APPROXIMATE IDENTITIES AND YOUNG TYPE INEQUALITIES IN VARIABLE
LEBESGUE–ORLICZ SPACES $L^{p(\cdot)}(\log L)^{q(\cdot)}$

Fumi-Yuki Maeda, Yoshihiro Mizuta and Takao Ohno

4-24 Furue-higashi-machi, Nishi-ku, Hiroshima 733-0872, Japan; fymaeda@h6.dion.ne.jp
Hiroshima University, Graduate School of Science, Department of Mathematics
Higashi-Hiroshima 739-8521, Japan; yomizuta@hiroshima-u.ac.jp
Hiroshima National College of Maritime Technology, General Arts
Higashino Oosakikamijima Toyotagun 725-0231, Japan; ohno@hiroshima-cmt.ac.jp

Abstract. Our aim in this paper is to deal with approximate identities in generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$. As a related topic, we also study Young type inequalities for convolution with respect to norms in such spaces.

1. Introduction

Following Cruz-Uribe and Fiorenza [2], we consider two variable exponents $p(\cdot): \mathbb{R}^n \to [1, \infty)$ and $q(\cdot): \mathbb{R}^n \to \mathbb{R}$, which are continuous functions. Letting $\Phi_{p(\cdot),q(\cdot)}(x,t) = \Phi_{p(x),q(x)}(log(c_0 + t))^{q(x)}$, we define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of all measurable functions $f$ on an open set $\Omega$ such that

$$\int_{\Omega} \Phi_{p(\cdot),q(\cdot)} \left( y, \left\{ \frac{|f(y)|}{\lambda} \right\} \right) dy < \infty$$

for some $\lambda > 0$; here we assume

(Φ) $\Phi_{p(\cdot),q(\cdot)}(x, \cdot)$ is convex on $[0, \infty)$ for every fixed $x \in \mathbb{R}^n$.

Note that (Φ) holds for some $c_0 \geq e$ if and only if there is a positive constant $K$ such that

$$K(p(x) - 1) + q(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

(see Appendix). Further, we see from (Φ) that $t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is nondecreasing in $t$.

We define the norm

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)},\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi_{p(\cdot),q(\cdot)} \left( y, \left\{ \frac{|f(y)|}{\lambda} \right\} \right) dy \leq 1 \right\}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$. Note that $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is a Musielak–Orlicz space [9]. Such spaces have been studied in [2, 8, 10]. In case $q(\cdot) = 0$ on $\mathbb{R}^n$, $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is denoted by $L^{p(\cdot)}(\Omega)$ [7]).

We assume throughout the article that our variable exponents $p(\cdot)$ and $q(\cdot)$ are continuous functions on $\mathbb{R}^n$ satisfying:

(p1) $1 \leq p_- := \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) =: p_+ < \infty$;
(p2) \(|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)}\) whenever \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^n\); 

(p3) \(|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}\) whenever \(|y| \geq |x|/2\); 

(q1) \(-\infty < q_- := \inf_{x \in \mathbb{R}^n} q(x) \leq \sup_{x \in \mathbb{R}^n} q(x) =: q_+ < \infty\); 

(q2) \(|q(x) - q(y)| \leq \frac{C}{\log(e + \log(e + 1/|x - y|))}\) whenever \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^n\) for a positive constant \(C\).

We choose \(p_0 \geq 1\) as follows: we take \(p_0 = p_-\) if \(t^{-p_-} \Phi_{p, q} (x, t)\) is uniformly almost increasing in \(t\); more precisely, if there exists \(C > 0\) such that \(s^{-p_-} \Phi_{p, q} (x, s) \leq C t^{-p_-} \Phi_{p, q} (x, t)\) whenever \(0 < s < t\) and \(x \in \mathbb{R}^n\). Otherwise we choose \(1 \leq p_0 < p_-\). Then note that \(t^{-p_0} \Phi_{p, q} (x, t)\) is uniformly almost increasing in \(t\) in any case.

Let \(\phi\) be an integrable function on \(\mathbb{R}^n\). For each \(t > 0\), define the function \(\hat{\phi}_t\) by \(\hat{\phi}_t (x) = t^{-n} \phi (x/t)\). Note that by a change of variables, \(\|\hat{\phi}_t\|_{L^1, \mathbb{R}^n} = \|\phi\|_{L^1, \mathbb{R}^n}\). We say that the family \(\{\phi_t\}\) is an approximate identity if \(\int_{\mathbb{R}^n} \phi (x) \, dx = 1\). Define the radial majorant of \(\phi\) to be the function

\[\hat{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|.\]

If \(\hat{\phi}\) is integrable, we say that the family \(\{\phi_t\}\) is of potential-type.

Cruz-Uribe and Fiorenza [1] proved the following result:

**Theorem A.** Let \(\{\phi_t\}\) be an approximate identity. Suppose that either

1. \(\{\phi_t\}\) is of potential-type, or  
2. \(\phi \in L^{(p, t)}(\mathbb{R}^n)\) and has compact support.

Then 

\[\sup_{0 < t \leq 1} \|\phi_t * f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}\]

and 

\[\lim_{t \to 0^+} \|\phi_t * f - f\|_{L^p(\mathbb{R}^n)} = 0\]

for all \(f \in L^p(\mathbb{R}^n)\).

Our aim in this note is to extend their result to the space \(L^p(\log L)^{q(\cdot)}(\Omega)\) of two variable exponents.

**Theorem 1.1.** Let \(\{\phi_t\}\) be a potential-type approximate identity. If \(f \in L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)\), then \(\{\phi_t * f\}\) converges to \(f\) in \(L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)\): 

\[\lim_{t \to 0} \|\phi_t * f - f\|_{L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)} = 0.\]

**Theorem 1.2.** Let \(\{\phi_t\}\) be an approximate identity. Suppose that \(\phi \in L^{(p_0, t)}(\mathbb{R}^n)\) and has compact support. If \(f \in L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)\), then \(\{\phi_t * f\}\) converges to \(f\) in \(L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)\):

\[\lim_{t \to 0} \|\phi_t * f - f\|_{L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)} = 0.\]

We show by an example that the conditions on \(\phi\) are necessary; see Remarks 3.5 and 3.6 below.

Finally, in Section 4, we give some Young type inequalities for convolution with respect to the norms in \(L^p(\log L)^{q(\cdot)}(\mathbb{R}^n)\).
2. The case of potential-type

Throughout this paper, let $C$ denote various positive constants independent of the variables in question.

Let us begin with the following result due to Stein [11].

**Lemma 2.1.** Let $1 \leq p < \infty$ and $\{\phi_t\}$ be a potential-type approximate identity. Then for every $f \in L^p(\mathbb{R}^n)$, $\{\phi_t \ast f\}$ converges to $f$ in $L^p(\mathbb{R}^n)$.

We denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$.

For a measurable set $E$, we denote by $|E|$ the Lebesgue measure of $E$.

The following is due to Lemma 2.6 in [8].

**Lemma 2.2.** Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_p(\cdot), q(\cdot)_{\mathbb{R}^n}} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$. Set

$$J = J(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy$$

and

$$L = L(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi_{p(\cdot), q(\cdot)}(y, f(y)) \, dy.$$ 

Then

$$J \leq C L^{1/p(x)} (\log(c_0 + L))^{-q(x)/p(x)},$$

where $C > 0$ does not depend on $x, r, f$.

Further we need the following result.

**Lemma 2.3.** Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ such that $(1 + |y|)^{-n-1} \leq f(y) \leq 1$ or $f(y) = 0$ for each $y \in \mathbb{R}^n$. Set

$$J = J(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy$$

and

$$L = L(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi_{p(\cdot), q(\cdot)}(y, f(y)) \, dy.$$ 

Then

$$J \leq C \{ L^{1/p(x)} + (1 + |x|)^{-n-1} \} ,$$

where $C > 0$ does not depend on $x, r, f$.

**Proof.** We have by Jensen’s inequality

$$J \leq \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)^{p(x)} \, dy \right)^{1/p(x)}$$

$$\leq \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2)} f(y)^{p(x)} \, dy \right)^{1/p(x)}$$

$$+ \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2)} f(y)^{p(x)} \, dy \right)^{1/p(x)}$$

$$= J_1 + J_2.$$
We see from (p3) that
\[
J_1 \leq C \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2)} f(y)^{p(y)} \, dy \right)^{1/p(x)}.
\]

Similarly, setting \( E_2 = \{ y \in \mathbb{R}^n : f(y) \geq (1 + |x|)^{-n-1} \} \), we see from (p3) that
\[
J_2 \leq C \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2) \setminus E_2} f(y)^{p(y)} \, dy \right)^{1/p(x)}
+ \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2) \setminus E_2} (1 + |x|)^{-p(x)(n+1)} \, dy \right)^{1/p(x)}
\leq C \left\{ \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)^{p(y)} \, dy \right)^{1/p(x)} + (1 + |x|)^{-(n+1)} \right\}.
\]

Since \( f(y) \leq 1 \), \( f(y)^{p(y)} \leq C\Phi_{p(q)}(f(y), f(y)) \). Hence, we have the required estimate. \( \square \)

By using Lemmas 2.2 and 2.3, we show the following theorem.

**Theorem 2.4.** If \( \{ \phi_t \} \) is of potential-type, then
\[
\| \phi_t * f \|_{\Phi_{p(q)}(\mathbb{R}^n)} \leq C \| \phi \|_{L^1(\mathbb{R}^n)} \| f \|_{\Phi_{p(q)}(\mathbb{R}^n)}
\]
for all \( t > 0 \) and \( f \in L^{p(q)}(\log L)^{q}(\mathbb{R}^n) \).

**Proof.** Suppose \( \| \hat{\phi} \|_{L^1(\mathbb{R}^n)} = 1 \) and take a nonnegative measurable function \( f \) on \( \mathbb{R}^n \) such that \( \| f \|_{\Phi_{p(q)}(\mathbb{R}^n)} \leq 1 \). Write
\[
f = f \chi_{\{ y \in \mathbb{R}^n : f(y) \geq 1 \}} + f \chi_{\{ y \in \mathbb{R}^n : (1 + |y|)^{-n-1} \leq f(y) \leq 1 \}} + f \chi_{\{ y \in \mathbb{R}^n : f(y) \leq (1 + |y|)^{-n-1} \}}
= f_1 + f_2 + f_3,
\]
where \( \chi_E \) denotes the characteristic function of a measurable set \( E \subset \mathbb{R}^n \).

Since \( \hat{\phi}_t \) is a radial function, we write \( \hat{\phi}(r) \) for \( \hat{\phi}_t(x) \) when \( |x| = r \). First note that
\[
|\phi_t * f(x)| \leq \int_{\mathbb{R}^n} \hat{\phi}_t(|x-y|) f_1(y) \, dy
= \int_0^\infty \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} f_1(y) \, dy \right) |B(x, r)| \, d(\hat{\phi}_t(r)),
\]
so that Jensen’s inequality and Lemma 2.2 yield
\[
\Phi_{p(q)}(x, |\phi_t * f_1(x)|)
\leq \int_0^\infty \Phi_{p(q)} \left( x, \frac{1}{|B(x, r)|} \int_{B(x, r)} f_1(y) \, dy \right) \, d(\hat{\phi}_t(r))
\leq C \int_0^\infty \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi_{p(q)}(y, f_1(y)) \, dy \right) \, d(\hat{\phi}_t(r))
= C(\hat{\phi}_t * g)(x),
\]
where \( g = \chi_{\{ y \in \mathbb{R}^n : f(y) \geq 1 \}} \).
where \( g(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y)) \). The usual Young inequality for convolution gives
\[
\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\hat{\phi}_t * f_1(x)|) \, dx \leq C \int_{\mathbb{R}^n} |\hat{\phi}_t * g(x)| \, dx
\]
\[
\leq C \|\hat{\phi}_t\|_{L^1,\mathbb{R}^n} \|g\|_{L^1,\mathbb{R}^n} \leq C.
\]

Similarly, noting that
\[
\frac{1}{|B(x,r)|} \int_{B(x,r)} f_2(y) \, dy \leq 1
\]
and applying Lemma 2.3, we derive the same result for \( f_2 \).

Finally, noting that \(|\hat{\phi}_t * f_3| \leq C \|\hat{\phi}_t\|_{L^1,\mathbb{R}^n} \leq C\), we obtain
\[
\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\hat{\phi}_t * f_3(x)|) \, dx \leq C \int_{\mathbb{R}^n} |\hat{\phi}_t * f_3(x)| \, dx
\]
\[
\leq C \|\hat{\phi}_t\|_{L^1,\mathbb{R}^n} \|f_3\|_{L^1,\mathbb{R}^n} \leq C,
\]
as required. \( \square \)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Given \( \varepsilon > 0 \), we find a bounded function \( g \) in \( L^{p(\cdot)(\log L)^{q(\cdot)}}(\mathbb{R}^n) \) with compact support such that \( \|f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} < \varepsilon \). By Theorem 2.4 we have
\[
\|\hat{\phi}_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}
\leq \|\hat{\phi}_t * (f - g)\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} + \|\hat{\phi}_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} + \|f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}
\leq C \varepsilon + \|\hat{\phi}_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n}.
\]
Since \( |\hat{\phi}_t * g| \leq \|g\|_{L^\infty,\mathbb{R}^n} \),
\[
\|\hat{\phi}_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C' \|\hat{\phi}_t * g - g\|_{L^1,\mathbb{R}^n} \to 0
\]
by Lemma 2.1. (Here \( C' \) depends on \( \|g\|_{L^\infty,\mathbb{R}^n} \). Hence
\[
\limsup_{t \to 0} \|\hat{\phi}_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq C \varepsilon,
\]
which completes the proof. \( \square \)

As another application of Lemmas 2.2 and 2.3, we can prove the following result, which is an extension of [4, Theorem 1.5] and [8, Theorem 2.7] (see also [6]).

Let \( Mf \) be the Hardy–Littlewood maximal function of \( f \).

**Proposition 2.5.** Suppose \( p_- > 1 \). Then the operator \( M \) is bounded from \( L^{p(\cdot)(\log L)^{q(\cdot)}}(\mathbb{R}^n) \) to \( L^{p(\cdot)(\log L)^{q(\cdot)}}(\mathbb{R}^n) \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbb{R}^n} \leq 1 \) and write \( f = f_1 + f_2 + f_3 \) as in the proof of Theorem 2.4. Take \( 1 < p_1 < p_- \) and apply Lemmas 2.2 and 2.3 with \( p(\cdot) \) and \( q(\cdot) \) replaced by \( p(\cdot)/p_1 \) and \( q(\cdot)/p_1 \), respectively. Then
\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_1(x)) \leq C[Mg_1(x)]^{p_1}
\]
and
\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_2(x)) \leq C \left\{ [Mg_1(x)]^{p_1} + (1 + |x|)^{-n-1} \right\},
\]
where \( g_1(y) = \Phi_{p(\cdot)/p_1,q(\cdot)/p_1}(y, f(y)) \). As to \( f_3 \), we have
\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_3(x)) \leq C[Mf_3(x)]^{p_1}.
\]
Then the boundedness of the maximal operator in \( L^{p_1}(\mathbb{R}^n) \) proves the proposition. \( \square \)
Remark 2.6. If $p_- > 1$, then the function $\Phi_{p(\cdot),q(\cdot)}$ is a proper $N$-function and our Proposition 2.5 implies that this function is of class $\mathcal{A}$ in the sense of Diening [5] (see [5, Lemma 3.2]). It would be an interesting problem to see whether “class $\mathcal{A}^*$” is also a sufficient condition or not for the boundedness of $M$ on $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

3. The case of compact support

We know the following result due to Zo [12]; see also [1, Theorem 2.2].

Lemma 3.1. Let $1 \leq p < \infty$, $1/p + 1/p' = 1$ and $\{\phi_t\}$ be an approximate identity. Suppose that $\phi \in L^p(\mathbb{R}^n)$ has compact support. Then for every $f \in L^p(\mathbb{R}^n)$, $\{\phi_t * f\}$ converges to $f$ pointwise almost everywhere.

Set

$$\overline{p}(x) = p(x)/p_0 \quad \text{and} \quad \overline{q}(x) = q(x)/q_0;$$

recall that $p_0 \in [1, p_-]$ is chosen such that $t^{-p_0}\Phi_{p(\cdot),q(\cdot)}(x, t)$ is uniformly almost increasing in $t$.

For a proof of Theorem 1.2, the following is a key lemma.

Lemma 3.2. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and let $\phi$ have compact support in $B(0, R)$ with $\|\phi\|_{L^{(p_0)'}, \mathbb{R}^n} \leq 1$. Set

$$F = F(x, t, f) = |\phi_t * f(x)|$$

and

$$G = G(x, t, f) = \int_{\mathbb{R}^n} |\phi_t(x - y)|\Phi_{\overline{p}(\cdot),\overline{q}(\cdot)}(y, f(y)) \, dy.$$ 

Then

$$F \leq CG^{1/\overline{p}(x)}(\log(c_0 + G))^{-\overline{q}(x)/\overline{p}(x)}$$

for all $0 < t \leq 1$, where $C > 0$ depends on $R$.

Proof. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and let $\phi$ have compact support in $B(0, R)$ with $\|\phi\|_{L^{(p_0)'}, \mathbb{R}^n} \leq 1$. By Hölder’s inequality, we have

$$G \leq \|\phi_t\|_{L^{(p_0)'}, \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Phi_{p(\cdot),q(\cdot)}(y, f(y)) \, dy\right)^{1/p_0} \leq t^{-n/p_0}.$$ 

First consider the case when $G \geq 1$. Since $G \leq t^{-n/p_0}$, for $y \in B(x, tR)$ we have by (p2)

$$G^{-p(y)} \leq G^{-p(x) + C/\log(e + (tR)^{-1})} \leq CG^{-p(x)}$$

and by (q2)

$$(\log(c_0 + G))^{q(y)} \leq C(\log(c_0 + G))^{q(x)}.$$
Hence it follows from the choice of \( p_0 \) that
\[
F \leq G^{1/p(x)} (\log(c_0 + G))^{-\overline{\tau}(x)/\overline{p}(x)} \int_{\mathbb{R}^n} |\phi_t(x - y)| \, dy
\]
\[
+ C \int_{\mathbb{R}^n} |\phi_t(x - y)| f(y) \left\{ \frac{f(y)}{G^{1/p(x)} (\log(c_0 + G))^{-\overline{\tau}(x)/\overline{p}(x)}} \right\}^{\overline{q}(y) - 1} \log(c_0 + f(y)) \log(c_0 + G^{1/p(x)} (\log(c_0 + G))^{-\overline{\tau}(x)/\overline{p}(x)}) \, dy
\]
\[
\leq C G^{1/p(x)} (\log(c_0 + G))^{-\overline{\tau}(x)/\overline{p}(x)}
\]
(cf. the proof of [8, Lemma 2.6]).
In the case \( G < 1 \), noting from the choice of \( p_0 \) that \( f(y) \leq C \Phi_{p(\cdot),q(\cdot)}(y, f(y)) \) for \( y \in \mathbb{R}^n \), we find
\[
F \leq C G \leq C G^{1/p(x)} \leq C G^{1/p(x)} (\log(c_0 + G))^{-\overline{\tau}(x)/\overline{p}(x)}.
\]
Now the result follows. \( \square \)

**Lemma 3.3.** Suppose that \( \|\phi\|_{L^{1,\mathbb{R}^n}} \leq 1 \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) with \( \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\cdot, \mathbb{R}^n)} \leq 1 \). Set
\[
I = I(x, t, f) = \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} |\phi_t(x - y)| f(y) \, dy
\]
and
\[
H = H(x, t, f) = \int_{\mathbb{R}^n} |\phi_t(x - y)| \Phi_{p(\cdot),q(\cdot)}(y, f(y)) \, dy.
\]
If \( A > 0 \) and \( H \leq H_0 \), then
\[
I \leq C (H^{1/p(x)} + |x|^{-A/p(x)})
\]
for \( |x| > 1 \) and \( 0 < t \leq 1 \), where \( C > 0 \) depends on \( A \) and \( H_0 \).

**Proof.** Suppose that \( \|\phi\|_{L^{1,\mathbb{R}^n}} \leq 1 \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) with \( \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\cdot, \mathbb{R}^n)} \leq 1 \).

Let \( |x| > 1 \). In the case \( H_0 \geq H \geq |x|^{-A} \) with \( A > 0 \), we have by (p3)
\[
H^{-p(y)} \leq C H^{-p(x)} \frac{C}{\log(e + |x|)} \leq C H^{-p(x)}
\]
for \( |y| \geq |x|/2 \). Hence we find by (\( \Phi \))
\[
I \leq C \left\{ H^{1/p(x)} + \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} |\phi_t(x - y)| f(y) \right\}^{\overline{q}(y) - 1} \frac{\log(c_0 + f(y))}{\log(c_0 + H^{1/p(x)})} \, dy
\]
\[
\leq C H^{1/p(x)}.
\]
Next note from (p3) that
\[
|x|^{p(y)} \leq |x|^{p(x) + C/\log(e + |x|)} \leq C |x|^{p(x)}
\]
for \(|y| \geq |x|/2\). Hence, when \(H \leq |x|^{-A}\), we obtain by (\(\Phi\))
\[
I \leq C \left\{ |x|^{-A/p(x)} + \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} |\phi_t(x-y)|f(y) \right\}
\[
\cdot \left\{ \frac{f(y)}{|x|^{-A/p(x)}} \right\}^{p(y)-1} \left\{ \frac{\log(c_0 + f(y))}{\log(c_0 + |x|^{-A/p(x)})} \right\}^{q(y)} dy
\]
\[
\leq C |x|^{-A/p(x)},
\]
which completes the proof. 

\(\Box\)

**Theorem 3.4.** Suppose that \(\phi \in L^{(p_0)/1}(\mathbb{R}^n)\) has compact support in \(B(0, R)\). Then
\[
\|\phi_t \ast f\|_{L^{(p_0)/1}(\mathbb{R}^n)} \leq C \|\phi\|_{L^{(p_0)/1}(\mathbb{R}^n)} \|f\|_{L^{(p_0)/1}(\mathbb{R}^n)}
\]
for all \(0 < t \leq 1\) and \(f \in L^{(p_0)/1}(\mathbb{R}^n)\), where \(C > 0\) depends on \(R\).

**Proof.** Let \(f\) be a nonnegative measurable function on \(\mathbb{R}^n\) such that \(\|f\|_{L^{(p_0)/1}(\mathbb{R}^n)} \leq 1\) and let \(\phi\) have compact support in \(B(0, R)\) with \(\|\phi\|_{L^{(p_0)/1}(\mathbb{R}^n)} \leq 1\). Write
\[
f = f \chi_{\{y \in \mathbb{R}^n : f(y) \geq 1\}} + f \chi_{\{y \in \mathbb{R}^n : f(y) < 1\}} = f_1 + f_2.
\]
We have by Lemma 3.2,
\[
|\phi_t \ast f_1(x)| \leq C (|\phi_t| \ast g(x))^{p_0/p(x)} (\log(c_0 + |\phi_t| \ast g(x)))^{-q(x)/p(x)},
\]
where \(g(y) = \Phi_{p_0/p(x)}(y, f(y)) = \Phi_{p_0/p(x)}(y, f(y))^{1/p_0}\), so that
\[
(3.1) \quad \Phi_{p_0/p(x)}(x, |\phi_t \ast f_1(x)|) \leq C (|\phi_t| \ast g(x))^{p_0}.
\]
Hence, since \(g \in L^{p_0}(\mathbb{R}^n)\), the usual Young inequality for convolution gives
\[
\int_{\mathbb{R}^n} \Phi_{p_0/p(x)}(x, |\phi_t \ast f_1(x)|) dx \leq C \int_{\mathbb{R}^n} (|\phi_t| \ast g(x))^{p_0} dx
\]
\[
\leq C (\|\phi_t\|_{L^{1}(\mathbb{R}^n)} \|g\|_{L^{p_0}(\mathbb{R}^n)})^{p_0} \leq C.
\]

Next we are concerned with \(f_2\). Write
\[
f_2 = f_2 \chi_{B(0, R)} + f_2 \chi_{\mathbb{R}^n \setminus B(0, R)} = f_2' + f_2''.
\]
Since \(|\phi_t \ast f_2(x)| \leq C\) on \(\mathbb{R}^n\), we have
\[
\int_{B(0, 2R)} \Phi_{p_0/p(x)}(x, |\phi_t \ast f_2(x)|) dx \leq C.
\]
Further, noting that \(\phi_t \ast f_2' = 0\) outside \(B(0, 2R)\), we find
\[
\int_{\mathbb{R}^n} \Phi_{p_0/p(x)}(x, |\phi_t \ast f_2'(x)|) dx \leq C.
\]
Therefore it suffices to prove
\[
\int_{\mathbb{R}^n \setminus B(0, R)} \Phi_{p_0/p(x)}(x, |\phi_t \ast f_2''(x)|) dx \leq C.
\]
Thus, in the rest of the proof, we may assume that \(0 \leq f < 1\) on \(\mathbb{R}^n\) and \(f = 0\) on \(B(0, R)\). Note that
\[
\int_{B(0, |x|/2)} \phi_t(x-y)f(y) dy = 0
\]
for $|x| > 2R$. Hence, applying Lemma 3.3, we have
\[|\phi_t * f(x)| \leq C(|\phi_t| * h(x) + |x|^{-A})\]
for $|x| > 2R$, where $h(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Thus the integration yields
\[
\int_{\mathbb{R}^n \setminus B(0, 2R)} |\phi_t * f(x)|^{p(x)} dx \leq C,
\]
which completes the proof.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Given $\varepsilon > 0$, choose a bounded function $g$ with compact support such that $\|f - g\|_{P_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \varepsilon$. As in the proof of Theorem 1.1, using Theorem 3.4 this time, we have
\[
\|\phi_t * f - f\|_{P_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\varepsilon + \|\phi_t * g - g\|_{P_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\]
Obviously, $g \in L^p(\mathbb{R}^n)$. Hence by Lemma 3.1, $\phi_t * g \to g$ almost everywhere in $\mathbb{R}^n$. Since there is a compact set $S$ containing all the supports of $\phi_t * g$,
\[
\|\phi_t * g - g\|_{P_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C'\|\phi_t * g - g\|_{L^{p+1}(\mathbb{R}^n)}
\]
with $C'$ depending on $|S|$, and the Lebesgue convergence theorem implies $\|\phi_t * g - g\|_{L^{p+1}(\mathbb{R}^n)} \to 0$ as $t \to \infty$. Hence
\[
\limsup_{t \to 0} \|\phi_t * f - f\|_{P_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\varepsilon,
\]
which completes the proof.

**Remark 3.5.** In Theorem 1.2 (and in Theorem A), the condition $\phi \in L^{(p+)}(\mathbb{R}^n)$ cannot be weakened to $\phi \in L^q(\mathbb{R}^n)$ for $1 \leq q < (p_+)'$. In fact, for given $p_1 > 1$ and $1 \leq q < (p_1)'$, we can find a smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that $p_- = p_1$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\phi \in L^q(\mathbb{R}^n)$ having compact support for which
\[
\|\phi * f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \infty.
\]
For this, let $a \in \mathbb{R}^n$ be a fixed point with $|a| > 1$ and let $p_2$ satisfy
\[
\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{q}.
\]
Then choose a smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that
\[
p(x) = p_1 \text{ for } x \in B(0, 1/2), \quad p(x) = p_2 \text{ for } x \in B(0, 1),
\]
p_+ = p_1 and $p(x) = \text{const. outside } B(0, |a| + 1)$. Take
\[
\phi_j = j^{n/q} \chi_{B(a, j^{-1})} \quad \text{and} \quad f_j = j^{n/p_1} \chi_{B(0, j^{-1})}, \quad j = 2, 3, \ldots.
\]
Then
\[
\|\phi_j\|_{L^{p_1}(\mathbb{R}^n)} = C < \infty \quad \text{and} \quad \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f_j\|_{L^{p_1, B(0, 1/2)}} = C < \infty.
\]
Note that if $x \in B(a, j^{-1})$, then
\[
\phi_j * f_j(x) = j^{n/q + n/p_1} |B(a, j^{-1}) \cap B(x, j^{-1})| \geq Cj^{n/q + n/p_1}j^{-n},
\]
so that
\[\int_{\mathbb{R}^n} \{\phi_j * f_j(x)\}^{p(x)} \, dx \geq \int_{B(a,j-1)} \{\phi_j * f_j(x)\}^{p(x)} \, dx
\geq C j^{p_2(n/q + (p_1 - n)/p_2)} j^{-n}
= C j^{p_2 n(1/q - 1/(p_1)' - 1/p_2)}.
\]

Now consider
\[\phi = \sum_{j=2}^{\infty} j^{-2} \phi_{2j} \quad \text{and} \quad f = \sum_{j=2}^{\infty} j^{-2} f_{2j}.
\]

Then \(\phi \in L^q(\mathbb{R}^n)\) and \(f \in L^{(\cdot)}(\mathbb{R}^n)\). On the other hand,
\[\int_{\mathbb{R}^n} \{\phi \ast f(x)\}^{p(x)} \, dx \geq j^{-4} \int_{\mathbb{R}^n} \{\phi_{2j} \ast f_{2j}(x)\}^{p(x)} \, dx
\geq C j^{-4} 2^{pn j(1/q - 1/(p_1)' - 1/p_2)} \to \infty
\]
as \(j \to \infty\). Hence, \(\|\phi \ast f\|_{L^{(\cdot)}(\mathbb{R}^n)} = \infty\).

**Remark 3.6.** Cruz-Uribe and Fiorenza [1] gave an example showing that it can occur
\[\limsup_{t \to 0} \|\phi_t \ast f\|_{L^{(\cdot)}(\mathbb{R}^n)} = \infty
\]
for \(f \in L^{(\cdot)}(\mathbb{R})\) when \(\phi\) does not have compact support.

By modifying their example, we can also find \(p(\cdot)\) and \(\phi \in L^{[p(\cdot)]'}(\mathbb{R}),\) whose support is not compact, such that
\[\|\phi \ast f\|_{L^{(\cdot)}(\mathbb{R})} \leq C \|f\|_{L^{(\cdot)}(\mathbb{R})}
\]
does not hold, namely there exists \(f_N \ (N = 1, 2, \ldots)\) such that \(\|f_N\|_{L^{(\cdot)}(\mathbb{R})} \leq 1\) and
\[\lim_{N \to \infty} \|\phi \ast f_N\|_{L^{(\cdot)}(\mathbb{R})} = \infty.
\]

For this purpose, choose \(p_1 > 1, p_2 > p_1\) and \(a > 1\) such that
\[-p_1/p_2 - ap_1 + 2 > 0,
\]
and let \(p(\cdot)\) be a smooth variable exponent on \(\mathbb{R}\) such that
\[p(x) = p_1 \text{ for } x \leq 0, \quad p(x) = p_2 \text{ for } x \geq 1
\]
and \(p_1 \leq p(x) \leq p_2\) for \(0 < x < 1\). Set \(\phi = \sum_{j=1}^{\infty} \chi_j\), where \(\chi_j = \chi_{[-j,-j+j^{-a}]}\). Then
\[\int_{\mathbb{R}} \phi(x)^q \, dx = \sum_{j=1}^{\infty} \int_{-j}^{-j+j^{-a}} \chi_j(x)^q \, dx = \sum_{j=1}^{\infty} j^{-a} \leq C(a) < \infty
\]
for any \(q > 0\). Further set \(f_N = N^{-1/p_2} \chi_{[1,N+1]}\). Note that for \(1 - j + j^{-a} < x < 0\) and \(j \leq N\)
\[\chi_j \ast f_N(x) \geq \int_{x+j-j^{-a}}^{x+j} \chi_j(x-y) f_N(y) \, dy = N^{-1/p_2} j^{-a},
\]
so that
\[
\int_{\mathbb{R}} \left\{ \phi * f_N(x) \right\}^{p(x)} dx \geq \int_{-\infty}^{0} \left\{ \sum_{j=1}^{\infty} \chi_j * f_N(x) \right\}^{p_1} dx
\]
\[
\geq \sum_{j=2}^{N} \int_{1-j-j^{-a}}^{0} \left\{ \chi_j * f_N(x) \right\}^{p_1} dx
\]
\[
\geq N^{-p_1/p_2} \sum_{j=2}^{N} j^{-ap_1} (j - j^{-a} - 1)
\]
\[
\geq CN^{-p_1/p_2-2p_1+2} \to \infty \quad (N \to \infty).
\]

4. Young type inequalities

Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if \( \phi \) satisfies the additional condition
\[(4.1) \quad |\phi(x - y) - \phi(x)| \leq \frac{|y|}{|x|^{n+1}} \quad \text{when} \quad |x| > 2|y|.
\]

Noting that this condition implies
\[
sup_{x, z \in B(0, 2^{n+1}) \setminus B(0, 2^n)} |\phi(x) - \phi(z)| \leq C2^{-nj},
\]
we see that \( \lim_{|x| \to \infty} \phi(x) = 0 \) since \( \phi \in L^1(\mathbb{R}^n) \) and
\[(4.2) \quad |\phi(x)| \leq C|x|^{-n}.
\]

if \( \phi \) satisfies (4.1). In this connection we show

**Theorem 4.1.** Let \( p_- > 1 \). Suppose that \( \phi \in L^1(\mathbb{R}^n) \cap L^{(p_0)'}(B(0, R)) \) and \( \phi \) satisfies (4.2) for \( |x| \geq R \). Then
\[
\|\phi * f\|_{\Phi(p_0)(\mathbb{R}^n)} \leq C(\|\phi\|_{L^1(\mathbb{R}^n)} + \|\phi\|_{L^{(p_0)'}(B(0, R))}) \|f\|_{\Phi(p_0)(\mathbb{R}^n)}
\]
for all \( f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \).

**Remark 4.2.** Theorem 4.1 does not imply an inequality
\[
\|\phi_t * f\|_{\Phi(p_0)(\mathbb{R}^n)} \leq C\|f\|_{\Phi(p_0)(\mathbb{R}^n)}
\]
with a constant \( C \) independent of \( t \in (0, 1] \) even if \( \phi \) satisfies (4.2) for all \( x \), because \( \{\|\phi_t\|_{L^{(p_0)'}(B(0, R))} \}_{0 < t \leq 1} \) is not bounded.

**Proof of Theorem 4.1.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{\Phi(p_0)(\mathbb{R}^n)} \leq 1 \). Suppose that \( \phi \) satisfies (4.2) for \( |x| \geq R \) and \( \|\phi\|_{L^1(\mathbb{R}^n)} + \|\phi\|_{L^{(p_0)'}(B(0, R))} \leq 1 \). Decompose \( \phi = \phi' + \phi'' \), where \( \phi' = \phi \chi_{B(0, R)} \). We first note by Theorem 1.2 that
\[
\|\phi' * f\|_{\Phi(p_0)(\mathbb{R}^n)} \leq C.
\]
Hence it suffices to show that
\[
\|\phi'' * f\|_{\Phi(p_0)(\mathbb{R}^n)} \leq C.
\]

For this purpose, write
\[
f = f \chi_{\{y \in \mathbb{R}^n : f(y) \geq 1\}} + f \chi_{\{y \in \mathbb{R}^n : f(y) < 1\}} = f_1 + f_2.
\]
as before. Then we have by (4.2) and $(\Phi)$
\[
|\phi'' \ast f_1(x)| \leq C \int_{\mathbb{R}^n \setminus B(x,R)} |x - y|^{-n} f_1(y) \, dy \\
\leq CR^{-n} \int_{\mathbb{R}^n} f_1(y) \, dy \\
\leq CR^{-n} \int_{\mathbb{R}^n} \Phi_{\nu,x}^+ f_1(y) \, dy \leq C.
\]
Noting that $|\phi'' \ast f_2| \leq 1$, we obtain
\[
\int_{B(0,R)} \Phi_{\nu,x}^+ f_2(x) \, dx \leq C.
\]
Next, let $h(y) = \Phi_{\nu,x}^+ f_2(x)$. Then
\[
|\phi''| \ast h(x) \leq CR^{-n} \int_{\mathbb{R}^n} h(y) \, dy \leq CR^{-n}.
\]
If $x \in \mathbb{R}^n \setminus B(0, R)$, then we have by (4.2) and Lemma 3.3
\[
|\phi'' \ast f(x)| \leq \int_{B(0,|x|/2)} |\phi''(x - y)| f(y) \, dy + \int_{\mathbb{R}^n \setminus B(0,|x|/2)} |\phi''(x - y)| f(y) \, dy \\
\leq C \left\{|x|^{-n} \int_{B(0,|x|/2)} f(y) \, dy + (|\phi''| \ast h(x))^{1/p(x)} + |x|^{-A/p(x)} \right\} \\
\leq C \left\{Mf(x) + (|\phi''| \ast h(x))^{1/p(x)} + |x|^{-A/p(x)} \right\}
\]
with $A > n$. Now it follows from Proposition 2.5 that
\[
\int_{\mathbb{R}^n \setminus B(0,R)} \Phi_{\nu,x}^+ f(x) \, dx \leq C \left\{\int_{\mathbb{R}^n \setminus B(0,R)} \Phi_{\nu,x}^+ f(x) \, dx \right\} \\
+ \int_{\mathbb{R}^n} |\phi| \ast h(x) \, dx + \int_{\mathbb{R}^n \setminus B(0,R)} |x|^{-A} \, dx \right\} \\
\leq C,
\]
as required. 

**Theorem 4.3.** Let $1 - p_- / p_+ \leq \theta < 1$, $1 < \tilde{p} < p_-$,
\[
\frac{1}{s} = 1 - \frac{\theta}{\tilde{p}} \quad \text{and} \quad \frac{1}{r(x)} = 1 - \frac{\theta}{p(x)}.
\]
Take $\nu = p_- / \tilde{p}$, if $t^{p_- / \tilde{p}} \Phi_{\nu,x}^+ f(t)$ is uniformly almost increasing in $t$; otherwise choose $1 \leq \nu < p_- / \tilde{p}$. Suppose that $\phi \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \cap L^{s'p'}(B(0,R))$ and $\phi$ satisfies
\[
|\phi(x)| \leq C|x|^{-n/s}
\]
for $|x| \geq R$. Then
\[
\|\phi \ast f\|_{L^s(\mathbb{R}^n)} \leq C(\|\phi\|_{L^1(\mathbb{R}^n)} + \|\phi\|_{L^s(\mathbb{R}^n)} + \|\phi\|_{L^{s'p'}(B(0,R))}) \|f\|_{L^{p'}(\log L)^{q'}(\mathbb{R}^n)}
\]
for all $f \in L^{p'}(\log L)^{q'}(\mathbb{R}^n)$. 


Proof. Suppose that \( \|\phi\|_{L^1(\mathbb{R}^n)} + \|\phi\|_{L^r(\mathbb{R}^n)} + \|\phi\|_{L^{r'},B(0,R)} \leq 1 \) and \( \phi \) satisfies \( |\phi(x)| \leq C|x|^{-n/s} \) for \( |x| \geq R \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{\Phi(\cdot,q(\cdot))\mathbb{R}^n} \leq 1 \), and decompose \( f = f_1 + f_2 \),

where \( f_1 = f \chi_{\{x \in \mathbb{R}^n : f(x) \geq 1\}} \). Let

\[
\frac{1}{r} = \frac{1 - \theta}{p_-} \quad \text{and} \quad \frac{1}{s_1} = \frac{1}{r} - \frac{1}{p_+}.
\]

By our assumption, \( s_1 \geq 1 \). It follows from Young’s inequality for convolution that

\[
\|\phi * f_2\|_{L^r(\mathbb{R}^n)} \leq \|\phi\|_{L^{s_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_+}(\mathbb{R}^n)}.
\]

Here note that \( 1 \leq s_1 < s \), so that \( \|\phi\|_{L^{s_1}(\mathbb{R}^n)} \leq \|\phi\|_{L^1(\mathbb{R}^n)} + \|\phi\|_{L^{r'}(\mathbb{R}^n)} \leq 1 \). Since \( 0 \leq f_2 < 1 \), \( \|f_2\|_{L^{p_+}(\mathbb{R}^n)} \leq C \|f\|_{\Phi(\cdot,q(\cdot))\mathbb{R}^n} \leq C \). Thus, noting that \( \|\phi * f_2\| \leq 1 \) and

\[
\frac{1}{r(x)} - \frac{1}{r} = \frac{1 - \theta}{p(x)} - \frac{1 - \theta}{p_-} \leq 0,
\]

we see that

\[
\|\phi * f_2\|_{\Phi_{r(\cdot),q(\cdot)}\mathbb{R}^n} \leq C \|\phi * f_2\|_{L^r(\mathbb{R}^n)} \leq C.
\]

On the other hand, we have by Hölder’s inequality

\[
|\phi * f_1(x)| \leq \left( \int_{\mathbb{R}^n} |\phi(x - y)|^s f_1(y)^{\theta} \, dy \right)^{1/(1 - \theta)/\tilde{p}} \left( \int_{\mathbb{R}^n} |\phi(x - y)|^s \, dy \right)^{(1 - \theta)/(1 - \theta)/\tilde{p}}
\]

\[
= \left( \int_{\mathbb{R}^n} |f_1(y)|^\theta \, dy \right)^{\theta/\tilde{p}} \left( \int_{\mathbb{R}^n} |\phi(x - y)|^s \, dy \right)^{(1 - \theta)/(1 - \theta)/\tilde{p}}.
\]

(4.3)

Noting that \( |\phi|^s \in L^1(\mathbb{R}^n) \cap L^{r'}(B(0,R)) \), \( |\phi|^s \) satisfies (4.2) for \( |x| \geq R \) and \( \|f_1^\theta\|_{\Phi(\cdot,p(\cdot))\mathbb{R}^n} \leq C \), we find by Theorem 4.1

\[
\|\phi^s * f_1^\tilde{p}\|_{\Phi_{r(\cdot),q(\cdot)}\mathbb{R}^n} \leq C.
\]

Since (4.4) implies

\[
\Phi_{r(\cdot),q(\cdot)}(x, \phi * f_1(x)) \leq C \Phi_{p(\cdot),q(\cdot)}(x, |\phi|^s * f_1^{p_1}(x)),
\]

it follows that

\[
\|\phi * f_1\|_{\Phi_{r(\cdot),q(\cdot)}\mathbb{R}^n} \leq C.
\]

Thus, together with (4.3), we obtain

\[
\|\phi * f\|_{\Phi_{r(\cdot),q(\cdot)}\mathbb{R}^n} \leq C,
\]

as required. \( \square \)

Remark 4.4. Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if \( \phi \) satisfies the additional condition (4.1).

If \( p_- > 1 \), this conjecture was shown to be true by Cruz-Uribe, Fiorenza, Martell and Pérez in [3], using an extrapolation theorem ([3, Theorem 1.3 or Corollary 1.11]). Using our Proposition 2.5, we can prove the following extension of [3, Theorem 1.3]:
Proposition 4.5. Let \( \mathscr{F} \) be a family of ordered pairs \((f, g)\) of nonnegative measurable functions on \( \mathbb{R}^n \). Suppose that for some \( 0 < p_0 < p^- \),
\[
\int_{\mathbb{R}^n} f(x)^{p_0} w(x) \, dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w(x) \, dx
\]
for all \((f, g) \in \mathscr{F}\) and for all \( A_1 \)-weights \( w \), where \( C_0 \) depends only on \( p_0 \) and the \( A_1 \)-constant of \( w \). Then
\[
\|f\|_{\mathcal{P}(\mathbb{R}^n)} \leq C \|g\|_{\mathcal{P}(\mathbb{R}^n)}
\]
for all \((f, g) \in \mathscr{F}\) such that \( g \in L^{p^*}(\log L)^{q^*}(\mathbb{R}^n) \).

Then, as in [3, p. 249], we can prove:

Theorem 4.6. Assume that \( p_- > 1 \). If \( \phi \) is an integrable function on \( \mathbb{R}^n \) satisfying (4.1), then
\[
\|\phi_t \ast f\|_{\mathcal{P}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{P}(\mathbb{R}^n)}
\]
for all \( t > 0 \) and \( f \in L^{p^*}(\log L)^{(q^*)}(\mathbb{R}^n) \). If, in addition, \( \int \phi(x) \, dx = 1 \), then
\[
\lim_{t \to 0} \|\phi_t \ast f - f\|_{\mathcal{P}(\mathbb{R}^n)} = 0.
\]

5. Appendix

For \( p \geq 1, q \in \mathbb{R} \) and \( c \geq e \), we consider the function
\[
\Phi(t) = \Phi(p, q, c; t) = t^p (\log(c + t))^q, \quad t \in [0, \infty).
\]
In this appendix, we give a proof of the following elementary result:

Theorem 5.1. Let \( X \) be a non-empty set and let \( p(\cdot) \) and \( q(\cdot) \) be real valued functions on \( X \) such that \( 1 \leq p(x) \leq p_0 < \infty \) for all \( x \in X \). Then, the following (1) and (2) are equivalent to each other:

1. There exists \( c_0 \geq e \) such that \( \Phi(p(x), q(x), c_0; \cdot) \) is convex on \([0, \infty)\) for every \( x \in X \);
2. There exists \( K > 0 \) such that \( K(p(x) - 1) + q(x) \geq 0 \) for all \( x \in X \).

This theorem may be well known; however, the authors fail to find any literature containing this result.

This theorem is a corollary to the following

Proposition 5.2. (1) If
\[
(1 + \log c)(p - 1) + q \geq 0,
\]
then \( \Phi \) is convex on \([0, \infty)\).

(2) Given \( p_0 > 1 \) and \( c \geq e \), there exists \( K = K(p_0, c) > 0 \) such that \( \Phi \) is not convex on \([0, \infty)\) whenever \( 1 \leq p \leq p_0 \) and \( q < -K(p - 1) \).

Proof. By elementary calculation we have
\[
\Phi''(t) = t^{p-2}(c + t)^{-2}(\log(c + t))^{q-2}G(t)
\]
with
\[
G(t) = p(p - 1)(c + t)^2(\log(c + t))^2 + 2pqt(c + t)\log(c + t) - qt^2\log(c + t) + q(q - 1)t^2
\]
for \( t > 0 \). \( \Phi(t) \) is convex on \([0, \infty)\) if and only if \( G(t) \geq 0 \) for all \( t \in (0, \infty) \).
(1) If \( q \geq 0 \), then
\[
G(t) \geq qt(2p(c + t) - t) \log(c + t) - qt^2 \geq qt(2pc + 2(p - 1)t) \geq 0
\]
for all \( t \in (0, \infty) \), so that \( \Phi \) is convex on \([0, \infty)\).
If \(-(1 + \log c)(p - 1) \leq q < 0\), then
\[
G(t) = \left\{ \sqrt{p-1}(c + t) \log(c + t) + \frac{q}{\sqrt{p-1}}t^2 - \frac{pq^2}{p-1}t^2 - qt^2 \log(c + t) + q(q - 1)t^2 \right\}
\geq (-q)t^2 \left( \frac{pq}{p-1} + \log c - (q - 1) \right) = (-q)t^2 \left( \frac{q}{p-1} + \log c + 1 \right) \geq 0
\]
for all \( t \in (0, \infty) \), so that \( \Phi \) is convex on \([0, \infty)\).

(2) If \( p = 1 \) and \( q < 0 \), then
\[
G(t) = qt((t + 2c) \log(c + t) + (q - 1)t) \to -\infty
\]
as \( t \to \infty \). Hence \( \Phi \) is not convex on \([0, \infty)\).

Next, let \( 1 < p \leq p_0 \) and \( q = -k(p - 1) \) with \( k > 0 \). Then
\[
\frac{G(t)}{p-1} = p((c + t) \log(c + t) - kt)^2 + k(\log(c + t) - k + 1)t^2 \leq p_0((c + t) \log(c + t) - kt)^2 + k(\log(c + t) - k + 1)t^2.
\]
Let \( \lambda = 1 - 1/(2p_0) \). Then \( 0 < \lambda < 1 \). If \( k > (\log c)/\lambda \), there is (unique) \( t_k > 0 \) such that \( \log(c + t_k) = \lambda k \). Note that \( t_k/k \to \infty \) as \( k \to \infty \). We have
\[
\frac{G(t_k)}{p-1} \leq p_0((c + t_k)\lambda k - kt_k)^2 + k(\lambda k - k + 1)t_k^2
= kt_k^2 \left\{ (p_0(1 - \lambda) - 1)(1 - \lambda)k + 1 - 2p_0c\lambda(1 - \lambda)\frac{k}{t_k} + p_0c^2\lambda^2\frac{k}{t_k^2} \right\}.
\]
Since \( p_0(1 - \lambda) - 1 = -1/2 \), it follows that there is \( K = K(c, p_0) > (\log c)/\lambda \) such that \( G(t_k) < 0 \) whenever \( k \geq K \). Hence \( \Phi \) is not convex if \( 1 < p \leq p_0 \) and \( q \leq -K(p - 1) \). \( \square \)

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References


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