COMPACT EMBDDEMENTS FOR SOBOLEV SPACES OF VARIABLE EXPONENTS AND EXISTENCE OF SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE $p(x)$-LAPLACIAN AND ITS CRITICAL EXPONENT

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Abstract. We give a sufficient condition for the compact embedding from $W^{k,p(x)}_0(\Omega)$ to $L^{q(x)}(\Omega)$ in case $\inf_{x \in \Omega} (Np(x)/(N-kp(x)) - q(x)) = 0$, where $\Omega$ is a bounded open set in $\mathbb{R}^N$. As an application, we find a nontrivial nonnegative weak solution of the nonlinear elliptic equation

$$-\text{div} \left( |\nabla u(x)|^{p(x)-2} \nabla u(x) \right) = |u(x)|^{q(x)-2} u(x) \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial \Omega.$$  

We also consider the existence of a weak solution to the problem above even if the embedding is not compact.

1. Introduction

In recent years, many authors have studied the generalized Lebesgue spaces; see [2, 5, 8–23, 26–29, 32]. First, let us recall some definitions. Following Orlicz [29] and Kováčik and Rákosník [22], for an open set $\Omega$ in $\mathbb{R}^N$ with $N \geq 1$ and a measurable function $p(\cdot): \Omega \to [1, \infty)$, we define the $L^{p(\cdot)}(\Omega)$-norm of a measurable function $f$ on $\Omega$ by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

and denote by $L^{p(\cdot)}(\Omega)$ the family of all measurable functions whose $L^{p(\cdot)}(\Omega)$-norms are finite. Further we denote by $W^{k,p(\cdot)}(\Omega)$ with $k \in \mathbb{N}$ the family of all measurable functions $u$ on $\Omega$ such that

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)} < \infty$$

and by $W^{k,p(\cdot)}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. 

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Recently, Kurata and the fourth author [23] posed the following problem: if a variable exponent \( q(\cdot) \) satisfies \( 2 < \text{ess inf}_{x \in \Omega} q(x) \leq \text{ess sup}_{x \in \Omega} q(x) \leq 2N/(N - 2) \) \((N \geq 3)\) and \( q(\cdot) \) is equal to \( 2N/(N - 2) \) at a point, then does the problem

\[
(1.1) \quad -\Delta u(x) = |u(x)|^{q(x)-2}u(x) \quad \text{in } \Omega \quad \text{and} \quad u(x) = 0 \quad \text{on } \partial \Omega
\]

have a positive solution? When \( q(\cdot) \) is a constant, problem (1.1) has been studied by many researchers. If \( q(\cdot) \) is a constant smaller than \( 2N/(N - 2) \), then the embedding from \( W^{1,2}_0(\Omega) \) to \( L^{q(\cdot)}(\Omega) \) is compact, and hence the existence of a positive solution to (1.1) is easily obtained by the standard mountain pass theorem. When \( q(\cdot) \) is \( 2N/(N - 2) \), problem (1.1) is quite interesting. If \( \Omega \) is star-shaped, then Pohozaev [31] showed that there is no solution. If \( \Omega \) has a nontrivial topology in the sense of \( \mathbb{Z}_2 \)-homology, then Bahri and Coron [3] showed that the problem has a positive solution; see also [7]. Even if \( \Omega \) is contractible, then, under some condition on the shape of \( \Omega \), Passaseo [30] obtained a positive solution. In the case when \( q(\cdot) \) is a variable exponent and \( q(\cdot) \) coincides with \( 2N/(N - 2) \) at a point in \( \Omega \), since the embedding of \( W^{1,2}_0(\Omega) \) to \( L^{q(\cdot)}(\Omega) \) may not be compact, the existence of positive solution to (1.1) is not trivial. Kurata and the fourth author showed that if there exist \( x_0 \in \Omega \), \( C_0 > 0 \), \( \eta > 0 \) and \( 0 < l < 1 \) such that \( \text{ess sup}_{x \in \Omega(\Omega)\setminus B_\eta(x_0)} q(x) < 2N/(N - 2) \) and

\[
(1.2) \quad q(x) \leq \frac{2N}{N - 2} - \frac{C_0}{(\log(1/|x - x_0|))^{\eta}} \quad \text{for almost every } x \in \Omega \cap B_\eta(x_0),
\]

then the embedding from \( W^{1,2}_0(\Omega) \) to \( L^{q(\cdot)}(\Omega) \) is compact; see [23, Theorem 2]. As an application of the compact embedding, they obtained a positive solution to (1.1).

Our first aim in this paper is to establish the compact embedding from \( W^{k,p(\cdot)}_0(\Omega) \) to \( L^{q(\cdot)}(\Omega) \) when \( q(\cdot) \) is an exponent satisfying a condition weaker than (1.2). As an application, we show the existence of a nontrivial nonnegative weak solution to the nonlinear elliptic equation

\[
(1.3) \quad \left\{ \begin{array}{ll}
-\text{div} \left( |\nabla u(x)|^{p(x)-2}\nabla u(x) \right) = |u(x)|^{q(x)-2}u(x) \quad & \text{in } \Omega, \\
|u(x)| = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]

Here \( u \) is called a weak solution of (1.3) if \( u \in W^{1,p(\cdot)}_0(\Omega) \) and

\[
\int_\Omega \left( |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) - |u(x)|^{q(x)-2}u(x)v(x) \right) \, dx = 0
\]

for all \( v \in W^{1,p(\cdot)}_0(\Omega) \). Our final goal is to find nontrivial nonnegative weak solutions to (1.3), even if the embedding might not be compact.

2. Preliminaries

Throughout this paper, we use the symbol \( C \) to denote various positive constants independent of the variables in question. We only use \( N \) as the dimension of the Euclidean space \( \mathbb{R}^N \) and we set \( B_r(x) = \{ y \in \mathbb{R}^N : |y - x| < r \} \) for \( x \in \mathbb{R}^N \) and \( r > 0 \). For a measurable subset \( E \) of \( \mathbb{R}^N \), we denote by \( |E| \) the Lebesgue measure of \( E \). For a measurable function \( u \), we set \( u^+ = \max\{u, 0\} \). Unless otherwise stated, we assume that \( N \geq 2 \) and \( \Omega \) is a bounded open set in \( \mathbb{R}^N \).
Compact embeddings for Sobolev spaces of variable exponents and nonlinear elliptic problems

A measurable function \( p(\cdot) : \Omega \to [1, \infty) \) is called a variable exponent on \( \Omega \). We set

\[
\begin{align*}
p_* &= \text{ess inf}_{x \in \Omega} p(x) \\
p^* &= \text{ess sup}_{x \in \Omega} p(x).
\end{align*}
\]

It is worth noting the next result, which follows readily from the definition of \( L^{p(\cdot)} \)-norm (see [17, Theorem 1.3]).

**Lemma 2.1.** If \( p(\cdot) \) is a variable exponent on \( \Omega \) satisfying \( 1 \leq p_* \leq p^* < \infty \), then

\[
\min \left\{ \| u \|_{L^{p_1}(\Omega)}, \| u \|_{L^{p_2}(\Omega)} \right\} \leq \int_\Omega |u(x)|^{p(x)} dx \leq \max \left\{ \| u \|_{L^{p_1}(\Omega)}, \| u \|_{L^{p_2}(\Omega)} \right\}.
\]

A variable exponent \( p(\cdot) \) is said to satisfy the log-Hölder condition on \( \Omega \) if

\[
|p(x) - p(y)| \leq \frac{C}{\log(1/|x - y|)} \quad \text{for each } x, y \in \Omega \text{ with } |x - y| < \frac{1}{2},
\]

where \( C \) is a positive constant. We set

\[
p^k_p(x) = \begin{cases} 
Np(x)/(N - kp(x)) & \text{if } 1 \leq p(x) < N/k, \\
\infty & \text{if } p(x) \geq N/k,
\end{cases}
\]

for each \( k \in \mathbb{N} \).

We know the following Sobolev inequality for functions in \( W^{1,p(\cdot)}_0(\Omega) \); see [20, Proposition 4.2 (1)].

**Lemma 2.2.** Let \( p(\cdot) \) be a variable exponent on \( \Omega \) satisfying the log-Hölder condition and \( 1 \leq p_* \leq p^* < \infty \). If \( p^* < N \), then there exists a constant \( C > 0 \) such that

\[
\| u \|_{L^{p^*_1}(\Omega)} \leq C \| \nabla u \|_{L^{p^*_p}(\Omega)}
\]

for \( u \in W^{1,p(\cdot)}_0(\Omega) \).

**Corollary 2.3.** Let \( p(\cdot) \) be as in the previous lemma. If \( p^* < N/k \) with \( k \in \mathbb{N} \), then there exists a constant \( C > 0 \) such that

\[
\| u \|_{L^{p^*_1}(\Omega)} \leq C \sum_{|\alpha|=k} \| D^\alpha u \|_{L^{p^*_p}(\Omega)}
\]

for \( u \in W^{k,p(\cdot)}_0(\Omega) \).

**Proof.** Assume \( p^* < N/k \) with \( k \in \mathbb{N} \). Let \( u \in W^{k,p(\cdot)}_0(\Omega) \) and let \( \ell \) be a positive integer with \( \ell \leq k \). Then we see from Lemma 2.2 that \( u \in W^{k-\ell,p^*_\ell(\cdot)}_0(\Omega) \), so that

\[
\| D^\alpha u \|_{L^{p^*_1}(\Omega)} \leq C \sum_{|\beta|=k-\ell+1} \| D^\beta u \|_{L^{p^*_\ell}(\Omega)}
\]

for \( |\alpha| = k - \ell \), where \( p^*_0(x) = p(x) \). This proves the required result.

\[
3. \text{Compact embeddings}
\]

In this section, we assume that \( p(\cdot) \) is a variable exponent on \( \Omega \) satisfying the log-Hölder condition and \( 1 \leq p_* \leq p^* < \infty \). For a set \( K \) in \( \mathbb{R}^N \), we define

\[
K(r) = \{ x \in \mathbb{R}^N : \delta_K(x) \leq r \} \quad \text{for } r > 0,
\]

where \( \delta_K(x) \) denotes the distance of \( x \) to \( K \).
First, as in [23], we show the following noncompact embedding from $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$.

**Proposition 3.1.** Let $x_{0} \in \Omega$ and $k \in \mathbb{N}$, and let $q(\cdot): \Omega \to [1, \infty)$ be a variable exponent on $\Omega$ such that there exist $C > 0$ and $\eta > 0$ satisfying

$$q(x) \geq p_{k}^{+}(x) - \frac{C}{\log(1/|x - x_{0}|)} \quad \text{for almost every } x \in \Omega \cap B_{\eta}(x_{0}).$$

If $p(x_{0}) < N/k$, then the embedding from $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact.

**Proof.** Assume $p(x_{0}) < N/k$. We may assume that $x_{0} = 0$ and $B_{1}(0) \subset \Omega$. Let $\psi \in C_{0}^{\infty}(\mathbb{R})$ be a function such that $0 \leq \psi(r) \leq 1$, $\psi(r) = 0$ for $r > 1$ and $\psi(r) = 1$ for $0 \leq r < 1/2$. Set

$$\psi_{n}(x) = n^{N/p_{k}^{(0)}} \psi(n|x|)$$

for each $n \in \mathbb{N}$. Then, for $n \geq 2$ and $0 \leq |\alpha| \leq k$, we note

$$\int_{\Omega} |D^{\alpha} \psi_{n}(x)|^{p(x)} dx \leq C \int_{B_{1/n}(0)} n^{(N/p_{k}^{(0)} + |\alpha|)p(0)} dx \leq C n^{(N/p_{k}^{(0)} + |\alpha|)(p(0) + C/\log n)} \int_{B_{1/n}(0)} dx \leq C$$

by the log-Hölder condition on $p(\cdot)$. Using (3.1), we have

$$\int_{\Omega} |\psi_{n}(x)|^{q(x)} dx \geq \int_{B_{1/(2n)}(0)} n^{Nq(x)/p_{k}^{(0)}} |\psi(n|x|)|^{q(x)} dx \geq C n^{N} \int_{B_{1/(2n)}(0)} dx = C > 0,$$

which implies that the embedding from $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact since $\int_{\Omega} |\psi_{n}(x)|^{p(x)} dx \to 0$ as $n \to \infty$. \hfill \Box

As a direct consequence, we have the following result:

**Corollary 3.2.** Let $K$ be a set in $\mathbb{R}^{N}$, and let $x_{0} \in K \cap \Omega$ and $k \in \mathbb{N}$. Let $q(\cdot): \Omega \to [1, \infty)$ be a variable exponent on $\Omega$ such that there exist $C > 0$ and $r > 0$ satisfying

$$q(x) \geq p_{k}^{+}(x) - \frac{C}{\log(1/\delta_{K}(x))} \quad \text{for almost every } x \in K(r) \cap \Omega.$$

If $p(x_{0}) < N/k$, then the embedding from $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact.

**Proof.** Assume $p(x_{0}) < N/k$. Since $\delta_{K}(x) \leq |x - x_{0}|$ for each $x \in \mathbb{R}^{N}$, we obtain the conclusion by the previous proposition. \hfill \Box

For the compact embeddings, we first give the following result.

**Proposition 3.3.** Assume that $p_{\ast} < N/k$ with some $k \in \mathbb{N}$. Let $q(\cdot)$ be a variable exponent on $\Omega$ such that $1 \leq q$, and

$$\text{ess inf}_{x \in \Omega} \left( p_{k}^{\ast}(x) - q(x) \right) > 0.$$  

Then the following hold.

(i) The embedding of $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.

(ii) If $\Omega$ satisfies the cone condition, then the embedding of $W_{0}^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.
The case (i) in the proposition is essentially a special case of [22, Theorem 3.8]; the case (ii) is a slight generalization of [14, Theorem 1.3] to the case \(1 \leq p_\ast\).

**Proof of Proposition 3.3.** We only give a proof of (ii), since (i) can be proved similarly. Assume that \(\Omega\) satisfies the cone condition. By (3.2), take \(\varepsilon > 0\) such that \(p_k^\varepsilon(x) - q(x) > 2\varepsilon > 0\) for almost every \(x \in \Omega\). Since \(p(\cdot)\) is uniformly continuous on \(\Omega\), one can find open balls \(\{B_j\}_{j=1}^l\) and \(\{\tilde{B}_j\}_{j=1}^l\) with \(l \in \mathbb{N}\) such that \(\overline{\Omega} \subset \bigcup_{i=1}^l B_i\), \(\overline{B}_j \subset \tilde{B}_j\) and

\[
\inf_{x \in \tilde{B}_j \cap \Omega} p_k^\varepsilon(x) - \varepsilon \geq \sup_{x \in B_j \cap \Omega} p_k^\varepsilon(x) - 2\varepsilon \geq \sup_{x \in B_j \cap \Omega} q(x) \quad \text{for each } j = 1, \ldots, l.
\]

Setting \(P_j = \inf_{x \in \tilde{B}_j \cap \Omega} p(x)\) and \(Q_j = \sup_{x \in \tilde{B}_j \cap \Omega} q(x)\), we see that \(Q_j < Np_j/(N - kp_j)\) and the embedding from \(\{u \in W^{k,p(\cdot)}(\Omega) : u = 0\text{ on } \Omega \setminus \tilde{B}_j\} \rightarrow W^{k,p(\cdot)}(\Omega)\) and the embedding from \(\{u \in L^{q_j}(\Omega) : u = 0\text{ on } \Omega \setminus \tilde{B}_j\} \rightarrow L^{q(\cdot)}(\Omega)\) are continuous. By the Rellich-Kondrachov theorem (see [1, Theorem 6.3]), \(W^{k,p(\cdot)}(\Omega)\) is compactly embedded into \(L^{q(\cdot)}(\Omega)\). Now, take \(\varphi_j \in C^1(\Omega; [0, 1])\) such that \(|\nabla \varphi_j| \leq C\) on \(\Omega\), \(\varphi_j = 1\) on \(\Omega \cap B_j\) and \(\varphi_j = 0\) on \(\Omega \setminus \tilde{B}_j\). It is easy to see that the linear operator \(u \mapsto \varphi_j u\) is continuous on \(W^{k,p(\cdot)}(\Omega)\). Noting \(\varphi_j u = 0\) on \(\Omega \setminus \tilde{B}_j\) for each \(u \in W^{k,p(\cdot)}(\Omega)\), we can infer that \(\{\varphi_j u : u \in W^{k,p(\cdot)}(\Omega)\}\) is compactly embedded into \(L^{q(\cdot)}(\Omega)\). Passing to subsequences repeatedly, we obtain the conclusion.

For a compact set \(K\) in \(\mathbb{R}^N\) and \(s \in [0, N]\), following Mattila [25], we say that the \((N - s)\)-dimensional upper Minkowski content of \(K\) is finite if

\[
|K(r)| \leq Cr^s \quad \text{for small } r > 0.
\]

Now we are concerned with the compact embedding from \(W^{k,p(\cdot)}_0(\Omega)\) to \(L^{q(\cdot)}(\Omega)\) when \(q(\cdot)\) and \(p_k^\varepsilon(\cdot)\) coincides on some part of \(\Omega\).

**Theorem 3.4.** Let \(\varphi(\cdot) : [1/r_0, \infty) \rightarrow (0, \infty)\) be a continuous function such that

(i) \(\varphi(r)/\log r\) is nonincreasing on \([1/r_0, \infty)\),

(ii) \(\varphi(r) \rightarrow \infty\) as \(r \rightarrow \infty\)

for some \(r_0 \in (0, 1/e)\). Let \(K\) be a compact set in \(\mathbb{R}^N\) whose \((N - s)\)-dimensional upper Minkowski content is finite for some \(s\) with \(0 < s \leq N\). Let \(k \in \mathbb{N}\) and let \(q(\cdot)\) be a variable exponent on \(\Omega\) such that

(iii) \(1 \leq q_\ast \leq q^\ast < \infty\),

(iv) \(\text{ess inf}_{\Omega \setminus K(r_0)} \left(p_k^\varepsilon(x) - q(x)\right) > 0\),

(v) \(q(x) \leq p_k^\varepsilon(x) - \varphi(1/\delta_K(x)) \frac{\log(1/\delta_K(x))}{\log(1/\delta_K(x))}\) for almost every \(x \in K(r_0) \cap \Omega\).

Then the embedding from \(W^{k,p(\cdot)}_0(\Omega)\) to \(L^{q(\cdot)}(\Omega)\) is compact.

**Proof.** Without loss of generality, we may assume \(\varphi(r)/\log r \rightarrow 0\) as \(r \rightarrow \infty\); otherwise, we have \(\text{ess inf}_{x \in \Omega}(p_k^\varepsilon(x) - q(x)) > 0\), so that the conclusion follows from Proposition 3.3 (i).

First, consider the case \(p^\ast < N/k\). Let us prove that

\[
(3.3) \quad \lim_{\varepsilon \rightarrow +0} \sup_{K(\varepsilon) \cap \Omega} \left\{ \int_{K(\varepsilon) \cap \Omega} |v(x)|^q dx : v \in W^{k,p(\cdot)}_0(\Omega), \|v\|_{W^{k,p(\cdot)}(\Omega)} \leq 1 \right\} = 0.
\]
For this purpose, take $\beta$ with $0 < \beta < s/(p^*)_k$. Let $\epsilon > 0$ such that $\epsilon^{-1} > 1/r_0$ and $
abla(1/\epsilon) \geq 1$. We set $\eta_n = \epsilon^{-\beta n}$ for each $n \in \mathbb{N}$. Then, by the assumptions on $\nabla$, we have for each $n \in \mathbb{N}$ and $x \in (K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega$,
\[
\frac{\eta_n^{q(x)} - p_k^*(x)}{\eta_n} \leq \frac{\nabla(1/\beta_n(x))}{\log(1/\eta_n^{1+1})} \leq \eta_n = \exp(-\beta n/(n+1))\nabla(1/\epsilon^{n+1}) \equiv A_n.
\]
Since
\[
|K(r) \cap \Omega| \leq Cr^s
\]
for all $r > 0$ by the boundedness of $\Omega$, we have
\[
\int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} \eta_n^{q(x)} dx \leq \eta_n^{(p^*)_k} \int_{K(\epsilon^n) \cap \Omega} dx \leq C\epsilon^{n(s-\beta(p^*)_k)}.
\]
Hence we have
\[
\int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} |v(x)|^{q(x)} dx \\
\leq \int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} |v(x)|^{q(x)} \left( \frac{|v(x)|}{\eta_n} \right)^{p_k^*(x)-q(x)} dx + \int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} \eta_n^{q(x)} dx \\
\leq A_n \int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} |v(x)|^{p_k^*(x)} dx + C\epsilon^{n(s-\beta(p^*)_k)},
\]
so that for each $n_0 \in \mathbb{N}$, we obtain
\[
\int_{K(\epsilon^n) \cap \Omega} |v(x)|^{q(x)} dx = \sum_{n=n_0}^{\infty} \int_{(K(\epsilon^n) \setminus K(\epsilon^{n+1})) \cap \Omega} |v(x)|^{q(x)} dx \\
\leq \left( \sup_{n \geq n_0} A_n \right) \int_{\Omega} |v(x)|^{p_k^*(x)} dx + C \sum_{n=n_0}^{\infty} \epsilon^{n(s-\beta(p^*)_k)}.
\]
Since $A_n \to 0$ as $n \to \infty$, $s - \beta(p^*)_k > 0$ and $\|v\|_{L^{p_k^*(\cdot)}(\Omega)} \leq C\|v\|_{W^{k,p(\cdot)}(\Omega)}$ for all $v \in W^{k,p(\cdot)}_0(\Omega)$ by Corollary 2.3, (3.3) is obtained by letting $n_0 \to \infty$.

Let $\{v_j\}$ be a bounded sequence in $W^{k,p(\cdot)}_0(\Omega)$. We may assume that it converges weakly to some $v \in W^{k,p(\cdot)}_0(\Omega)$. By Proposition 3.3 (ii), the embedding from $W^{k,p(\cdot)}(B)$ to $L^{q(\cdot)}(B)$ is compact for each ball $B \subset \Omega$ such that $\text{ess inf}_{x \in B} (p_k^*(x) - q(x)) > 0$. Let $n \in \mathbb{N}$. Since $\Omega \setminus K(2^{-n})$ is a bounded open set in $\mathbb{R}^N$, there exists a finite family of balls contained in $\mathbb{R}^N \setminus K(2^{-n-1})$ whose union contains $\Omega \setminus K(2^{-n})$. Since $\text{ess inf}_{x \in \Omega \setminus K(2^{-n-1})} (p_k^*(x) - q(x)) > 0$, we can find a subsequence $\{v_{jn,n}\}$ of $\{v_j\}$ such that $v_{jn,n} \to v$ in $L^{q(\cdot)}(\Omega \setminus K(2^{-n}))$ as well as almost everywhere on $\Omega \setminus K(2^{-n})$. Using the diagonal method, we can find a subsequence $\{v_{jn}\}$ such that $v_{jn} \to v$ in $L^{q(\cdot)}(\Omega \setminus K(\epsilon))$ for each small $\epsilon > 0$ and $v_{jn} \to v$ almost everywhere on $\Omega$. It follows that
\[
\lim_{n \to \infty} \int_{\Omega} |v_{jn}(x) - v(x)|^{q(x)} dx \\
= \lim_{n \to \infty} \left( \int_{K(\epsilon) \cap \Omega} |v_{jn}(x) - v(x)|^{q(x)} dx + \int_{\Omega \setminus K(\epsilon)} |v_{jn}(x) - v(x)|^{q(x)} dx \right) \\
= \lim_{n \to \infty} \int_{K(\epsilon) \cap \Omega} |v_{jn}(x) - v(x)|^{q(x)} dx,
\]
for each small $\varepsilon > 0$, which together with (3.3) implies that \( \|v_{j_n} - v\|_{L^{q}(\Omega)} \to 0 \) as \( n \to \infty \).

Next consider the general case. We choose $\varepsilon_0 > 0$ such that
\[
q^* \leq N(N/k - \varepsilon_0)/(k\varepsilon_0) - \varphi(1/r_0)/\log(1/r_0).
\]

We set $p_{\varepsilon_0}(x) = \min\{p(x), N/k - \varepsilon_0\}$. Since the embedding from $W^{k,p(\cdot)}_0(\Omega)$ to $W^{k,p_{\varepsilon_0}(\cdot)}_0(\Omega)$ is bounded, we can apply the first considerations to obtain the required result. \( \Box \)

As a special case of Theorem 3.4, we have the following corollary, which gives an extension of [23, Theorem 2]. We put $\log^r r = \log r$ and $\log^{n+1} r = \log(\log^n r)$, inductively.

**Corollary 3.5.** Let $k \in \mathbb{N}$ and let $q(\cdot)$ be a variable exponent on $\Omega$ such that $1 \leq q_r \leq q^* < \infty$. Suppose there exist $x_0 \in \Omega$, $C > 0$, $n \in \mathbb{N}$ and small $r_0 > 0$ such that
\[
\text{ess inf}_{x \in \Omega \cap B_{r_0}(x_0)} \left( p^*_{k}(x) - q(x) \right) > 0
\]
and
\[
q(x) \leq p^*_{k}(x) - C \frac{\log^n(1/|x - x_0|)}{\log(1/|x - x_0|)} \quad \text{for almost every } x \in B_{r_0}(x_0).
\]

Then the embedding from $W^{k,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.

4. Existence of a solution to (1.3): compact embedding case

In this section, we assume that $p(\cdot)$ is a variable exponent on $\Omega$ satisfying the log-Hölder condition and $1 < p_\ast \leq p^* < N$. Further let $q(\cdot)$ be a variable exponent on $\Omega$ such that $p^* < q_r \leq q(\cdot) \leq p^*_r(\cdot)$ for almost every $x \in \Omega$.

As an application of Theorem 3.4, we show an existence result of nontrivial nonnegative weak solutions to (1.3) as follows.

**Theorem 4.1.** Assume that the embedding from $W^{1,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. Then there exists a nontrivial nonnegative weak solution of (1.3).

In the case of $\text{ess inf}_{x \in \Omega}(p^*_1(x) - q(x)) > 0$, Fan and Zhang obtained such a result in [15, Theorem 4.7]. Although $q(\cdot)$ can be equal to $p^*_1(\cdot)$ at some points, the proof in [15] also works in our case with minor changes since we consider the case that the embedding from $W^{1,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. However, for the reader’s convenience, we give a proof of our theorem.

Let $X$ be a Banach space. We say that $u \in X$ is a critical point of $I \in C^1(X; \mathbb{R})$ if the Fréchet derivative $I'(u)$ of $I$ at $u$ is zero. We say that $\{u_n\} \subset X$ is a Palais–Smale sequence for $I$ if $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ in the dual space of $X$. We also say that $I$ satisfies the Palais–Smale condition if every Palais–Smale sequence for $I$ has a convergent subsequence.

We consider a functional $I : W^{1,p(\cdot)}_0(\Omega) \to \mathbb{R}$ defined by
\[
I(u) = \int_\Omega \left( \frac{1}{p(x)} |\nabla u(x)|^{p(x)} - \frac{1}{q(x)} u_+(x)^{q(x)} \right) \, dx \quad \text{for } u \in W^{1,p(\cdot)}_0(\Omega).
\]
The Gâteaux derivative $I'(u)$ of $I$ at $u \in W^{1,p_(\cdot)}_0(\Omega)$ is given by
\[
\langle I'(u), v \rangle = \lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = \int_\Omega \left( |\nabla u(x)|^{p(x) - 2} \nabla u(x) \nabla v(x) - u^+(x)^{q(x) - 1} v(x) \right) \, dx
\]
for each $v \in W^{1,p_(\cdot)}_0(\Omega)$. By the Vitali convergence theorem, we see that $I'$ is continuous from $W^{1,p_(\cdot)}_0(\Omega)$ to its dual space $(W^{1,p_(\cdot)}_0(\Omega))^\prime$, and hence $I \in C^1(W^{1,p_(\cdot)}_0(\Omega); \mathbb{R})$.

The following is essentially due to Boccardo and Murat [4, Theorem 2.1].

**Proposition 4.2.** Let $\{u_n\} \subset W^{1,p_(\cdot)}_0(\Omega)$ be a Palais–Smale sequence for $I$. Then $\{u_n\}$ is bounded in $W^{1,p_(\cdot)}_0(\Omega)$. Further there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in W^{1,p_(\cdot)}_0(\Omega)$ such that $\nabla u_{n_k}(x)$ converges to $\nabla u(x)$ for almost every $x \in \Omega$.

**Proof.** Setting $\beta = \sup_{n \in \mathbb{N}} I(u_n)$, we have
\[
\int_\Omega \left( \frac{1}{p^+} |\nabla u_n(x)|^{p(x)} - \frac{1}{q_*} u_n^+(x)^{q(x)} \right) \, dx \leq I(u_n) \leq \beta \quad \text{for all } n \in \mathbb{N}.
\]
Since $I'(u_n) \to 0$ as $n \to \infty$ in $(W^{1,p_(\cdot)}_0(\Omega))^\prime$, we have
\[
\int_\Omega \left( |\nabla u_n(x)|^{p(x)} - u_n^+(x)^{q(x)} \right) \, dx = \langle I'(u_n), u_n \rangle \geq -\|u_n\|_{W^{1,p_(\cdot)}_0(\Omega)}
\]
for each large positive integer $n$. Subtracting (4.2) divided by $q_*$ from (4.1) gives
\[
\left( \frac{1}{p^+} - \frac{1}{q_*} \right) \int_\Omega |\nabla u_n(x)|^{p(x)} \, dx \leq \beta + \frac{1}{q_*} \|u_n\|_{W^{1,p_(\cdot)}_0(\Omega)} \leq C(\|\nabla u_n\|_{L^p(\Omega)} + 1);
\]
we used Lemma 2.2 in the second inequality. Thus Lemma 2.1 gives
\[
\|\nabla u_n\|_{L^p(\Omega)} + 1 \geq C \min \left\{ \|\nabla u_n\|_{L^p(\Omega)}^{p^-}, \|\nabla u_n\|_{L^p(\Omega)}^{p^+} \right\},
\]
so that $\{u_n\}$ is bounded in $W^{1,p_(\cdot)}_0(\Omega)$. Hence, passing to a subsequence, we may assume that $\{u_{n_k}\}$ converges weakly to some $u \in W^{1,p_(\cdot)}_0(\Omega)$ and $\{u_{n_k}(x)\}$ converges to $u(x)$ for almost every $x \in \Omega$. For $\eta > 0$, let $T_\eta : \mathbb{R} \to \mathbb{R}$ be a function such that
\[
T_\eta(t) = \begin{cases} 
0 & \text{for } |t| \leq \eta, \\
\eta t/|t| & \text{for } |t| > \eta.
\end{cases}
\]
Since $\{T_\eta(u_n - u)\}$ converges weakly to $0$ in $W^{1,p_(\cdot)}_0(\Omega)$ and $\{u_n\}$ is bounded in $L^{q(\cdot)}(\Omega)$ by Lemma 2.2, we have
\[
\lim_{n \to \infty} \int_\Omega \left( |\nabla u_n(x)|^{p(x)} - |\nabla u(x)|^{p(x)} \right) \nabla \left( T_\eta(u_n(x) - u(x)) \right) \, dx = \lim_{n \to \infty} \int_\Omega u_n^+(x)^{q(x) - 1} T_\eta(u_n(x) - u(x)) \, dx \leq C\eta,
\]
where $C > 0$ is a constant which is independent of $\eta > 0$. We set
\[
\rho_n(x) = \left( |\nabla u_n(x)|^{p(x)} - |\nabla u(x)|^{p(x)} \right) \nabla (u_n(x) - \nabla u(x)).
\]
We note that $\rho_n \geq 0$ almost everywhere for each $n \in \mathbb{N}$. Further we set
\[
E_n = \{ x \in \Omega : |u_n(x) - u(x)| \leq \eta \}, \quad F_n = \{ x \in \Omega : |u_n(x) - u(x)| > \eta \}. 
\]
for each $n \in \mathbb{N}$. We fix $\theta \in (0, 1)$. Since
\[
\int_\Omega \rho_n(x)^\theta \, dx \leq \left( \int_{E_n} \rho_n(x) \, dx \right)^\theta |E_n|^{1-\theta} + \left( \int_{F_n} \rho_n(x) \, dx \right)^\theta |F_n|^{1-\theta}
\] for each $n \in \mathbb{N}$, $|F_n| \to 0$ and $\{\rho_n\}$ is bounded in $L^1(\Omega)$, we have
\[
\lim_{n \to \infty} \int_\Omega \rho_n(x)^\theta \, dx \leq (C\eta)^\theta |\Omega|^{1-\theta}.
\]
Letting $\eta \to 0$, we have $\int_\Omega \rho_n(x)^\theta \, dx \to 0$. Thus we may assume $\{\rho_n(x)\}$ converges to 0 for almost every $x \in \Omega$. Since $p_\ast > 1$, we see that a subsequence of $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$ for almost every $x \in \Omega$. \hfill \Box

**Lemma 4.3.** Suppose the embedding from $W^{1,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. Then the functional $I$ satisfies the Palais-Smale condition.

**Proof.** Let $\{u_n\} \subset W^{1,p(\cdot)}_0(\Omega)$ be a Palais–Smale sequence for $I$. By the previous proposition, we may assume that $\{u_n\}$ converges weakly to some $u \in W^{1,p(\cdot)}_0(\Omega)$, and $\{\nabla u_n(x)\}$ and $\{\nabla u_n(x)\}$ converge to $u(x)$ and $\nabla u(x)$ almost every $x \in \Omega$, respectively. Since $\langle I'(u_n), u \rangle \to 0$, the Vitali convergence theorem implies that
\[
\int_\Omega |\nabla u(x)|^{p(x)} \, dx = \int_\Omega u^+(x)^{q(x)} \, dx.
\]
This equality together with $\langle I'(u_n), u_n \rangle \to 0$ and the compact embedding assumption give
\[
\lim_{n \to \infty} \int_\Omega |\nabla u_n(x)|^{p(x)} \, dx = \lim_{n \to \infty} \int_\Omega u_n^+(x)^{q(x)} \, dx = \int_\Omega \nabla u(x)^{p(x)} \, dx.
\]
(4.3)
Now, we consider the function
\[
w_n(x) = 2^{p_\ast - 1} \left( |\nabla u_n(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) - |\nabla u_n(x) - \nabla u(x)|^{p(x)}.
\]
Since $w_n(x) \geq 0$ for almost every $x \in \Omega$, we see from Fatou’s lemma and (4.3) that
\[
2^{p_\ast} \int_\Omega |\nabla u(x)|^{p(x)} \, dx \geq \lim_{n \to \infty} \int_\Omega |\nabla u_n(x) - \nabla u(x)|^{p(x)} \, dx
\]
so that
\[
\lim_{n \to \infty} \int_\Omega |\nabla u_n(x) - \nabla u(x)|^{p(x)} \, dx = 0.
\]
Hence we see that $\{u_n\}$ converges strongly to $u$ in $W^{1,p(\cdot)}_0(\Omega)$. \hfill \Box

We recall the following variant of the mountain pass theorem; see e.g., [34].

**Theorem 4.4.** Let $X$ be a Banach space and let $I$ be a $C^1$ functional on $X$ such that $I(0) = 0$,
\[
(i) \quad \text{there exist positive constants } \kappa, r > 0 \text{ such that } I(u) \geq \kappa \text{ for all } u \in X \text{ with } \|u\| = r, \text{ and}
\]
\[
(ii) \quad \text{there exists an element } v \in X \text{ such that } I(v) < 0 \text{ and } \|v\| > r.
\]
Define
\[ c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \]
where
\[ (4.4) \quad \Gamma = \{ \gamma \in C([0, 1]; X) : \gamma(0) = 0, I(\gamma(1)) < 0, \|\gamma(1)\| > r \}. \]

Then \( c > 0 \) and for each \( \varepsilon > 0 \), there exists \( u \in X \) such that \( |I(u) - c| \leq \varepsilon \) and \( \|I'(u)\| \leq \varepsilon \).

Now we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** First we find \( r > 0 \) such that
\[ (4.5) \inf\{ I(u) : u \in W^{1,p} \Gamma(\Omega), \|u\|_{W^{1,p}(\Omega)} = r \} > 0. \]

Taking \( r > 0 \) so small, by Lemma 2.2, we have \( \|\nabla u\|_{L^p(\Omega)} \leq 1 \) and \( \|u\|_{L^q(\Omega)} \leq 1 \) for all \( u \in W^{1,p} \Gamma(\Omega) \) with \( \|u\|_{W^{1,p}(\Omega)} = r \). Then for each \( u \in W^{1,p} \Gamma(\Omega) \) with \( \|u\|_{W^{1,p}(\Omega)} = r \), we have
\[ \int_\Omega u^+(x)^q(x) dx \leq \|u\|_{L^q(\Omega)}^q \leq C\|\nabla u\|_{L^p(\Omega)}^q \leq C\|\nabla u\|_{L^p(\Omega)}^q \]
by Lemmas 2.1 and 2.2, so that
\[ I(u) \geq \frac{1}{p^*} \|\nabla u\|_{L^p(\Omega)}^{p^*} - \frac{C}{q^*} \|\nabla u\|_{L^p(\Omega)}^{q^*}. \]

Since \( p^* < q^* \), we have (4.5) if \( r > 0 \) is small.

Next we prove \( I(tu) \to -\infty \) as \( t \to \infty \) for \( u \in W^{1,p} \Gamma(\Omega) \) with \( u^+ \neq 0 \). In fact, if \( u \in W^{1,p} \Gamma(\Omega) \) such that \( u^+ \neq 0 \), then we see that
\[ I(tu) \leq t^{p^*} \int_\Omega \frac{1}{p(x)} \|\nabla u(x)\|_{L^p(\Omega)}^{p(x)} dx - t^{q^*} \int_\Omega \frac{1}{q(x)} u^+(x)^q(x) dx \to -\infty \]
as \( t \to \infty \), since \( p^* < q^* \).

Now the required result follows from Lemma 4.3 and Theorem 4.4. \( \square \)

As a direct consequence of Theorem 4.1, we have the following:

**Corollary 4.5.** Suppose all hypotheses in Theorem 3.4 hold for \( k = 1 \). Then there exists a nontrivial nonnegative weak solution of (1.3).

5. Existence of a solution to (1.3): noncompact embedding case

Our final aim is to deal with the existence result of a nontrivial nonnegative weak solution to (1.3) in the case that the embedding may not be compact.

For real sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n = b_n + o(1) \) or \( a_n \leq b_n + o(1) \) if \( \lim_n (a_n - b_n) = 0 \) or \( \lim_n (a_n - b_n) \leq 0 \), respectively.

**Proposition 5.1.** Let \( p(\cdot) \) be a log-Hölder continuous function on \( \Omega \) with \( 1 < p_\# \leq p^* < N \) and let \( q(\cdot) \) be a measurable function on \( \Omega \) such that \( p^* < q_\# \leq q(x) \leq N \).
\( p_1^+(x) \) for almost every \( x \in \Omega \). Assume \( \inf_{u \in \mathcal{N}_f} I(u) < \inf_{u \in \mathcal{N}_f} J(u) \), where

\[
I(u) = \int_{\Omega} \left( \frac{1}{p(x)} |\nabla u(x)|^{p(x)} - \frac{1}{q(x)} u^+(x)^{q(x)} \right) \, dx \quad \text{for } u \in W^{1,p_1}(\Omega),
\]

\[
J(u) = \int_{\Omega} \left( \frac{1}{p(x)} |\nabla u(x)|^{p(x)} - \frac{1}{p_1^+(x)} u^+(x)^{p_1^+(x)} \right) \, dx \quad \text{for } u \in W^{1,p_1}(\Omega),
\]

\[
\mathcal{N}_f = \left\{ u \in W^{1,p_1}(\Omega) \setminus \{0\} : \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx = \int_{\Omega} u^+(x)^{q(x)} \, dx \right\},
\]

\[
\mathcal{N}_f = \left\{ u \in W^{1,p_1}(\Omega) \setminus \{0\} : \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx = \int_{\Omega} u^+(x)^{p_1^+(x)} \, dx \right\}.
\]

Then problem \((1.3)\) has a nontrivial nonnegative weak solution.

**Proof.** We set \( c = \inf_{u \in \mathcal{N}_f} I(u) \), and define \( \Gamma \) by \((4.4)\) with \( X = W^{1,p_1}(\Omega) \). Along the similar lines as those in the proof of Theorem 4.1, we can easily see that \( \Gamma \neq \emptyset \), \( \mathcal{N}_f \neq \emptyset \), \( \mathcal{N}_f \neq \emptyset \) and \((4.5)\) holds for small \( r > 0 \).

First we show

\[
(5.1) \quad c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)).
\]

Let \( u \in \mathcal{N}_f \). For \( \alpha_a > 1 \) large enough, consider the path \( \gamma_a \in \Gamma \) defined by \( \gamma_a(t) = t\alpha_a u \) for \( t \in [0,1] \). Since \( I(u) = \max_{0 \leq t \leq 1} I(\gamma_a(t)) \), we have

\[
c \geq \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)).
\]

On the other hand, let \( \gamma \in \Gamma \). Then

\[
\int_{\Omega} (|\nabla \gamma(1)|^{p(x)} - (\gamma(1))^+ q(x)) \, dx < 0.
\]

As in the proof of Theorem 4.1, we find a small \( t > 0 \) satisfying

\[
\int_{\Omega} (|\nabla \gamma(t)|^{p(x)} - (\gamma(t))^+ q(x)) \, dx > 0.
\]

By the intermediate value theorem, there exists \( t \in (0,1) \) such that \( \gamma(t) \in \mathcal{N}_f \), which implies \( c \leq \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \). Thus \((5.1)\) holds.

Now, in view of Theorem 4.4, \( c > 0 \). Moreover there exists \( \{u_n\} \subset W^{1,p_1}(\Omega) \) such that \( I(u_n) \to c \) and \( I'(u_n) \to 0 \) in \((W^{1,p_1}(\Omega))'\). By Proposition 4.2 and \( c > 0 \), we find a constant \( C > 0 \) such that

\[
(5.2) \quad \frac{1}{C} \leq \int_{\Omega} |\nabla u_n(x)|^{p(x)} \, dx \leq C \quad \text{for large } n \in \mathbb{N}.
\]

Here we may assume that \( \{u_n\} \) converges weakly to some \( u \in W^{1,p_1}(\Omega) \); further \( \{u_n(x)\} \) and \( \{\nabla u_n(x)\} \) converge to \( u(x) \) and \( \nabla u(x) \) for almost every \( x \in \Omega \), respectively. Then it follows that \( I'(u) = 0 \). If we show that \( u \neq 0 \), then \( u \) is a nontrivial nonnegative weak solution of \((1.3)\).

On the contrary, suppose \( u = 0 \). Since \( I(u_n) \to c > 0 \) and \( I'(u_n) \to 0 \), taking a subsequence if necessary, we may assume \( u_n^+ \neq 0 \) for all \( n \in \mathbb{N} \). Then for each \( n \in \mathbb{N} \), there exists a unique \( t_n \in (0,\infty) \) such that

\[
\int_{\Omega} |\nabla (t_n u_n(x))|^{p(x)} \, dx = \int_{\Omega} (t_n u_n(x)^{p_1^{+}(x)}) \, dx,
\]
i.e., \( t_n u_n \in \mathcal{N} \). We will show \( t_n \leq 1 + o(1) \). On the contrary, if there exists \( \varepsilon > 0 \) such that \( t_n \geq 1 + \varepsilon \) for all \( n \in \mathbb{N} \), then
\[
t_n^{q^*} \int \Omega |\nabla u_n(x)|^{p(x)} dx \geq \int \Omega |\nabla (t_n u_n(x))|^{p(x)} dx = \int \Omega (t_n u_n^+ (x))^{p_1^*(x)} dx \geq t_n^{q^*} \int \Omega u_n^+ (x)^{p_1^*(x)} dx
\]
for all \( n \in \mathbb{N} \). Using Lebesgue’s convergence theorem, we have
\[
\int \Omega |\nabla u_n(x)|^{p(x)} dx = \int \Omega u_n^+ (x)^{q(x)} dx + o(1)
\]
\[
= \int \{x \in \Omega : u_n(x) \leq 1\} u_n^+ (x)^{q(x)} dx + \int \{x \in \Omega : u_n(x) > 1\} u_n^+ (x)^{q(x)} dx + o(1)
\]
\[
\leq \int \Omega \min\{u_n^+ (x), 1\} dx + \int \Omega u_n^+ (x)^{p_1^*(x)} dx + o(1)
\]
\[
\leq \int \Omega u_n^+ (x)^{p_1^*(x)} dx + o(1).
\]
Hence it follows that
\[
\int \Omega |\nabla u_n(x)|^{p(x)} dx \geq t_n^{q^*} \int \Omega u_n^+ (x)^{p_1^*(x)} dx \geq (1 + \varepsilon)^{q^* - p^*} \int \Omega u_n^+ (x)^{p_1^*(x)} dx \geq (1 + \varepsilon)^{q^* - p^*} \left( \int \Omega |\nabla u_n(x)|^{p(x)} dx + o(1) \right),
\]
which together with (5.2) yields a contradiction. Thus we have shown \( t_n \leq 1 + o(1) \).

On the other hand, for each \( n \in \mathbb{N} \), take a unique number \( s_n > 0 \) such that
\[
\int t_n u_n(x) = \int (s_n u_n(x))^q dx = \int (s_n u_n^+ (x))^q dx
\]
i.e., \( s_n u_n \in \mathcal{N} \). We see easily that \( I(s_n u_n) = \max_{s > 0} I(s u_n) \) for each \( n \in \mathbb{N} \). By (5.2), (5.3) and \( \langle I'(u_n), u_n \rangle = o(1) \), we infer that \( s_n = 1 + o(1) \), so that
\[
I(u_n) = I(s_n u_n) + o(1) = \max_{s \geq 0} I(s u_n) + o(1) \geq I(t_n u_n) + o(1).
\]
Let \( \varepsilon \in (0, 1) \). Then, noting
\[
\int_{\{x \in \Omega : q(x) \leq p_1^*(x) - \varepsilon\}} (t_n u_n^+ (x))^q dx \leq \int \Omega \min\{t_n u_n^+ (x), 1\} dx + \int \Omega (t_n u_n^+ (x))^{p_1^*(x) - \varepsilon} dx
\]
we obtain
\[
c = I(u_n) + o(1) \geq I(t_n u_n) + o(1)
\]
\[
\geq \int \Omega \left( \frac{1}{p(x)} |\nabla (t_n u_n(x))|^{p(x)} - \frac{1}{p_1^*(x) - \varepsilon} (t_n u_n^+ (x))^{p_1^*(x)} \right) dx + o(1)
\]
\[
= J(t_n u_n) + \int \Omega \left( \frac{1}{p_1^*(x) - \varepsilon} - \frac{1}{p_1^*(x) - \varepsilon} \right) (t_n u_n^+ (x))^{p_1^*(x)} dx + o(1) \geq \inf_{v \in \mathcal{N}_J} J(v) - C \varepsilon,
\]
where \( C \) is a constant which is independent of \( \varepsilon \in (0, 1) \). Since \( \varepsilon \in (0, 1) \) is arbitrary, we conclude that \( c \geq \inf_{v \in \mathcal{N}_J} J(v) \), which contradicts our assumption. Hence it follows that \( u \neq 0 \), as required. \( \square \)
We denote by $\mathcal{D}^{1,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ by the norm $\|\nabla u\|_{L^p(\mathbb{R}^N)}$ in $C_0^\infty(\mathbb{R}^N)$.

**Theorem 5.2.** Let $p(\cdot): \mathbb{R}^N \to \mathbb{R}$ be a log-Hölder continuous function with $1 < p_* \leq p^* < N$, and let $q(\cdot): \mathbb{R}^N \to \mathbb{R}$ be a measurable function such that $p^* < q_* \leq q(x) \leq p^*(x)$ for almost every $x \in \mathbb{R}^N$. Assume that $\mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N)$ is continuously embedded into $L^{p^*}(\mathbb{R}^N)$, i.e., there exists a constant $C > 0$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N).$$

Assume also that there exist a measurable subset $D$ of $\mathbb{R}^N$ and a number $q_0$ such that

$$\lim_{R \to \infty} \left| \{ x \in B_1(0) : R x \in D \} \right| < \left| B_1(0) \right|,$$

and $p/(N + p_* - p) < q_0 < Np/(N - p)$, and $\text{ess sup}_{x \in \mathbb{R}^N \setminus D} q(x) \leq q_0$, where $p = \lim_{x \to -\infty} p(x)$. Then there exists $R > 0$ such that for each bounded open set $\Omega$ in $\mathbb{R}^N$ which contains $B_R(0)$, problem (1.3) has a nontrivial nonnegative weak solution.

**Proof.** We set

$$J_{\mathbb{R}^N}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{p(x)}|\nabla u(x)|^{p(x)} - \frac{1}{p^*_1(x)}u^+(x)^{p^*_1(x)} \right) \, dx \quad \text{for } u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N),$$

$$\mathcal{N}_{\mathbb{R}^N} = \left\{ u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u(x)|^{p(x)} \, dx = \int_{\mathbb{R}^N} u^+(x)^{p^*_1(x)} \, dx \right\}.$$ 

By Lemma 2.1 we have for $u \in \mathcal{N}_{\mathbb{R}^N}$

$$\min \left\{ \|\nabla u\|_{L^p(\mathbb{R}^N)}^{p_*}, \|\nabla u\|_{L^{p^*}(\mathbb{R}^N)}^{p^*} \right\} \leq \int_{\mathbb{R}^N} |\nabla u(x)|^{p(x)} \, dx$$

$$= \int_{\mathbb{R}^N} u^+(x)^{p^*_1(x)} \, dx \leq \max \left\{ \|u^+\|_{L^{p^*_1}(\mathbb{R}^N)}^{p^*_1}, \|u^+\|_{L^{p^*_1}(\mathbb{R}^N)}^{p^*_1} \right\},$$

which together with (5.4) implies that

$$\inf_{u \in \mathcal{N}_{\mathbb{R}^N}} \|\nabla u\|_{L^p(\mathbb{R}^N)} > 0.$$

Hence we infer that

$$\inf_{u \in \mathcal{N}_{\mathbb{R}^N}} J_{\mathbb{R}^N}(u) > 0.$$

Choose any $p_0$ such that

$$1 < p_0 < \frac{Np_0}{N + p_* - p_0} < q_0 < \frac{Np_0}{N - p_0}.$$

Let $\bar{u}_1 \in W_0^{1,p_0}(B_1(0))$ be a weak solution of the problem

$$\begin{cases}
-\text{div} \left( |\nabla u(x)|^{p_0-2} \nabla u(x) \right) = u(x)^{q_0-1} & \text{in } B_1(0), \\
u(x) > 0 & \text{in } B_1(0), \\
u(x) = 0 & \text{on } \partial B_1(0).
\end{cases}$$

According to [24, Theorem 1] or [33, Proposition 2.1], we see that $\bar{u}_1 \in C^{1,\beta}(\overline{B_1(0)})$ for some $\beta \in (0, 1)$. Hence, for each $R > 0$, $\bar{u}_R(x) = R^{-p_0/(q_0-p_0)}\bar{u}_1(x/R)$ is a weak
solution of (5.7). Take $R_1 > 0$ such that $\max_{|x| \leq R} \tilde{u}_R(x) \leq 1$ for $R \geq R_1$. For each $R > 0$, there exists a unique $t_R \in (0, \infty)$ such that

$$\int_{B_R(0)} |\nabla (t_R \tilde{u}_R(x))|^{p(x)} \, dx = \int_{B_R(0)} |t_R \tilde{u}_R(x)|^{q(x)} \, dx.$$ 

From (5.5), we find $\delta > 0$ and $R_2 \geq R_1$ such that

$$\{x \in B_1(0) : Rx \in D\} \leq |B_1(0)| - \delta \quad \text{for each } R \geq R_2.$$

We will show $\{t_R : R \geq R_2\}$ is bounded. If $t_R > 1$ with $R \geq R_2$, then we have

$$t_R^{p^*} \int_{B_R(0)} |\tilde{u}_R(x)|^{p(x)} \, dx \geq t_R \int_{B_R(0) \setminus D} |\tilde{u}_R(x)|^{q_0} \, dx \geq t_R^{p^*} \int_{B_R(0) \setminus D} |\tilde{u}_R(x)|^{q_0} \, dx$$

$$= t_R^{p^*} \left( \int_{B_R(0)} |\tilde{u}_R(x)|^{q_0} \, dx - \int_{B_R(0) \cap D} |\tilde{u}_R(x)|^{q_0} \, dx \right),$$

which implies

$$t_R^{p^* - p^*} \leq \frac{\int_{B_1(0)} R^{\frac{q_0(p_0 - p(R_x))}{q_0 - p_0}} |\tilde{u}_1(x)|^{p(R_x)} \, dx}{\int_{B_1(0)} |\tilde{u}_1(x)|^{q_0} \, dx - \sup \{ \int_A |\tilde{u}_1(x)|^{q_0} \, dx : A \subset B_1(0), |A| \leq |B_1(0)| - \delta \}}.$$

Let $r_0 > 0$ such that $p(x) > p_0$ for all $x \in \mathbb{R}^N$ with $|x| \geq r_0$. By (5.6) and the boundedness of $|\nabla \tilde{u}_1|$, we have for $R \geq r_0$,

$$\int_{B_1(0)} R^{\frac{q_0(p_0 - p(R_x))}{q_0 - p_0}} |\nabla \tilde{u}_1(x)|^{p(R_x)} \, dx \leq C \left( \int_{|x| < r_0 / R} R^{\frac{q_0(p_0 - p(R_x))}{q_0 - p_0}} \, dx \right.$$

$$\left. + \int_{r_0 / R \leq |x| \leq 1} R^{\frac{q_0(p_0 - p(R_x))}{q_0 - p_0}} \, dx \right) \leq C \left( R^{\frac{q_0(p_0 - p_0)}{q_0 - p_0}} \left( \frac{r_0}{R} \right)^N + 1 \right) \leq C,$$

where each $C$ is a positive constant which is independent of $R$. Hence we insist that $\{t_R : R \geq R_2\}$ is bounded. Then we have

$$\int_{B_R(0)} \left( \frac{1}{p(x)} |\nabla (t_R \tilde{u}_R(x))|^{p(x)} - \frac{1}{q(x)} |t_R \tilde{u}_R(x)|^{q(x)} \right) \, dx \leq C \int_{B_R(0)} |\nabla \tilde{u}_R(x)|^{p(x)} \, dx$$

$$= C \int_{B_1(0)} R^{\frac{q_0(p_0 + q_0 - p(R_x))}{q_0 - p_0}} |\nabla \tilde{u}_1(x)|^{p(R_x)} \, dx \leq C \left( R^{-\frac{q_0 p_0}{q_0 - p_0}} r_0^N + R^{-\frac{q_0 p_0}{q_0 - p_0} + N} \right) \to 0$$

as $R \to \infty$. Hence we can find $R \geq R_2$ satisfying

$$\int_{B_R(0)} \left( \frac{1}{p(x)} |\nabla (t_R \tilde{u}_R(x))|^{p(x)} - \frac{1}{q(x)} |t_R \tilde{u}_R(x)|^{q(x)} \right) \, dx < \inf_{v \in \mathcal{A}_{R_N}} J_{R_N}(v).$$

Now, let $\Omega$ be any bounded open set which contains $B_R(0)$. Extending $\tilde{u}_R$ on $\Omega$ with $\tilde{u}_R(x) = 0$ for $x \in \Omega \setminus B_R(0)$, we have $\tilde{u}_R \in W_0^{1,p}(\Omega)$. Letting $I$, $J$, $\mathcal{A}_I$ and $\mathcal{A}_J$ be as in the previous proposition, we have

$$\inf_{v \in \mathcal{A}_J} I(v) \leq \inf_{v \in \mathcal{A}_J} J_{R_N}(v) \leq \inf_{v \in \mathcal{A}_J} J(v).$$

Hence problem (1.3) has a nontrivial nonnegative weak solution on $\Omega$ by the proposition.

Finally, we give a sufficient condition for (5.4). We recall the following result, which is a special case of [6, Theorem 1.8].
Lemma 5.3. Let \( p(\cdot) : \mathbb{R}^N \to \mathbb{R} \) be a log-Hölder continuous function which satisfies \( 1 < p_\ast \leq p^\ast < N \) and
\[
|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}
\]
for each \( x, y \in \mathbb{R}^N \) with \( |y| \geq |x| \).
Then the fractional integral operator
\[
u \mapsto \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-1}} \, dy
\]
is bounded from \( L^{p(\cdot)}(\mathbb{R}^N) \) to \( L^{p^\ast(\cdot)}(\mathbb{R}^N) \).

Corollary 5.4. Let \( p(\cdot) : \mathbb{R}^N \to \mathbb{R} \) be as in the previous lemma, and let \( D, q_0 \) and \( q(\cdot) \) be as in Theorem 5.2. Then there exists \( R > 0 \) such that for each bounded open set \( \Omega \) in \( \mathbb{R}^N \) which contains \( B_R(0) \), problem (1.3) has a nontrivial nonnegative weak solution.

Proof. Using the previous lemma, we can show that \( \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N) \) is continuously embedded into \( L^{p^\ast(\cdot)}(\mathbb{R}^N) \) by similar lines as those in [35, p. 88]. Hence we obtain the conclusion by Theorem 5.2. \( \square \)

References

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