A SPACE OF PROJECTIONS ON THE BERGMAN SPACE

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Abstract. We define a set of projections on the Bergman space $A^2$, which is parameterized by an affine subset of a Banach space of holomorphic functions in the disk and which includes the classical Forelli–Rudin projections.

1. Introduction

Recall that the Bergman projection of $L^2(D)$ onto the holomorphic Bergman space $A^2 = L^2(D) \cap H(D)$, where $H(D)$ denotes the space of holomorphic functions in the unit disk, is given by

$$P_\varphi(z) = \int_D \frac{\varphi(w)}{(1 - z \bar{w})^2} dA(w),$$

where $dA$ is the normalized Lebesgue measure in the disk. Recall also the family of Forelli–Rudin projections parameterized by $\alpha > -1$

$$P_\alpha \varphi(z) = \int_D (\alpha + 1) \left( \frac{1 - |w|^2}{1 - z \bar{w}} \right)^\alpha \frac{\varphi(w)}{(1 - z \bar{w})^2} dA(w).$$

These are the orthogonal projections of the weighted $L^2(D,(1 - |w|^\alpha) dA(w))$ onto $H(D) \cap L^2(D,(1 - |w|) dA(w))$. It is well known (see [6, Th. 7.1.4]) that $P_\alpha$ is a continuous projection of $L^2(D)$ onto $A^2$, for each $\alpha > -1/2$.

Since

$$\left\{ \frac{1 - |w|^2}{1 - z \bar{w}}, z, w \in D \right\} \subset D_1$$

where $D_1 = \{ z : |z - 1| < 1 \}$, we may replace the function $g_\alpha(\zeta) = (\alpha + 1)\zeta^\alpha$ in the definition of $P_\alpha$ by any holomorphic function $g$ on $D_1$ to obtain an operator $T_g$ mapping the space $C_c(D)$ of compactly supported continuous functions defined on $D$ into $A^2$. An equivalent formulation of the operators defined this way was given by Bonet, Engliš and Taskinen in [1] to construct continuous projections in weighted $L^\infty$ spaces of $D$ into $H(D)$. The purpose of this paper is to study the space $\mathcal{P}$ of all holomorphic functions $g \in D_1$, for which the corresponding operator $T_g$ can be extended continuously to $L^2(D)$. In particular we study the set $\mathcal{P}_0$ of those functions $g \in \mathcal{P}$ that define continuous projections on $A^2$. For notational convenience we will translate the functions in $\mathcal{P}$ to the unit disk $D$.

doi:10.5186/aasfm.2010.3512

2000 Mathematics Subject Classification: Primary 46E20.

Key words: Bergman spaces, projection.

First author partially supported by the spanish grant MTM2008-04594/MTM.

Second author partially supported by Conacyt-DAIC U48633-F.
We will prove that $\mathcal{P}$ is a Banach space when we define the norm of $g \in \mathcal{P}$ as the operator norm of the operator $T_g$ and that $\Phi(g) = \int_0^1 g(r) \, dr$ defines a bounded linear functional in $\mathcal{P}^*$. We give an analytic description of the elements of $\mathcal{P}$ and show that if $g \in \mathcal{P}$ then either $T_g$ is identically zero on $A^2$ or it is a multiple of a continuous projection onto $A^2$, implying that $\mathcal{P}_0 = \Phi^{-1}(\{1\})$ is a closed affine subspace of $\mathcal{P}$.

As usual, for each $z \in D$, $\phi_z$ will denote by $\phi_z$ the Möbius transform $\phi_z(w) = \frac{z-w}{1-z\bar{w}}$ which satisfies $(\phi_z)^{-1} = \phi_z$ and $\phi'_z(w) = -\frac{1-|z|^2}{(1-z\bar{w})^2}$. Throughout this paper we will write

$\psi_z(w) = \frac{1-|w|^2}{1-z\bar{w}}$

and

$H = \{z \in C : \text{Re}(z) > 1/2\}$.

Clearly the mapping $z \to \frac{1}{1-z}$ is a bijection of $D$ onto $H$, and

(1) $\psi_z(w) = 1 - \bar{w}\phi_w(z)$.

2. A space of projections on $A^2$

Let us start by presenting our new definitions and spaces of projections.

Definition 1. Let $g$ be holomorphic in $D$. We define

$T_g \varphi(z) = \int_D g(\bar{w}\phi_w(z)) \varphi(w) \frac{dA(w)}{(1-z\bar{w})^2},$

for any $\varphi \in C_c(D)$. We denote by $\mathcal{P}$ (resp. $\mathcal{P}_0$) the space of holomorphic functions $g \in \mathcal{H}(D)$ such that $T_g$ extends continuously to $L^2(D)$ (resp. $T_g$ is a projection on the Bergman space $A^2$). We provide the space $\mathcal{P}$ with the norm $\|g\|_{\mathcal{P}} = \|T_g\|_{L^2(D) \rightarrow L^2(D)}$.

Remark 2. In [1] it was introduced, for each $F$ holomorphic in $H$ the operator

$S_F \varphi(z) = \int_D F \left( \frac{1-z\bar{w}}{1-|w|^2} \right) \varphi(w) \frac{dA(w)}{(1-|w|^2)^2}.$

We have $T_g = S_F$, with $F(\eta) = \frac{1}{\eta} g(1 - \frac{1}{\eta})$. We will say that such $F \in \mathcal{P}$ (resp. $\mathcal{P}_0$) if $g \in \mathcal{P}$ (resp. $\mathcal{P}_0$).

Example 3. Let $g_\alpha(z) = (\alpha + 1)(1-z)^\alpha$ for every $\alpha > -1$. Then $g_\alpha \in \mathcal{P}_0$ for $\alpha > -1/2$. In fact by (1) we have $T_{g_\alpha} = P_\alpha$, which is a bounded projection from $L^2(D)$ into $A^2$ if and only if $\alpha > -1/2$.

Example 4. If $P(z) = \sum_{k=0}^N a_k z^k$ is a polynomial then $P \in \mathcal{P}$. Moreover, $P \in \mathcal{P}_0$ if and only if $\sum_{k=0}^N \frac{a_k}{(k+1)} = \int_0^1 P(r) \, dr = 1$.

Proof. Write $P(z) = \sum_{k=0}^N b_k (1-z)^k$ where $b_k = (-1)^k \frac{P(k)(1)}{k!}$. Hence

$T_P = \sum_{k=0}^N \frac{b_k}{(k+1)} P_k.$
This shows that $T_P \in \mathcal{P}$ and $\|P\|_{\mathcal{P}} \leq \sum_{k=0}^{N} \frac{b_k}{(k+1)^{4}} \|P_k\|$. On the other hand $T_P \in \mathcal{P}_0$ if and only if $\sum_{k=0}^{N} \frac{b_k}{(k+1)^{4}} = 1$. Notice now that $\sum_{k=0}^{N} \frac{b_k}{(k+1)^{4}} = \int_0^1 P(r) \, dr$ to conclude the proof.

Example 5. If $g \in \mathcal{H}(\mathbb{D})$ is such that $(1 - z)^{-\alpha} g(z)$ is bounded for some $\alpha > -1/2$ then $g \in \mathcal{P}$ and $\|g\|_{\mathcal{P}} \leq C \sup_{|z| < 1} |(1 - z)^{-\alpha} g(z)|$. In particular the space of bounded holomorphic functions $H^\infty(\mathbb{D})$ is contained in $\mathcal{P}$ and $\|f\|_{\mathcal{P}} \leq C \|f\|_{\infty}$.

Proof. Use the fact that $P^\alpha \varphi(z) = \int_D \frac{(1 - |w|^2)^\alpha}{1 - z \bar{w}} \varphi(w) \, dA(w)$ also defines a bounded operator on $L^2(\mathbb{D})$ (see [5, Theorem 1.9]).

Proposition 6. Let $g : \{z : |z - 1| < 2\} \to \mathbb{C}$ be holomorphic such that $g(z) = \sum_{n=1}^{\infty} a_n (1 - z)^n$ for $|z - 1| < 2$. If $n_{\infty} \frac{\alpha}{(n + 1)^{5/4}} < \infty$, then $g \in \mathcal{P}$ and

$$\|g\|_{\mathcal{P}} \leq C \sum_{n=0}^{\infty} \frac{2^n |a_n|}{(n + 1)^{5/4}}.$$  

Moreover, $g \in \mathcal{P}_0$ if and only if $\sum_{n=0}^{\infty} \frac{a_n}{n+1} = 1$.

Proof. Indeed, the norm $\|P_n\| = \frac{\sqrt{(2n)!}}{n!}$ (see [2, 3]). Then for $\varphi \in C_c(\mathbb{D})$

$$T_g \varphi(z) = \sum_{n=1}^{\infty} \frac{a_n}{(n + 1)^{5/4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \varphi(z),$$

and

$$\|g\|_{\mathcal{P}} \leq \sum_{n=0}^{\infty} \frac{a_n}{n+1} \frac{2^n}{(n + 1)^{5/4}}.$$  

Finally observe that, from Stirling's formula, $\frac{\sqrt{(2n)!}}{n!} \sim \frac{2^n}{(n + 1)^{n+\frac{1}{2}}}$. To conclude the result note that $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} < \infty$ and

$$T_g \varphi(z) = \sum_{n=1}^{\infty} \frac{a_n}{(n + 1)^{5/4}} \varphi(z),$$

for $\varphi \in A^2$.

Example 7. Let $h_\beta(z) = A_\beta (1 + z)^{-\beta}$ for $\beta > 0$ where $A_\beta = \frac{1 - \beta}{2 - \beta + 1}$ if $\beta \neq 1$ and $A_1 = (\log 2)^{-1}$. Then $h_\beta \in \mathcal{P}_0$ for $0 < \beta < 5/4$.

Proof. Since $\frac{1}{(1-w)^{\beta}} = \sum_{n=0}^{\infty} \beta_n w^n$ for $\beta > 0$, $|w| < 1$, where $\beta_n \sim (n + 1)^{\beta - 1}$, we have

$$h_\beta(z) = \frac{A_\beta}{2^{\beta}(1 - (1 - z)/2)^\beta} = \sum_{n=0}^{\infty} A_\beta 2^{-(n+\beta)} \beta_n (1 - z)^n.$$  

Now Proposition 6 implies $h_\beta \in \mathcal{P}$.

Note that

$$1 = \int_1^2 A_\beta s^{-\beta} \, ds = \int_0^1 h_\beta(r) \, dr = \sum_{n=0}^{\infty} \frac{A_\beta 2^{-(n+1)} \beta_n}{n + 1}.$$  

Apply again Proposition 6 to finish the proof.
Let us now give some necessary conditions that functions \( g \) in \( \mathcal{P} \) should satisfy.

**Theorem 8.** If \( g \in \mathcal{P} \), then

\[
\sup_{z \in D} \left\{ \int_D |g(\bar{w}\phi_w(z))|^2 dA(w) \right\}^{1/2} \leq 2 \|g\|_{\mathcal{P}},
\]

(2)

\[
\left( \int_0^1 |g(r)|^2 dr \right)^{1/2} \leq 2 \|g\|_{\mathcal{P}},
\]

(3)

\[
\left( \int_0^1 \left( \int_D \frac{|g(ru)|^2}{|1 - ru|^4} dA(u) \right)(1 - r^2)^2 r dr \right)^{1/2} \leq 2 \|g\|_{\mathcal{P}},
\]

(4)

**Proof.** If \( g \in \mathcal{P} \) and \( \varphi \in C_c(D) \) one has \( T_g \varphi \in A^2 \). Hence for each \( z \in D \)

\[ |T_g \varphi(z)| \leq \frac{\|T_g \varphi\|_2}{(1 - |z|)} \leq \frac{\|g\|_{\mathcal{P}} \|\varphi\|_2}{(1 - |z|)}. \]

Therefore

\[ \int_D g(\bar{w}\phi_w(z)) dA(w) \leq \frac{\|g\|_{\mathcal{P}} \|\varphi\|_2}{(1 - |z|)}. \]

Then by duality,

\[
\left\{ \int_D \left| g(\bar{w}\phi_w(z)) \right|^2 dA(w) \right\}^{1/2} \leq \frac{\|g\|_{\mathcal{P}} \|\varphi\|_2}{(1 - |z|)} \leq \frac{2 \|g\|_{\mathcal{P}}}{(1 - |z|^2)}. \]

(5)

Let us show the following formula:

\[ \overline{\phi_z(u)} \phi_{\phi_z(u)}(z) = u \phi_u(z). \]

(6)

Indeed, since

\[ 1 - |\phi_z(u)|^2 = \frac{(1 - |z|^2)(1 - |u|^2)}{|1 - \bar{z}u|^2}, \]

then

\[ \psi_z(\phi_z(u)) = \frac{1 - |\phi_z(u)|^2}{1 - \overline{\phi_z(u)} z} = \frac{(1 - |u|^2)}{(1 - \bar{z}u)} = \frac{\psi_z(u)}{\phi_u(z)}. \]

(7)

Now (6) follows from (1) and (7)

\[
\overline{\phi_z(u)} \phi_{\phi_z(u)}(z) = 1 - \psi_z(\phi_z(u)) = u \phi_u(z).
\]

(8)

Changing the variable \( u = \phi_z(w) \) in (5) and using (6) we obtain

\[
\left\{ \int_D \left| g(\bar{w}\phi_w(z)) \right|^2 dA(w) \right\}^{1/2} \leq 2 \|f\|_{\mathcal{P}}.
\]

(9)

Now replacing \( u \) and \( \bar{z} \) by \( \bar{w} \) and \( z \) respectively the inequality (2) is achieved.

Part (3) follows selecting \( z = 0 \) in (2).
Part (4) follows from (2) replacing the supremum by an integral over $D$ and changing the variable $u = \phi_w(z)$,

\[
\int_D \int_D |g(\bar{\phi}_w(z))|^2 \, dA(w) \, dA(z) = \int_D \left( \int_D \frac{|g(\bar{\psi}_u(z))|^2}{1 - |\psi_u|^2} \, dA(u) \right) (1 - |w|^2)^2 \, dA(w) \\
= \int_D \left( \int_D \frac{|g(|\psi_u|u)|^2}{1 - |\psi_u|^2} \, dA(u) \right) (1 - |w|^2)^2 \, dA(w) \\
= \int_0^1 \left( \int_D \frac{|g(ru)|^2}{1 - |ru|^2} \, dA(u) \right) (1 - r^2)^2 \, dr. \quad \square
\]

Remark 9. $(\mathcal{P}, \| \cdot \|_{\mathcal{P}})$ is a normed space and $\Phi(g) = \int_0^1 g(r) \, dr \in \mathcal{P}^\ast$. Indeed, the only condition which needs a proof is the fact that $\|g\|_{\mathcal{P}} = 0$ implies $g = 0$. It follows from (3) that if $\|g\|_{\mathcal{P}} = 0$, then $g(r) = 0$ for $0 < r < 1$. Hence by analytic continuation, $g(z) = 0$ for $z \in D$. Notice also that (3) implies $\|\Phi\| \leq 2$.

Remark 10. The space $\mathcal{P}$ is not invariant under under rotations. Given $\theta \in [0, 2\pi)$ denote $R_{\theta}(f)(z) = f(\bar{e}^{i\theta}z)$ for $f \in H(D)$. Observe that $R_{\theta}T_g(\varphi) = T_g(R_{\theta}\varphi)$. However, “$T_g$ is bounded in $L^2(D)$” does not imply $T_{R_{\theta}g}$ is bounded in $L^2(D)$”. For instance, the function $g(z) = (1 + z)^{-1/2}$ belongs to $\mathcal{P}$, but by (3), its reflection $g(z) = (1 - z)^{-1/2} \notin \mathcal{P}$.

Let us now also give some necessary conditions to belong to the class $\mathcal{P}_0$.

Theorem 11. If $g \in \mathcal{P}_0$ then

\[
\int_D g(u\bar{\phi}_w(z))\psi(u) \, dA(u) = \psi(0)
\]

for all $\psi \in A_2$ and $z \in D$. In particular,

(i) If $g \in \mathcal{P}_0$ then $\int_0^1 g(r) \, dr = 1$.

(ii) Let $S_2 = \{ \bar{z}(1 - |z|^2) \varphi(z) : \varphi \in A^2 \}$. If $g \in \mathcal{P}_0$ and $g' \in \mathcal{P}$ then $S_2 \subset \ker(T_{g'})$.

Proof. Assume

\[
\int_D g(\bar{\phi}_w(z)) \frac{\varphi(w)}{(1 - \bar{w}z)^2} \, dA(w) = \varphi(z)
\]

for all $\varphi \in A^2$. Given $\psi \in A^2$ and $z \in D$, consider $\varphi(w) = \psi(\phi_z(w)) \frac{(1 - |\psi_z|^2)^2}{(1 - \bar{w}z)^2}$. Clearly $\varphi \in A_2$ and $\|\varphi\|_2 = (1 - |\psi|^2)\|\psi\|_2$. From the assumption,

\[
\int_D g(\bar{\phi}_w(z))\psi(\phi_z(w)) \frac{(1 - |\psi_z|^2)^2}{|1 - \bar{w}z|^4} \, dA(w) = \psi(0).
\]

for all $\psi \in A^2$ and $z \in D$.

Now changing the variable $u = \phi_z(w)$, and using (6), one gets

\[
\int_D g(u\bar{\phi}_z(w))\psi(u) \, dA(u) = \psi(0)
\]

for all $\psi \in A_2$ and $z \in D$. Finally changing $u$ by $\bar{w}$ one obtains

\[
\int_D g(\bar{\psi}_w(z))\psi(\bar{w}) \, dA(w) = \psi(0)
\]

for all $\psi \in A_2$ and $z \in D$. (i) follows selecting $\psi = 1$ and $z = 0$ in (10).
Differentiating in (10) with respect to $z$ one obtains
\[
\int_D g'(\bar{w}\varphi_w(z)) \frac{-\bar{w}(1-|w|^2)}{(1-\bar{w}z)^2} \psi(\bar{w}) \, dA(w) = T_{g'}(\psi_1) = 0
\]
where $\varphi_1(u) = -\bar{u}(1-|u|^2)\varphi(\bar{u})$. Hence (ii) is finished. \hfill $\square$

Let us now show that $(\mathcal{P}, || \cdot ||_\mathcal{P})$ is complete. For such a purpose, let us define $h_z : D \to H$ by
\[
h_z(w) = \frac{1}{\psi_z(w)} = \frac{1-z\bar{w}}{1-|w|^2},
\]
and let us mention that
\[
D_1 = \{ \frac{1-|w|^2}{1-z\bar{w}} : z, w \in D \} = \{ \psi_z(w) : z, w \in D \}.
\]

**Lemma 12.** For every $\xi \in H$, there exist $0 \leq \alpha < 1$ and $w \in D$ such that $\xi = h_\alpha(w)$ and $h_\alpha$ is an diffeomorphism of a neighborhood $U$ of $w$ onto an open neighborhood of $\xi$.

**Proof.** For $0 \leq r, \alpha < 1$ fixed,
\[(11)\]
\[
h_\alpha(re^{i\theta}) = \frac{1}{1-r^2} - \frac{r\alpha}{1-r^2} e^{-i\theta}
\]
describes the circle $C_{r,\alpha}$ centered at the complex number $\frac{1}{1-r^2}$ with radius $\frac{r\alpha}{1-r^2}$. Let $\xi \in H$. To prove that $\xi \in h_\alpha(D)$ it is enough to see that $\xi \in C_{r,\alpha}$ for some $0 \leq r, \alpha < 1$. Let
\[(12)\]
\[
\beta = \frac{1}{r^2} \left[ (1-r^2)^2 |\xi|^2 + 1 - 2(1-r^2) \text{Re}\xi \right] = \frac{|(1-r^2)|\xi|^2 - 1|^2}{r^2}.
\]
It is clear that $\beta \geq 0$ and $\beta < 1 \Leftrightarrow (1-r^2)|\xi|^2 + 1 < 2\text{Re}\xi$.

Also, since $\xi \in H$, we have for some $\varepsilon > 0$ that $2\text{Re}\xi > 1 + \varepsilon$. Hence if $|\xi|^2 < \frac{\varepsilon}{(1-r^2)}$ then $\beta < 1$. We conclude that there exists $r_0$ for which $0 \leq \beta < 1$ provided $r_0 < r < 1$. Then if $r_0 < r < 1$ and $\alpha = \sqrt{\beta}$ we have $0 \leq \alpha < 1$ and
\[
\left| \frac{\xi - 1}{1-r^2} \right| = \frac{r\alpha}{1-r^2},
\]
that is $\xi \in C_{r,\alpha}$. Hence there exists $\theta_r$ and $0 \leq \alpha_r < 1$ such that $h_{\alpha_r}(re^{i\theta_r}) = \xi$.

To find $\theta_r$ explicitly, we let $\varphi_r = \pi - \theta_r$. From (11) we can write
\[
\xi = \frac{1}{1-r^2} + \frac{r\alpha_r}{1-r^2} e^{i\varphi_r}.
\]
Hence $\varphi_r$ is the argument of $\xi$ in polar coordinates centered at the complex number $\frac{1}{1-r^2}$. Then if $\frac{1}{1-r^2} \geq \text{Re}(\xi)$,
\[
\sin \theta_r = \sin \varphi_r = \frac{\text{Im}(\xi)}{r\alpha_r} (1-r^2)
\]
\[(13)\]
\[
\cos \theta_r = -\cos \varphi_r = \frac{(1-r^2)}{r\alpha_r} \left( \frac{1}{1-r^2} - \text{Re}(\xi) \right) = \frac{1-(1-r^2)}{r\alpha_r} \text{Re}(\xi).
\]

Now we will prove that possibly except for a finite number of values of $r \geq r_0$, the jacobian matrix $Dh_{\alpha_r}(re^{i\theta_r})$ is not singular, where $\alpha_r$ and $\theta_r$ are chosen so that
$h_{\alpha}(re^{i\theta}) = \xi$ as before. To this end, it is enough to see that the set of values of $r$ for which the vectors

$$\left( \frac{\partial h_{\alpha}}{\partial \rho}(pe^{i\theta}) \right)_{\rho=r} \quad \text{and} \quad \frac{1}{r} \left( \frac{\partial h_{\alpha}}{\partial \theta}(re^{i\theta}) \right)_{\theta=\theta_r}$$

are linearly dependent is finite.

We have

$$\frac{\partial h_{\alpha}}{\partial \rho}(pe^{i\theta}) = \left( \frac{2\rho}{(1-\rho^2)^2} - \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2} \cos \theta, \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2} \sin \theta \right),$$

$$\frac{1}{\rho} \left( \frac{\partial h_{\alpha}}{\partial \theta}(re^{i\theta}) \right) = \left( \frac{\alpha}{(1-\rho^2)} \sin \theta, \frac{\alpha}{(1-\rho^2)} \cos \theta \right),$$

and the jacobian of $h_{\alpha}$

$$J_{h_{\alpha}}(pe^{i\theta}) = \det \left[ \frac{\partial h_{\alpha}}{\partial \rho}(pe^{i\theta}) \frac{1}{\rho} \frac{\partial h_{\alpha}}{\partial \theta}(pe^{i\theta}) \right]$$

$$= \det \begin{bmatrix} \frac{2\rho}{(1-\rho^2)^2} & \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2} \cos \theta \\ \frac{\alpha}{(1-\rho^2)} \sin \theta & \frac{\alpha}{(1-\rho^2)} \cos \theta \end{bmatrix}$$

$$= \frac{\alpha}{(1-\rho^2)^3} \left( 2\rho \cos \theta - \alpha(1+\rho^2) \right).$$

If $2r \cos \theta_r - \alpha_r(1+r^2) = 0$, then multiplying this equation by $\alpha_r r^2$ we obtain

$$2r^2 \alpha_r r \cos \theta_r - \alpha_r^2 r^2 (1+r^2) = 0.$$

However, from (12) and (13) we see that $2r^2 \alpha_r r \cos \theta_r - \alpha_r^2 r^2 (1+r^2)$ is a polynomial of degree 6 in the variable $r$. We conclude that the vectors in (14) are linearly dependent for six values of $r$ at the most and the proof of the lemma is complete. \(\square\)

**Theorem 13.** $\mathcal{P}$ is a Banach space.

**Proof.** Let $g \in \mathcal{P}$. We have by Theorem 8 that

$$\sup_{z \in \mathcal{D}} \left\{ \int_{\mathcal{D}} |g(\bar{w}\phi_w(z))|^2 \, dA(w) \right\}^{1/2} \leq 2 \|g\|_{\mathcal{P}}.$$  

Fix $\xi \in \mathcal{D}$. Since $\psi_z = 1/h_z$, the local invertibility statement of Lemma 12 holds for the family of functions $1 - \psi_z$ taking $\xi \in \mathcal{D}$, namely, there exist $\alpha \in (0, 1)$, $w_\xi \in \mathcal{D}$ and open neighborhoods $U$ and $V$ of $\xi$ and $w_\xi$ respectively, such that $1 - \psi_z$ is a diffeomorphism of $V$ into $U$.

Hence

$$\left\{ \int_{\mathcal{D}} |g(u)|^2 \, dA(u) \right\}^{1/2} = \left\{ \int_{\mathcal{D}} |g(1 - \psi_{\alpha}(w))|^2 \, |J\psi_{\alpha}(w)| \, dA(w) \right\}^{1/2} \leq C(\xi) \left\{ \int_{\mathcal{D}} |g(\bar{w}\phi_w(\alpha))|^2 \, dA(w) \right\}^{1/2} \leq C(\xi) \|g\|_{\mathcal{P}}.$$
It follows that
\[ \left\{ \int_K |g(u)|^2 \, dA(u) \right\}^{1/2} \leq C_K \|g\|_\mathcal{P}, \]
for every compact set \( K \subset D \). This implies that
\begin{equation}
(18) \quad \sup_{u \in K} |g(u)| \leq \|g\|_\mathcal{P} C_K'.
\end{equation}
If \( \{g_n\} \) is a Cauchy sequence in \( \mathcal{P} \), we have by (18) that \( \{g_n\} \) converges uniformly on compact sets of \( D \) to a holomorphic function \( g \).

Let us show that \( g \in \mathcal{P} \) and \( \|g_n - g\|_\mathcal{P} \to 0 \). Note first that for each \( \varphi \in C_c(D) \) we have
\[ T_g \varphi(z) \to T_g \varphi(z), \quad z \in D. \]
Using the fact \( \sup_{n \in \mathbb{N}} \|g_n\|_\mathcal{P} = M < \infty \) and Fatou’s lemma one gets
\[ \|T_g \varphi\|_2^2 \leq \liminf_{n \to \infty} \|T_{g_n} \varphi\|_2^2 \leq M \|\varphi\|_2^2. \]
Hence \( g \in \mathcal{P} \). On the other hand, given \( \varepsilon > 0 \) there exists \( n_0 \) such that
\[ \|T_{g_n} \varphi - T_{g_m} \varphi\|_2 \leq \|g_n - g_m\|_\mathcal{P} < \varepsilon \]
for \( m, n \geq n_0 \) and \( \|\varphi\|_2 = 1 \). Applying Fatou’s lemma again we conclude that
\[ \|T_{g_n} \varphi - T_g \varphi\|_2 \leq \varepsilon \]
for \( n \geq n_0 \). Therefore \( g_n \to g \) in \( \mathcal{P} \).

\[ \square \]

3. Main results

Let us now describe the norm in \( \mathcal{P} \) in a more explicit way. We shall use the formulation of the space given in [1].

**Theorem 14.** Let \( g \in \mathcal{H}(D) \) and put \( F(\xi) = \frac{1}{\xi} g(1 - \frac{1}{\xi}) \). Then \( g \in \mathcal{P} \) if and only if
\[ \sup_j \frac{1}{j! \sqrt{j + 1}} \left( \int_1^\infty [(x - 1)x^j |xF^{(j)}(x)|^2 \, dx \right)^{1/2} < \infty. \]

**Proof.** We use the expression
\[ T_g \varphi(z) = \int_D F \left( \frac{1 - z \overline{w}}{1 - |w|^2} \right) \varphi(w) \frac{dA(w)}{(1 - |w|^2)^2}. \]
Consider the space \( M \) of functions of the form
\[ \varphi = \sum_{\text{finite}} \varphi_j(r) e^{ij\theta}, \]
with \( \varphi_j \in L^2((0,1), r \, dr) \). Then \( M \) is a dense subspace of \( L^2(D) \).

For \( z \in D \) and \( 0 \leq r < 1 \) fixed, let \( f(\zeta) = F \left( \frac{1 - rz \overline{\zeta}}{1 - r^2} \right) \), which is holomorphic on \( \overline{D} \). We have
\[ f(\zeta) = F \left( \frac{1 - rz \overline{\zeta}}{1 - r^2} \right) = \sum_{j \geq 0} \frac{1}{j!} \left( \frac{-rz}{1 - r^2} \right)^j F^{(j)} \left( \frac{1}{1 - r^2} \right) \zeta^j, \quad |\zeta| \leq 1. \]
Then for $g \in M$,
\[
\int_0^{2\pi} f(r e^{i\theta}) \varphi(r e^{i\theta}) \frac{d\theta}{2\pi} = \sum_{j \geq 0} \varphi_j(r) \frac{(-1)^j}{j!} \left( \frac{r}{1-r^2} \right)^j F^{(j)}(\frac{1}{1-r^2}) z^j,
\]
Hence
\[
T_g(\varphi)(z) = \sum_{j \geq 0} \gamma_j(\varphi_j) \sqrt{j+1} z^j,
\]
where $\gamma_j$ is the functional in $L^2((0,1), r \, dr)$ defined by
\[
\gamma_j(\varphi) = \frac{(-1)^j}{\sqrt{j+1}!} \int_0^1 \varphi(r) \left( \frac{r}{1-r^2} \right)^j F^{(j)}(\frac{1}{1-r^2}) \frac{r}{(1-r^2)^2} \, dr.
\]
Using the normalized Lebesgue measure $dA$, the set $\{\sqrt{j+1} z^j\}$ is an orthonormal basis for $A^2$, so we conclude that $T_g$ is bounded in $L^2(D)$ if and only if
\[
\left\| (\gamma_j(\varphi_j))_{j \geq 0} \right\|_{L^2} \leq C \|\varphi\|_{L^2(D)} = C \left( \sum_j \int |\varphi_j(r)|^2 \, r \, dr \right)^{1/2}.
\]
Using duality, this will hold if and only if
\[
\sup_{j \geq 0} \frac{1}{\sqrt{j+1}!} \left( \int_0^1 \left( \frac{r}{1-r^2} \right)^{2j} \left| F^{(j)}(\frac{1}{1-r^2}) \right|^2 \frac{r \, dr}{(1-r^2)^4} \right)^{1/2} < \infty.
\]
Making the change of variables $x = \frac{1}{1-r^2}$, the integrals above equal
\[
\frac{1}{2} \int_1^{\infty} \left[ (x-1)x^j |xF^{(j)}(x)|^2 \right] \, dx
\]
and the proof is complete. \hfill \Box

We can now give an alternative proof of a well known result.

**Corollary 15.** $P_\alpha$ is bounded on $L^2(D)$ if and only if $\alpha > -1/2$.

**Proof.** Consider $g_\alpha(z) = (1 - z)^\alpha$. Assume first that $g_\alpha \in \mathcal{P}$. Then (3) in Theorem 8 implies that $\int_0^1 (1-r)^{2\alpha} \, dr < \infty$ and therefore $\alpha > -1/2$.

Assume now that $\alpha > -1/2$. Since $F_\alpha(\xi) = \xi^{-m}$ with $m = 2+\alpha$ and $2m-3 > 0$, one has for $j \geq 0$ that
\[
F^{(j)}_\alpha(x) = (-1)^j m(m+1) \cdots (m+j)x^{-(m+j)} = (-1)^j \frac{\Gamma(m+j)}{\Gamma(m)} x^{-(m+j)}.
\]
Therefore
\[
\int_1^{\infty} \left[ (x-1)x^j |xF^{(j)}_\alpha(x)|^2 \right] \, dx = \int_1^{\infty} \left( 1 - \frac{1}{x} \right)^j (x^{j+1} F^{(j)}_\alpha(x))^2 \, dx
\]
\[
= \left( \frac{\Gamma(m+j)}{\Gamma(m)} \right)^2 \int_1^{\infty} \left( 1 - \frac{1}{x} \right)^j x^{-2m+4} \, dx
\]
\[
= \left( \frac{\Gamma(m+j)}{\Gamma(m)} \right)^2 \int_0^1 (1-r)^j r^{2m-4} \, dr
\]
\[
= \left( \frac{\Gamma(m+j)}{\Gamma(m)} \right)^2 B(2m-3, j+1).
\]
Using $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ one concludes that

$$\frac{1}{(j!)^2(j+1)} \int_1^\infty \frac{[(x-1)x]^j}{(m,j)^{2j(j+1)}} \, dx = \frac{B(2m-3, j+1)}{B(m,j)^{2j(j+1)}}.$$  

Finally since for $p$ fixed, $B(p,j) \sim j^{-p}$ one obtains

$$\frac{B(2m-3, j+1)}{B(m,j)^{2j(j+1)}} \sim 1. \quad \square$$

**Example 16.** In Example 7 it was shown that, for $0 < \beta < 5/4$, $g(z) = (1 + z)^{-\beta} \in \mathcal{P}$ (which corresponds to $F(\xi) = \frac{\xi^{\beta-2}}{(2\xi-1)^\beta}$). Let us show, for instance, that $g(z) = (1 + z)^{-2} \notin \mathcal{P}$. In this case $F(\xi) = \frac{1}{(2\xi-1)^2}$ and

$$F^{(j)}(\xi) = \frac{(-1)^j(j+1)!2^j}{(2\xi-1)^{2+j}}.$$  

Since $\frac{1}{2} \leq x - 1 \leq x$ for $x \geq 2$ we have

$$\left( \int_2^\infty (x(x-1))^j |xF^{(j)}(x)|^2 \, dx \right)^{1/2} \sim 2^j(j+1)! \left( \int_2^\infty \frac{x^{2j+2}}{(2x-1)^{4+2j}} \, dx \right)^{1/2} \sim 2^j(j+1)!.$$  

Hence the condition in Theorem 14 does not hold.

The conditions

$$\sup_{j \geq 0} \frac{1}{j!} \int_1^\infty |(x-1)^j F^{(j)}(x)| \, dx < \infty, \tag{21}$$

$$\lim_{x \to \infty} x^{j+1} F^{(j)}(x) = 0 \tag{22}$$

were introduced in [1]. These conditions imply that on the space of all the holomorphic functions $\phi$ such that $S_F \phi$ is well defined, the operator $S_F$ is a constant multiple of the identity. Now we will see that (21) and (22) hold for every $g \in \mathcal{P}$ which allows to show the following result.

**Theorem 17.** Let $g \in \mathcal{P}$ and $c_0 = \int_0^1 g(r) \, dr$. Then

$$T_g(\phi) = c_0 \phi, \quad \phi \in A^2.$$  

**Proof.** Let us notice first that $(x-1)^j F^{(j)}(x) \in L^1([1, \infty), dx)$ for $j \geq 0$. Indeed,

$$\int_1^\infty |x-1|^j |F^{(j)}(x)| \, dx = \int_1^\infty |x(x-1)|^j |xF^{(j)}(x)| \frac{dx}{x^{j+1}}$$

$$\leq \left( \int_1^\infty (x(x-1))^j |xF^{(j)}(x)|^2 \, dx \right)^{1/2} \left( \int_1^\infty \frac{(x(x-1))^j}{x^{2j+2}} \, dx \right)^{1/2}$$

$$= \left( \int_1^\infty (x(x-1))^j |xF^{(j)}(x)|^2 \, dx \right)^{1/2} \left( \int_0^1 (1-r)^j \, dr \right)^{1/2}$$

$$= \frac{1}{\sqrt{j+1}} \left( \int_1^\infty |x(x-1)|^j |xF^{(j)}(x)|^2 \, dx \right)^{1/2} \leq C j! \|g\|_{\mathcal{P}}.$$  

Applying (19) in Theorem 14 to $\phi(z) = \sum_{j=0}^N a_j z^j$ one obtains

$$T_g \phi = \sum_{j=0}^N c_j a_j z^j \tag{23},$$
and 
\[ c_j = \frac{(-1)^j}{j!} \int_1^\infty (x-1)^j F^{(j)}(x) \, dx, \]
where \( c_j \) is well defined. As in [1, Th. 1] we have by integration by parts
\[ c_j - c_{j+1} = \frac{(-1)^j}{(j+1)!} \lim_{x \to \infty} (1-x)^{j+1} F^{(j)}(x). \]

Let us now show that \( \lim_{x \to \infty} (1-x)^{j+1} F^{(j)}(x) = 0 \). Note first that \((x-1)^{j+1} F^{(j)}(x) \in L^2([1, \infty), dx)\) for \( j \geq 0 \). Indeed
\[ \int_1^\infty |(x-1)^{j+1} F^{(j)}(x)|^2 \, dx \leq \int_1^\infty |x(x-1)^j |x F^{(j)}(x)|^2 \, dx \leq C(j+1)(j!)^2. \]
In particular \((x-1)^j F^{(j)}(x) \in L^2([1, \infty), dx)\) for \( j \geq 1 \). From Cauchy–Schwarz and the previous estimates one has that if \( f_j(x) = [(x-1)^j F^{(j)}(x)]^2 \), then \( (f_j)' \in L^1([1, \infty)) \) for every \( j \geq 0 \). Therefore writing
\[ |(x-1)^j F^{(j)}(x)|^2 = \int_1^x (f_j)'(y) \, dy \]
we see that the \( \lim_{x \to \infty} (x-1)^j F^{(j)}(x) \) exists and by (24) it vanishes for all \( j \).
Hence (23) becomes \( T_g(\varphi) = c_0 \varphi \) where
\[ c_0 = \int_1^\infty F(x) \, dx = \int_1^\infty g(1-\frac{1}{x}) \frac{dx}{x^2} = \int_0^1 g(r) \, dr. \]

**Corollary 18.** Let \( g \in \mathcal{P} \). Then \( A^2 \subset \text{Ker} \, T_g \) if and only if \( \int_0^1 g(r) \, dr = 0 \).

**Corollary 19.** Let \( \Phi(g) = \int_0^1 g(r) \, dr \) for \( g \in \mathcal{P} \). Then \( \mathcal{P}_0 = \Phi^{-1}(\{1\}) \).

**Corollary 20.** Let \( g \in \mathcal{P} \). If \( T_g \) is not identically zero in \( A^2 \) then there exists \( \lambda \neq 0 \) and \( g_0 \in \mathcal{P}_0 \) such that \( g = \lambda g_0 \).

**References**


Received 30 January 2009