z_\nu}}{r}\right) - \frac{1}{2} \sum_{|w_\nu|\leq r''} \left(\frac{r}{w_\nu} - \frac{\overline{w_\nu}}{r}\right).$$

From (3.29), (3.53), and (3.57) we conclude that

$$\begin{aligned}
\left|c_1^A(r,F)\right| &\leq \frac{(1+\tilde{\varepsilon})}{2} r^\rho \left(\frac{10^4}{M}\left(\frac{\rho}{\rho-1}\right) + \tilde{\varepsilon}\right) \\
&< \left(\frac{1}{1.99}\right)\left(\frac{3\pi}{10^5}\right)\left(\frac{r^\rho}{\rho-1}\right) < (9.88)(10^{-5}) \left|c_1(r,F)\right|.
\end{aligned}$$

From Lemma 11, (3.24), and (3.38), we conclude there exists a continuous argument of $c_1^B(r,F)$ such that

$$\left|\arg c_1^B(r,F) - \beta(r,f)\right| < \frac{\pi}{36} + \frac{\pi}{150}.$$

The above estimate on $|c_1^A(r,F)|$ implies there exists a continuous argument $\beta(r,F)$ of $c_1(r,F)$ such that

$$\left|\arg c_1^B(r,F) - \beta(r,F)\right| < \sin^{-1}\left((9.98)(10^{-5})\right) < 10^{-4}.$$

We conclude that

$$|\beta(r,f) - \beta(r,F)| < \frac{\pi}{36} + \frac{\pi}{150} + 10^{-4} < \frac{\pi}{36} + \frac{\pi}{135}. \tag{3.58}$$

Let $\tilde{\alpha}(r)$ be an argument of $c_1(r,H)$. By (3.56) we have

$$\begin{aligned}
&(|c_1(r,H)| - |c_1(r,F)|)^2 + 2\,|c_1(r,H)|\,|c_1(r,F)|\,(1 - \cos(\tilde{\alpha}(r) - \beta(r,F))) \\
&= |c_1(r,G)|^2 < \left(\frac{3}{4}(1+\tilde{\varepsilon}) r^\rho\right)^2,
\end{aligned}$$

implying

$$1-\cos(\tilde{\alpha}(r)-\beta(r,F)) < \frac{9(1+\tilde{\varepsilon})^2 r^{2\rho}}{32\,|c_1(r,H)|\,|c_1(r,F)|}.$$

From (3.55) and (3.57) we conclude

$$1-\cos(\tilde{\alpha}(r)-\beta(r,F)) < \frac{9(\rho-1)^2}{32(.48)^2} < \frac{3(\rho-1)^2}{2},$$

and hence

$$|\tilde{\alpha}(r)-\beta(r,F)| < 2(\rho-1) \tag{3.59}$$

for an appropriate choice of $\tilde{\alpha}(r)$.

We define $Q(re^{i\theta})$ by

$$\log\left|H(re^{i\theta})\right| = N(r,0,H)-N(r,\infty,H)+2\,|c_1(r,H)|\cos(\theta+\tilde{\alpha}(r))+Q(re^{i\theta}).$$

From (3.59) we have

$$\frac{d}{d\theta}(2\,|c_1(r,H)|\cos(\theta+\tilde{\alpha}(r)))\begin{cases} < -\,|c_1(r,H)|\,, & \left|\theta+\beta(r,F)-\frac{\pi}{2}\right| < \frac{\pi}{4}, \\ > |c_1(r,H)|\,, & \left|\theta+\beta(r,F)+\frac{\pi}{2}\right| < \frac{\pi}{4}. \end{cases} \tag{3.60}$$

From (3.52) we have for all θ that

$$\left|\frac{d}{d\theta}Q(re^{i\theta})\right| < (1+\tilde{\varepsilon})\sum_{m=2}^{\infty}\frac{m}{m-\rho}2^{\rho-m}r^{\rho} < 4r^{\rho},$$

which combined with (3.55) and (3.60) yields

$$\frac{d}{d\theta}\log\left|H(re^{i\theta})\right|\begin{cases} < -\,|c_1(r,H)|\,/2, & \left|\theta+\beta(r,F)-\frac{\pi}{2}\right| < \frac{\pi}{4}, \\ > |c_1(r,H)|\,/2, & \left|\theta+\beta(r,F)+\frac{\pi}{2}\right| < \frac{\pi}{4}. \end{cases} \tag{3.61}$$

Consider a factor $E_1\left(\frac{z}{z_\nu}\right)$ of $G(z)$, where of course $\frac{r}{2} < |z_\nu| \le 2r$ and

$$|\arg z_\nu + \beta(|z_\nu|\,,f)| < \frac{\pi}{36}.$$

If $\left|\theta+\beta(r,F)\pm\frac{\pi}{2}\right| < \frac{\pi}{4}$, then

$$\frac{\pi}{4} < |\theta+\beta(r,F)| < \frac{3\pi}{4}.$$

We have by Lemma 11 and (3.58)

$$\begin{aligned} |\theta-\arg z_\nu| &\ge |\theta+\beta(r,F)| - |\beta(r,f)-\beta(r,F)| \\ &\quad - |\beta(|z_\nu|\,,f)-\beta(r,f)| - |\beta(|z_\nu|\,,f)+\arg z_\nu| \\ &\ge \frac{\pi}{4}-\left(\frac{\pi}{36}+\frac{\pi}{135}\right)-\frac{\pi}{150}-\frac{\pi}{36} > \frac{\pi}{6} \end{aligned}$$

and similarly

$$|\theta-\arg z_\nu| < \pi.$$

Thus

$$\left|re^{i\theta}-z_\nu\right| \ge \frac{r}{2}\left(2\sin\frac{\pi}{12}\right) > \frac{r}{4}.$$

We conclude for the two intervals of θ in question that

$$\left|\frac{d}{d\theta}\log\left|E_1\left(\frac{re^{i\theta}}{z_\nu}\right)\right|\right| = \left|\operatorname{Im}\frac{(re^{i\theta})^2}{z_\nu(re^{i\theta}-z_\nu)}\right| \leq 8.$$

Applying a similar analysis to the poles of G, we conclude that

$$\left|\frac{d}{d\theta}\log\left|G(re^{i\theta})\right|\right| \leq 8n(2r) < 8(1+\tilde{\varepsilon})2^\rho r^\rho$$

if $\left|\theta+\beta(r,F)\pm\frac{\pi}{2}\right| < \frac{\pi}{4}$. From (3.55) and (3.61) we conclude

$$\frac{d}{d\theta}\log\left|F(re^{i\theta})\right| \begin{cases} < -\left|c_1(r,H)\right|/4, & \left|\theta+\beta(r,F)-\frac{\pi}{2}\right| < \frac{\pi}{4}, \\ > \left|c_1(r,H)\right|/4, & \left|\theta+\beta(r,F)+\frac{\pi}{2}\right| < \frac{\pi}{4}. \end{cases} \tag{3.62}$$

We now apply Lemma 1 with $g = F$. Let

$$S_b(r) = \left\{\theta \in \left[-\beta(r,f)-\frac{\pi}{2}, -\beta(r,f)+\frac{3\pi}{2}\right) : \left|F_b(re^{i\theta})\right| > r^\rho\right\}.$$

(Here the meaning of F_b is that specified in Lemma 1.) We have from (2.1) and (2.8) that

$$\frac{m(S_b(r))}{2\pi}r^\rho \leq \frac{1}{2\pi}\int\limits_{S_b(r)}\left|F_b(re^{i\theta})\right|d\theta \leq \left(\frac{m(S_b(r))}{2\pi}\right)^{1/2} 12\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^\rho,$$

implying

$$m(S_b(r)) < 300\pi(\tilde{\varepsilon})^2. \tag{3.63}$$

By Lemma 1 for all θ we have

$$\left|F_a(re^{i\theta}) - 2\left|c_1(r,F)\right|\cos(\theta+\beta(r,F))\right| < 3N(r),$$

and thus from (3.57) and (3.58) we conclude for every

$$\theta \in \left(-\beta(r,f)-\frac{\pi}{2}-\left(\frac{\pi}{36}+\frac{\pi}{120}\right), -\beta(r,f)-\frac{\pi}{2}-\left(\frac{\pi}{36}+\frac{\pi}{130}\right)\right) - S_b(r)$$

that $\log\left|F(re^{i\theta})\right| < 0$ and for every

$$\theta \in \left(-\beta(r,f)-\frac{\pi}{2}+\left(\frac{\pi}{36}+\frac{\pi}{130}\right), -\beta(r,f)-\frac{\pi}{2}+\left(\frac{\pi}{36}+\frac{\pi}{120}\right)\right) - S_b(r)$$

that $\log\left|F(re^{i\theta})\right| > 0$. Combined with (3.63), these inequalities establish the existence of the required θ_1. The existence of θ_2 is established similarly. The uniqueness of solutions of $\log\left|F(re^{i\theta})\right| = 0$ in the specified intervals follows from (3.58) and (3.62).

We now turn our attention to the second conclusion of Lemma 12. Let $D_r^1 = [\theta_1,\theta_2)$, $D_r^2 = [\theta_2,\theta_1+2\pi)$, and $D_r^3 = \left\{\theta\in[\theta_1,\theta_1+2\pi) : \log\left|F(re^{i\theta})\right| > 0\right\}$. Suppose $\theta \in D_r^1 - D_r^3$. In view of the monotonicity of $\log\left|F(re^{i\theta})\right|$ implied by (3.62), we have

$$\cos(\theta+\beta(r,F)) \geq \frac{\sqrt{2}}{2}.$$

From Lemma 1 applied with $g = F$ and (3.57) we deduce

$$F_a(re^{i\theta}) \geq 2\left|c_1(r,F)\right|\cos(\theta+\beta(r,F)) - 3N(r) > \sqrt{2}\left|c_1(r,F)\right| - 3\,N(r) > 0.$$

Since $\theta \notin D_r^3$, we evidently have

$$\left|F_b(re^{i\theta})\right| > \left|\log\left|F(re^{i\theta})\right|\right|,$$

implying

$$\frac{1}{2\pi}\int\limits_{D_r^1 - D_r^3} \left|\log\left|F(re^{i\theta})\right|\right| d\theta \le \frac{1}{2\pi}\int\limits_{D_r^1 - D_r^3} \left|F_b(re^{i\theta})\right| d\theta. \tag{3.64}$$

Now suppose $\theta \in D_r^2 \cap D_r^3$. As above we have

$$\cos(\theta + \beta(r, F)) \le -\frac{\sqrt{2}}{2}.$$

We conclude

$$F_a\left(re^{i\theta}\right) \le 2\left|c_1(r,F)\right| \cos(\theta+\beta(r,F)) + 3N(r) < -\sqrt{2}\left|c_1(r,F)\right| + 3\,N(r) < 0.$$

Since $\theta \in D_r^3$, we have

$$\left|F_b(re^{i\theta})\right| > \left|\log\left|F(re^{i\theta})\right|\right|,$$

and

$$\frac{1}{2\pi}\int\limits_{D_r^2 \cap D_r^3} \left|\log\left|F(re^{i\theta})\right|\right| d\theta \le \frac{1}{2\pi}\int\limits_{D_r^2 \cap D_r^3} \left|F_b(re^{i\theta})\right| d\theta. \tag{3.65}$$

Finally we have from Lemma 1, (3.64), and (3.65) that

$$\begin{aligned}
\left\|\log\left|F(re^{i\theta})\right| - Y(re^{i\theta})\right\|_1 &= \frac{1}{2\pi}\int\limits_{D_r^1} \left|\log\left|F(re^{i\theta})\right| - \log^+\left|F(re^{i\theta})\right|\right| d\theta \\
&\quad + \frac{1}{2\pi}\int\limits_{D_r^2} \left|\log\left|F(re^{i\theta})\right| + \log^-\left|F(re^{i\theta})\right|\right| d\theta \\
&= \frac{1}{2\pi}\int\limits_{D_r^1 - D_r^3} \left|\log\left|F(re^{i\theta})\right|\right| d\theta + \frac{1}{2\pi}\int\limits_{D_r^2 \cap D_r^3} \left|\log\left|F(re^{i\theta})\right|\right| d\theta \\
&\le \left\|F_b(re^{i\theta})\right\|_1 \le \left\|F_b(re^{i\theta})\right\|_2 < 12\,\tilde{\varepsilon}N(r),
\end{aligned}$$

finishing the proof of Lemma 12. □

For simplicity of notation for $r > R_4$ we let $D_r = D_r^1$ and $\widetilde{D}_r = D_r^2$. Let $\gamma(r)$ denote the midpoint of D_r. Evidently from Lemma 12

$$|\beta(r, f) + \gamma(r)| < \frac{\pi}{36} + \frac{\pi}{120}. \tag{3.66}$$

We next establish a suitable analogue of (1.3), with F playing the role of f in that inequality. We outline the argument before presenting the details. We initially suppose $m(D_r) \ge \pi$. In view of Lemma 12, the contribution of a factor $E_1\left(\dfrac{z}{z_\nu}\right)$ or $1/E_1\left(\dfrac{z}{w_\nu}\right)$ to $T(r, F)$ is essentially the integral over D_r of the logarithm of the modulus of that factor evaluated at $z = re^{i\theta}$. For the purpose of this overview, we identify the contribution of such a Weierstrass factor to $T(r, F)$ with this integral over D_r. For all zeros $z_\nu = |z_\nu|e^{i\alpha_\nu}$ of F with $|\alpha_\nu - \gamma(r)|$ small, Lemma 9 implies that the choice of α_ν maximizing the contribution of $E_1\left(\dfrac{z}{z_\nu}\right)$ to $T(r,F)$ is $\alpha_\nu = \gamma(r)$. Using Lemma 7 we show below that a pole $w_\nu = |w_\nu|e^{i\beta_\nu}$ with $|\beta_\nu + \pi - \gamma(r)|$ small makes a smaller contribution to $T(r,F)$ than would a zero $z_\nu = |w_\nu|e^{i(\beta_\nu+\pi)}$, whose contribution by the above remarks is in turn dominated by that of a zero at $|w_\nu|e^{i\gamma(r)}$. By Lemma 11 and (3.66), since F has no outlying zeros or poles, the above discussion accounts for all zeros and poles of F with modulus between r'' and r'. Once the contribution to $T(r,F)$ of the Weierstrass factors associated with the remaining zeros and poles of F is taken into account by relatively routine methods, we obtain (3.90), an inequality asserting that $T(r,F)$ is effectively dominated by the characteristic of a canonical product with zeros of argument $\gamma(r)$ at the moduli of the zeros and poles of F.

Let

$$R_5 = R_4'\left(\frac{10^6}{\tilde{\varepsilon}}\right)^{\left(\frac{1}{\rho-1}\right)}. \tag{3.67}$$

Suppose $r > R_5$. (Recall in particular this implies $r > r_0$, where r_0 is specified in (3.1) and (3.2).) Suppose temporarily that

$$m(D_r) \geq \pi. \tag{3.68}$$

We note by Lemma 12 that

$$m(D_r) < \pi + 2\left(\frac{\pi}{36} + \frac{\pi}{120}\right). \tag{3.69}$$

If

$$\left\{R_4' < t < r : |\beta(t,f) - \beta(r,f)| \geq \frac{\pi}{12}\right\} = \emptyset,$$

let $t_1 = R_4'$. Otherwise let

$$t_1 = \sup\left\{R_4' < t < r : |\beta(t,f) - \beta(r,f)| \geq \frac{\pi}{12}\right\}. \tag{3.70}$$

Let

$$t_2 = \inf\left\{t > r : |\beta(t,f) - \beta(r,f)| \geq \frac{\pi}{12}\right\}, \tag{3.71}$$

where we interpret $\inf \emptyset = +\infty$. From Lemma 11 and (3.29) it is evident that $t_1 < r'' < r' < t_2$. Define intervals $I_1 = (0, t_1'']$, $I_2 = (t_1'', t_1]$, $I_3 = (t_1, t_2]$, $I_4 = (t_2, t_2']$, and $I_5 = (t_2', +\infty)$, where I_4 and I_5 are empty if $t_2 = +\infty$. For $1 \leq j \leq 5$, let Z_j be the set of zeros z_ν of F for which $|z_\nu| \in I_j$, and let W_j be the set of poles of F for which $|w_\nu| \in I_j$.

For $1 \leq j \leq 5$, define

$$F_j(z) = \frac{\displaystyle\prod_{z_\nu \in Z_j} E_1\left(\frac{z}{z_\nu}\right)}{\displaystyle\prod_{w_\nu \in W_j} E_1\left(\frac{z}{w_\nu}\right)}$$

and

$$L_j(z) = \prod_{z_\nu \in Z_j} E_1\left(\frac{z}{|z_\nu|\, e^{i\gamma(r)}}\right) \cdot \prod_{w_\nu \in W_j} E_1\left(\frac{z}{|w_\nu|\, e^{i\gamma(r)}}\right).$$

We also define

$$G_3(z) = \prod_{z_\nu \in Z_3} E_1\left(\frac{z}{z_\nu}\right) \cdot \prod_{w_\nu \in W_3} E_1\left(-\frac{z}{w_\nu}\right).$$

If any of the sets Z_j or W_j is empty, the corresponding product is interpreted to be 1.

Suppose $w_\nu \in W_3$. From (3.38), (3.66), (3.70), and (3.71) we have for an appropriate choice of the argument that

$$\begin{aligned} |\arg w_\nu + \pi - \gamma(r)| &\le |\arg w_\nu + \pi + \beta(|w_\nu|, f)| \\ &\quad + |\beta(r,f) - \beta(|w_\nu|, f)| + |\beta(r,f) + \gamma(r)| \\ &\le \pi/36 + \pi/12 + \pi/36 + \pi/120 < \pi/6. \end{aligned} \tag{3.72}$$

Set $\varphi(z) = E_1\left(-\dfrac{z}{w_\nu}\right)$ and $\psi(z) = 1/E_1\left(\dfrac{z}{w_\nu}\right)$. In view of (3.66), (3.69), and (3.72), we may apply Lemma 7 with $a = -w_\nu$ to conclude that

$$\frac{1}{2\pi}\int\limits_{D_r} \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta \ge \frac{1}{2\pi}\int\limits_{\tilde{D}_r} \log\left|\varphi(re^{i\theta})\right| - \log\left|\psi(re^{i\theta})\right| d\theta.$$

Summing over all $w_\nu \in W_3$, we conclude that

$$\begin{aligned} &\frac{1}{2\pi}\int\limits_{D_r} \log\left|G_3(re^{i\theta})\right| - \log\left|F_3(re^{i\theta})\right| d\theta \\ &\ge \frac{1}{2\pi}\int\limits_{\tilde{D}_r} \log\left|G_3(re^{i\theta})\right| - \log\left|F_3(re^{i\theta})\right| d\theta. \end{aligned} \tag{3.73}$$

If z_ν is a zero of G_3, i.e., if either $z_\nu \in Z_3$ or $-z_\nu \in W_3$, by the argument establishing (3.72) we have

$$|\arg z_\nu - \gamma(r)| < \pi/6,$$

and from Lemma 9 and Jensen's Theorem we conclude by summing over all zeros z_ν of G_3 that

$$\begin{aligned} &\frac{1}{2\pi}\int\limits_{D_r} \log\left|L_3(re^{i\theta})\right| - \log\left|G_3(re^{i\theta})\right| d\theta - \frac{1}{2\pi}\int\limits_{\tilde{D}_r} \log\left|L_3(re^{i\theta})\right| - \log\left|G_3(re^{i\theta})\right| d\theta \\ &= \frac{1}{\pi}\int\limits_{D_r} \log\left|L_3(re^{i\theta})\right| - \log\left|G_3(re^{i\theta})\right| d\theta \ge 0. \end{aligned}$$

Combined with (3.73), this implies

$$\begin{aligned} &\frac{1}{2\pi}\int\limits_{D_r} \log\left|L_3(re^{i\theta})\right| - \log\left|F_3(re^{i\theta})\right| d\theta \\ &\ge \frac{1}{2\pi}\int\limits_{\tilde{D}_r} \log\left|L_3(re^{i\theta})\right| - \log\left|F_3(re^{i\theta})\right| d\theta. \end{aligned} \tag{3.74}$$

We now suppose $t_2 < +\infty$ and consider $z_\nu = r_\nu e^{i\alpha_\nu}$ in Z_4. By (3.66), (3.71), Lemma 11, and the definition of F we have for an appropriate choice of $\arg z_v$ that

$$
\begin{aligned}
|\arg z_\nu - \gamma(r)| &= |\arg z_\nu + \beta(|z_\nu|, f) \\
&\quad -\beta(|z_\nu|, f) + \beta(t_2, f) - \beta(t_2, f) + \beta(r, f) - \beta(r, f) - \gamma(r)| \\
&\geq |\beta(t_2, f) - \beta(r, f)| - (|\arg z_\nu + \beta(|z_\nu|, f)| \\
&\quad + |\beta(|z_\nu|, f) - \beta(t_2, f)| + |\beta(r, f) + \gamma(r)|) \\
&> \frac{\pi}{12} - \left(\frac{\pi}{36} + \frac{\pi}{150} + \frac{\pi}{36} + \frac{\pi}{120}\right) > \frac{\pi}{80}.
\end{aligned} \tag{3.75}
$$

Evidently

$$|\arg z_v - \gamma(r)| < \frac{\pi}{12} + \frac{\pi}{36} + \frac{\pi}{150} + \frac{\pi}{36} + \frac{\pi}{120} < \frac{\pi}{6}.$$

Recall that $z_\nu \,\varepsilon\, Z_4$ and thus r_ν is much larger than r. Upon setting $\delta = (m(D_r) - \pi)/2 < \frac{\pi}{36} + \frac{\pi}{120}$ in (2.12), we obtain

$$
\begin{aligned}
&\frac{1}{2\pi} \int\limits_{D_r} \log \left| E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\gamma(r)}}\right)\right| - \log \left| E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\alpha_\nu}}\right)\right| d\theta \\
&\geq \frac{1}{4\pi}\left(\frac{r}{r_\nu}\right)^2 \left(1 - \cos\frac{2\pi}{80}\right) \sin 2\delta + \frac{1}{9\pi}\left(\frac{r}{r_\nu}\right)^3 \left(1 - \cos\frac{3\pi}{80}\right) \cos 3\delta \\
&\quad - \frac{2}{\pi} \sum_{m=4}^{\infty} \frac{1}{m^2} \left(\frac{r}{r_\nu}\right)^m > \frac{1 - \cos\frac{3\pi}{80}}{10\pi} \left(\frac{r}{r_\nu}\right)^3,
\end{aligned}
$$

where in the last step we use $r/r_\nu < r/r' < 10^{-4/(\rho-1)}$.

Suppose $w_\nu = s_\nu e^{i\eta_\nu} \in W_4$. As in (3.75) and the observation immediately thereafter, we have for an appropriate choice of the argument

$$\frac{\pi}{80} < |\arg w_\nu + \pi - \gamma(r)| < \frac{\pi}{6}, \tag{3.76}$$

and we conclude as before from (2.12) that

$$\frac{1}{2\pi} \int\limits_{D_r} \log \left| E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\gamma(r)}}\right)\right| + \log \left| E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\eta_\nu}}\right)\right| d\theta \geq \frac{\left(1 - \cos\frac{3\pi}{80}\right)}{10\pi} \left(\frac{r}{s_\nu}\right)^3.$$

Summing over all $z_\nu \in Z_4$ and $w_\nu \in W_4$ we conclude

$$
\begin{aligned}
&\frac{1}{2\pi} \int\limits_{D_r} \log \left| L_4(re^{i\theta})\right| - \log \left| F_4(re^{i\theta})\right| d\theta \\
&\geq \frac{\left(1 - \cos\frac{3\pi}{80}\right)}{10\pi} \int\limits_{t_2}^{t_2'} \left(\frac{r}{t}\right)^3 dn(t) > \frac{\left(1 - \cos\frac{3\pi}{80}\right)}{30\pi} r^\rho \left(\frac{t_2}{r}\right)^{\rho-3} \\
&> \left(\frac{.006}{30\pi}\right) r^\rho \left(\frac{t_2}{r}\right)^{\rho-3}.
\end{aligned} \tag{3.77}
$$

Suppose $z_\nu = r_\nu e^{i\alpha_\nu} \in Z_5$ and $w_\nu = s_\nu e^{i\eta_\nu} \in W_5$. From (2.12) we have

$$\frac{1}{2\pi} \int\limits_{D_r} \log \left| E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\gamma(r)}}\right)\right| - \log \left| E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\alpha_\nu}}\right)\right| d\theta$$

$$\geq \frac{1}{4\pi}\left(\frac{r}{r_\nu}\right)^2 (1-\cos 2(\gamma(r)-\alpha_\nu))\sin 2\delta \tag{3.78}$$
$$+\frac{1}{9\pi}\left(\frac{r}{r_\nu}\right)^3 (1-\cos 3\,(\gamma(r)-\alpha_\nu))\cos 3\delta$$
$$-\frac{2}{\pi}\sum_{m=4}^{\infty}\frac{1}{m^2}\left(\frac{r}{r_\nu}\right)^m > -\frac{1}{4\pi}\left(\frac{r}{r_\nu}\right)^4.$$

Likewise we have

$$\frac{1}{2\pi}\int_{D_r}\log\left|E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\gamma(r)}}\right)\right| + \log\left|E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\eta_\nu}}\right)\right| d\theta > -\frac{1}{4\pi}\left(\frac{r}{s_\nu}\right)^4. \tag{3.79}$$

From (3.78) and (3.79) we deduce that

$$\frac{1}{2\pi}\int_{D_r}\log\left|L_5(re^{i\theta})\right| - \log\left|F_5(re^{i\theta})\right| d\theta \geq -\frac{1}{4\pi}\int_{t_2'}^{\infty}\left(\frac{r}{t}\right)^4 dn(t) \tag{3.80}$$
$$> -\frac{(1+\tilde{\varepsilon})}{\pi(4-\rho)} r^\rho \left(\frac{t_2'}{r}\right)^{\rho-4}.$$

From (3.77), (3.80), the fact that $t_2/t_2' < 10^{-4/(\rho-1)}$, and Jensen's Theorem we conclude

$$\frac{1}{2\pi}\int_{D_r}\left|(L_4L_5)(re^{i\theta})\right| - \log\left|(F_4F_5)(re^{i\theta})\right| d\theta > 0 \tag{3.81}$$
$$> \frac{1}{2\pi}\int_{\tilde{D}_r}\log\left|(L_4L_5)(re^{i\theta})\right| - \log\left|(F_4F_5)(re^{i\theta})\right| d\theta.$$

We now suppose $t_1 > R_4'$, i.e., there exists $t \in (R_4', r)$ with $|\beta(t,f) - \beta(r,f)| \geq \frac{\pi}{12}$. As observed immediately after (3.71), we must have $t_1 < r''$ or, equivalently, $t_1' < r$. For $z_\nu = r_\nu e^{i\alpha_\nu} \in Z_2$, the argument establishing (3.75) holds in this case as well, and (2.13) yields

$$\frac{1}{2\pi}\int_{D_r}\log\left|E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\gamma(r)}}\right)\right| - \log\left|E_1\left(\frac{re^{i\theta}}{r_\nu e^{i\alpha_\nu}}\right)\right| d\theta$$
$$\geq \frac{1}{\pi}\left(\frac{r}{r_\nu}\right)\left(1-\cos\frac{\pi}{80}\right)\cos\delta - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{r_\nu}{r}\right)^m > (.00024)\frac{r}{r_\nu},$$

where we have used $r_\nu/r \leq t_1/r < t_1/t_1' < 10^{-4/(\rho-1)}$. For $w_\nu = s_\nu e^{i\eta_\nu}$ in W_2, (3.76) holds as before and from (2.13) we have

$$\frac{1}{2\pi}\int_{D_r}\log\left|E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\gamma(r)}}\right)\right| + \log\left|E_1\left(\frac{re^{i\theta}}{s_\nu e^{i\eta_\nu}}\right)\right| d\theta > (.00024)\frac{r}{s_\nu}.$$

Thus

$$\begin{aligned}(3.82)\qquad &\frac{1}{2\pi}\int\limits_{D_r} \log\left|L_2(re^{i\theta})\right| - \log\left|F_2(re^{i\theta})\right| d\theta > (.00024)\int\limits_{t_1''}^{t_1}\left(\frac{r}{t}\right) dn(t)\\ &> (.00023)\left(\frac{r^\rho}{\rho-1}\right)\left(\frac{t_1}{r}\right)^{\rho-1}.\end{aligned}$$

The moduli of the zeros and poles of F_1 (equivalently the zeros of L_1) are much smaller than r. Applying rough estimates from (2.6) to the Fourier coefficients $c_m(r, F_1)$ and $c_m(r, L_1)$ and noting that $t_1'' > R_4 \geq R_0$, we have as in (3.53)

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{D_r} \left|\log\left|L_1(re^{i\theta})\right| - \log\left|F_1(re^{i\theta})\right|\right| d\theta\\ &\leq \left\|\log\left|L_1(re^{i\theta})\right| - \log\left|F_1(re^{i\theta})\right|\right\|_\infty \leq (2.1)\int\limits_{b}^{t_1''}\frac{r}{t}dn(t)\\ (3.83)\qquad &< (2.1)(1+\tilde{\varepsilon})r^\rho\left(\left(\frac{t_1''}{r}\right)^{\rho-1} + \frac{R_0^\rho}{br^{\rho-1}} + \left(\frac{1}{\rho-1}\right)\left(\frac{t_1''}{r}\right)^{\rho-1}\right)\\ &< (2.1)(1+\tilde{\varepsilon})r^\rho\left(\frac{t_1}{r}\right)^{\rho-1}\left(\frac{10^4}{M}\right)\left(\frac{\rho}{\rho-1}+\tilde{\varepsilon}\right)\\ &< (.00021)\left(\frac{r^\rho}{\rho-1}\right)\left(\frac{t_1}{r}\right)^{\rho-1}.\end{aligned}$$

The combination of (3.82) and (3.83) yields

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{D_r} \log\left|(L_1L_2)(re^{i\theta})\right| - \log\left|(F_1F_2)(re^{i\theta})\right| d\theta\\ &> 2(10^{-5})\left(\frac{1}{\rho-1}\right)\left(\frac{r}{t_1}\right)t_1^\rho\\ &> 2\left(N(r,0,F_1F_2) + N(r,\infty,F_1F_2)\right) \geq 2N\left(r,\infty,F_1F_2\right),\end{aligned}$$

where we have used $r/t_1 > r/r'' > 10^{4/(\rho-1)}$.

From Jensen's Theorem we have

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{D_r} \log\left|(L_1L_2)(re^{i\theta})\right| - \log\left|(F_1F_2)(re^{i\theta})\right| d\theta\\ &+\frac{1}{2\pi}\int\limits_{\widetilde{D}_r} \log\left|(L_1L_2)(re^{i\theta})\right| - \log\left|(F_1F_2)(re^{i\theta})\right| d\theta = 2N(r,\infty,F_1F_2).\end{aligned}$$

Rearranging, we obtain in the case $t_1 > R_4'$ that

$$\frac{1}{2\pi}\int\limits_{D_r} \log\left|(L_1L_2)(re^{i\theta})\right| - \log\left|(F_1F_2)(re^{i\theta})\right| d\theta$$

$$-\frac{1}{2\pi}\int\limits_{\widetilde{D}_r}\log\left|(L_1L_2)(re^{i\theta})\right|-\log\left|(F_1F_2)(re^{i\theta})\right|d\theta \tag{3.84}$$
$$=\frac{1}{\pi}\int\limits_{D_r}\log\left|(L_1L_2)(re^{i\theta})\right|-\log\left|(F_1F_2)(re^{i\theta})\right|d\theta-2N(r,\infty,F_1F_2)$$
$$\geq 2N(r,\infty,F_1F_2)\geq 0.$$

Now suppose $t_1=R_4'$, i.e., $|\beta(t,f)-\beta(r,f)|<\frac{\pi}{12}$ for $R_4'<t<r$. If $z_\nu\in Z_2$ and $w_\nu\in W_2$, then as in (3.72)

$$\begin{aligned}&|\arg w_\nu+\pi-\gamma(r)|\leq|\arg w_\nu+\pi+\beta(|w_\nu|,f)|\\&\quad+|\beta(t_1,f)-\beta(|w_\nu|,f)|+|\beta(r,f)-\beta(t_1,f)|+|\beta(r,f)+\gamma(r)|\\&<\frac{\pi}{36}+\frac{\pi}{150}+\frac{\pi}{12}+\frac{\pi}{36}+\frac{\pi}{120}<\frac{\pi}{6},\end{aligned}$$

and

$$|\arg z_\nu-\gamma(r)|<\frac{\pi}{6}$$

by similar reasoning. We may thus repeat the argument leading from (3.72) to (3.74) to conclude

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{D_r}\log\left|L_2(re^{i\theta})\right|-\log\left|F_2(re^{i\theta})\right|d\theta\\&\geq\frac{1}{2\pi}\int\limits_{\widetilde{D}_r}\log\left|L_2(re^{i\theta})\right|-\log\left|F_2(re^{i\theta})\right|d\theta.\end{aligned}\tag{3.85}$$

Estimate (3.83) holds just as before, and we have

$$\begin{aligned}&\left\|\log\left|L_1(re^{i\theta})\right|-\log\left|F_1(re^{i\theta})\right|\right\|_\infty\leq(2.1)\int_b^{t_1''}\frac{r}{t}dn(t)\\&<(2.1)(1+\tilde{\varepsilon})r^\rho\left(\frac{R_4'}{r}\right)^{\rho-1}\left(\frac{10^4}{M}\right)\left(\frac{\rho}{\rho-1}+\tilde{\varepsilon}\right)<\tilde{\varepsilon}r^\rho,\end{aligned}\tag{3.86}$$

where we have used (3.67).

From (3.85) and (3.86) we conclude in the case $t_1=R_4'$ that

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_{D_r}\log\left|(L_1L_2)(re^{i\theta})\right|-\log\left|(F_1F_2)(re^{i\theta})\right|d\theta\\&\geq\frac{1}{2\pi}\int\limits_{\widetilde{D}_r}\log\left|(L_1L_2)(re^{i\theta})\right|-\log\left|(F_1F_2)(re^{i\theta})\right|d\theta-\tilde{\varepsilon}r^\rho.\end{aligned}\tag{3.87}$$

The combination of (3.74), (3.81), (3.84), and (3.87) yields in all cases that

$$\frac{1}{2\pi}\int\limits_{D_r}\log\left|L(re^{i\theta})\right|-\log\left|F(re^{i\theta})\right|d\theta$$

$$\geq \frac{1}{2\pi} \int\limits_{\widetilde{D}_r} \log \left|L(re^{i\theta})\right| - \log \left|F(re^{i\theta})\right| d\theta - \tilde{\varepsilon} r^{\rho}, \tag{3.88}$$

where $L = L_1 L_2 L_3 L_4 L_5$. This is a preliminary form of the desired analogue of (1.3), with L and F in (3.88) playing the roles of $\hat{f}$ and f respectively in (1.3).

Let

$$\tilde{h}(re^{i\theta}) = \begin{cases} \log \left|L(re^{i\theta})\right| - \log \left|F(re^{i\theta})\right| + 2\,\tilde{\varepsilon} r^{\rho}, & \theta \in D_r, \\ \log \left|L(re^{i\theta})\right| - \log \left|F(re^{i\theta})\right|, & \theta \in \widetilde{D}_r. \end{cases}$$

We note from (3.88) that

$$\frac{1}{2\pi} \int\limits_{D_r} \tilde{h}(re^{i\theta})\, d\theta \geq \frac{1}{2\pi} \int\limits_{\widetilde{D}_r} \tilde{h}(re^{i\theta})\, d\theta. \tag{3.89}$$

We now combine Lemma 10 and Lemma 12 with $V = Y$, $W = \tilde{h}$, $A = D_r = [\theta_1, \theta_2)$, and $B = [\theta_2, \theta_1 + 2\pi)$. We are permitted to apply Lemma 10 in view of (3.89), and we conclude

$$\begin{aligned} &\|\log|F|\|_1 - 12\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho} \leq \|\log|F|\|_1 - \|Y - \log|F|\|_1 \\ &\leq \|Y\|_1 \leq \left\|Y + \tilde{h}\right\|_1 < 2\tilde{\varepsilon} r^{\rho} + \|Y + \log|L| - \log|F|\|_1 \\ &\leq \|\log|L|\|_1 + \|Y - \log|F|\|_1 + 2\,\tilde{\varepsilon} r^{\rho} \\ &\leq \|\log|L|\|_1 + 14\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho}. \end{aligned}$$

Thus we have

$$\begin{aligned} 2T(r,F) - N(r) &= \|\log|F|\|_1 \leq \|\log|L|\|_1 + 26\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho} \\ &= 2T(r,L) - N(r) + 26\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho}, \end{aligned}$$

or

$$T(r,F) \leq T(r,L) + 13\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho}. \tag{3.90}$$

If (3.68) fails, i.e., if $m(\widetilde{D}_r) > \pi$, we repeat the above argument starting with (3.68), replacing D_r by $\widetilde{D}_r$, $F(z)$ by $1/F(z)$, and $L(z)$ by $L(-z)$. Instead of (3.90), we obtain the inequality

$$T\left(r,\, 1/F(z)\right) \leq T\left(r,\, L(-z)\right) + 13\,\tilde{\varepsilon}(1+\tilde{\varepsilon})r^{\rho},$$

which is equivalent to (3.90) by the first fundamental theorem. Thus (3.90) holds in all cases, and is the desired analogue of (1.3).

We apply (3.90) for all r in an interval $[R_A, R_B]$ where $R_A > R_5$ and R_B/R_A is large enough that (3.42) and (3.45) are satisfied. In view of (2.1), (3.1), (3.2), (3.3ii), (3.47), and (3.90), we have

$$\frac{1+\varepsilon}{k_L(\rho)} \log \frac{R_B}{R_A} < \int\limits_{R_A}^{R_B} \frac{T(r,f)}{N(r)} \frac{dr}{r} < (1+2\tilde{\varepsilon})\rho \left(\int\limits_{R_A}^{R_B} \frac{T(r,F)}{r^{\rho+1}} dr + \frac{2\,\varepsilon}{5k_L(\rho)} \log \frac{R_B}{R_A} \right)$$

$$
\begin{aligned}
&< (1+2\tilde{\varepsilon}) \left(\int_{R_A}^{R_B} \rho \frac{T(r,L)}{r^{\rho+1}} dr + \left(\frac{2\,\varepsilon\rho}{5k_L(\rho)} + 13\,\tilde{\varepsilon}(1+\tilde{\varepsilon})\rho \right) \log \frac{R_B}{R_A} \right) \\
&< (1+2\tilde{\varepsilon})^2 \left(\frac{1+\frac{\varepsilon}{2}}{k_L(\rho)} + \frac{2\varepsilon\rho}{5k_L(\rho)} + 13\,\tilde{\varepsilon}(1+\tilde{\varepsilon})\rho \right) \log \frac{R_B}{R_A} \\
&< \frac{1+\varepsilon}{k_L(\rho)} \log \frac{R_B}{R_A}.
\end{aligned}
$$

This is the desired contradiction.

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