LACUNARY SERIES AND $Q_K$ SPACES ON THE UNIT BALL

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Abstract. In this paper, we establish a necessary condition for a kind of lacunary series on the unit ball to be in $Q_K$. As a consequence, we prove a necessary and sufficient condition for that $Q_K$ coincides with the Bloch space. In the case of $Q_p$ spaces we show that the condition, which is similar to that obtained by Hu, is also sufficient. This is a generalization of the result of Aulaskari, Xiao and Zhao for $Q_p$ spaces on the unit disk.

1. Introduction

Let $B^m$ denote the unit ball of $\mathbb{C}^m$, $S$ the boundary of $B^m$. For $z = (z_1, \cdots, z_m)$ and $a = (a_1, \cdots, a_m)$ in $B^m$, let $\langle z, a \rangle = z_1 \overline{a}_1 + \cdots + z_m \overline{a}_m$ and $|z| = \langle z, z \rangle^{1/2}$. The group of Möbius transformations of $B^m$ is denoted by $\text{Aut}(B^m)$. For $a \in B$, let $\phi_a \in \text{Aut}(B^m)$ be the Möbius transformation which satisfies $\phi_a(0) = a$ and $\phi_a^{-1} = \phi_a$.

By $H(B^m)$ we denote the collection of all holomorphic functions on $B^m$. For $f \in H(B^m)$ and $z = (z_1, \cdots, z_m) \in B^m$, let $\nabla f(z) = \left( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_m} \right)$ denote the complex gradient of $f$, and

$$\mathcal{R}f(z) = \sum_{j=1}^{m} z_j \frac{\partial f}{\partial z_j}$$

denote the radial derivative of $f$. The invariant gradient $\tilde{\nabla} f(z)$ of $f$ is defined by $\tilde{\nabla} f(z) = \nabla (f \circ \phi_z)(0)$. $\tilde{\nabla} f(z)$ and $\nabla f(z)$ are related by ([11])

$$|\tilde{\nabla} f(z)|^2 = (1 - |z|^2)(|\nabla f(z)|^2 - |\mathcal{R} f(z)|^2).$$

Let $\nu$ denote the Lebesgue measure on $\mathbb{C}^m = \mathbb{R}^{2m}$, so normalized that $\nu(B^m) = 1$ and $\sigma$ the normalized surface measure on $S$ so that $\sigma(S) = 1$. Let

$$d\tau(z) = \frac{d\nu(z)}{(1 - |z|^2)^{m+1}},$$

which is Möbius invariant.

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The Möbius invariant Green function of $B^m$ is defined by $G(z, a) = g(|\phi_a(z)|)$, where
\[ g(r) = \frac{m+1}{2m} \int_r^1 (1-t^2)^{m-1} t^{-2m+1} \, dt. \]

For $m > 1$, we have
\[ C_m^{-1} (1-r^2)^m r^{-2(m-1)} \leq g(r) \leq C_m (1-r^2)^m r^{-2(m-1)}, \]
where $C_m$ is a constant depending on $m$ only.

The notion of the spaces $Q_p$ was first considered for holomorphic functions defined on the unit disk $D$ of the complex plane $[3, 5, 6, 16, 17]$ and, later, generalized to hyperbolic Riemann surfaces and the unit ball of $C^m$ $[2, 4, 7, 12]$. More general spaces $Q_K$ was introduced and investigated on the unit disk in $[9]$, and on the unit ball $B^m$ of $C^m$ in $[18]$.

Let $K(t)$, $0 < t < \infty$, be a non-negative and non-decreasing function, which is not equal to 0 identically. The Banach space $Q_K$ is defined by
\[
Q_K = \left\{ f \in H(B^m) : \sup_{a \in B^m} \left| \int_{B^m} |\nabla f(z)|^2 K(G(z, a)) \, d\tau(z) < \infty \right. \}.
\]

When $K(t) = t^p$, $0 < p < \infty$, $Q_K$ is denoted by $Q_p$. For $m > 1$, it is proved in $[12]$ that $Q_p$ contains non-constant functions if and only if $(m-1)/m < p < m/(m-1)$.

For $Q_K$ spaces $[18]$, this condition becomes
\[ \int_0^1 \frac{t^{2m-1}}{(1-t^2)^m} K(g(r)) \, dr < \infty. \]

In this paper, $K(t)$ always denotes a function, formulated as above, and satisfies (1.3).

A function $f \in H(B^m)$ is called a Bloch function if
\[ \|f\|_B = \sup_{z \in B^m} |\nabla f(z)| < \infty. \]

It is obvious that $(1-|z|^2)|\mathcal{B} f(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\nabla f(z)|$ for $z \in B^m$. Further, Timoney $[14]$ proved that
\[ \|f\|_B \leq C \sup_{z \in B^m} (1-|z|^2)|\mathcal{B} f(z)|, \]
where $C$ is an absolute constant. The class of all Bloch functions is called the Bloch space and denoted by $\mathcal{B}$. $Q_K$ is always a subspace of $\mathcal{B}$. In $[18]$, it is proved that $Q_K = \mathcal{B}$ if
\[ \int_0^1 \frac{t^{2m-1}}{(1-t^2)^m} K(g(r)) \, dr < \infty. \]

In the case $m = 1$ (see $[9]$), (1.5) is a necessary and sufficient condition for $Q_K = \mathcal{B}$.

Lacunary series have been involved in the study of $Q_p$ spaces. For a lacunary series $f(z) = \sum_{n=0}^\infty a_n z^{2^n}$ on the unit disk, $0 < p \leq 1$, to be in $Q_p$, Aulaskari, Xiao and Zhao $[6]$ gave a necessary and sufficient condition: $f \in Q_p$ if and only if $\sum_{n=0}^\infty 2^{n(1-p)} |a_n|^2 < \infty$. This result was extended to $Q_p$ spaces on the unit ball by Hu $[10]$ and, recently, to $Q_K$ spaces on the unit disk by Wulan and Zhu $[15]$. The purpose of this paper is to generalize the necessary condition to $Q_K$ spaces on the unit ball for a kind of lacunary series. As an application, we show that the condition...
We have

The denoted $N$ is defined by $\delta > 0$ and $n$ is a positive integer, then

$$d(\zeta, \xi) = \left( 1 - |\langle \zeta, \xi \rangle|^2 \right)^{1/2} \quad \text{for} \quad \zeta, \xi \in S.$$  

The $d$-ball $E_{\delta}(\zeta)$ with radius $\delta$ and center $\zeta \in S$ is defined by

$$E_{\delta}(\zeta) = \{ \xi \in S : d(\zeta, \xi) < \delta \}.$$  

A set $\Gamma \subset S$ is said to be $d$-separated by $\delta > 0$, if $d$-balls with radius $\delta$ and centers at points of $\Gamma$ are pairwise disjoint. The following lemma was proved in [8].

**Lemma 1.** If $\Gamma \subset S$ is $d$-separated by $\delta > 0$ and $n$ is a positive integer, then

$$\sum_{\zeta \in \Gamma} |\langle \zeta, \xi \rangle|^n \leq 1 + \sum_{k=1}^{\infty} (k + 2)^{2m-2} e^{-k^2 \delta^2 n/2} \quad \text{for} \quad \xi \in S.$$  

Let $N$ be a positive integer and $\Gamma_N \subset S$ be the set of all points $\zeta = (\zeta_1, \ldots, \zeta_m)$ defined by

$$\zeta_1 = \sin \left( \frac{(N + n_1)}{6N} \right), \quad \zeta_m = e^{i l_{m-1} \pi / (2N)} \cos \left( \frac{(N + n_1)}{6N} \right) \ldots \cos \left( \frac{(N + n_{m-1})}{6N} \right),$$

$$\zeta_k = e^{i l_{k-1} \pi / (2N)} \cos \left( \frac{(N + n_1)}{6N} \right) \ldots \cos \left( \frac{(N + n_{k-1})}{6N} \right) \sin \left( \frac{(N + n_k)}{6N} \right), \quad 2 \leq k \leq m - 1,$$

where $n_1, l_1, \ldots, n_{m-1}, l_{m-1}$ are integers between 1 and $N$. Note that $\Gamma_N$ contains $N^{2(m-1)}$ different points.

**Lemma 2.** $d(\zeta, \xi) \geq 2^{-m}/N$, if $\zeta, \xi \in \Gamma_N$ and $\zeta \neq \xi$.

**Proof.** Let $\zeta$ and $\xi$ be two distinct points in $\Gamma_N \subset S$, which are defined by $n_1, l_1, \ldots, n_{m-1}, l_{m-1}$, and $n'_1, l'_1, \ldots, n'_{m-1}, l'_{m-1}$, respectively. For $k = 1, \ldots, m - 1$, denote

$$\alpha_k = \frac{(N + n_k)}{6N}, \quad \alpha'_k = \frac{(N + n'_k)}{6N}, \quad \beta_k = \frac{l_k}{2N}, \quad \beta'_k = \frac{l'_k}{2N}.$$  

We have

$$\langle \zeta, \xi \rangle = \sin \alpha_1 \sin \alpha'_1 + e^{i(\beta_1 - \beta'_1)} \cos \alpha_1 \cos \alpha'_1 \sin \alpha_2 \cos \alpha'_2 \ldots \cos \alpha_{m-1} \cos \alpha'_{m-1}$$

$$+ \sum_{k=2}^{m-1} e^{i(\beta_k - \beta'_k)} \cos \alpha_1 \cos \alpha'_1 \sin \alpha_{k-1} \sin \alpha'_k \sin \alpha_{k+1} \cos \alpha'_{k+1}.$$  

We distinguish two cases.
(i) There exists a $j$ such that $1 \leq j \leq m - 1$ and $n_j \neq n'_j$. Then,
\[
|\langle \zeta, \xi \rangle| \leq \sin \alpha_1 \sin \alpha'_1 \cos \alpha_1 \cos \alpha'_1 \cos \alpha_m \cos \alpha'_m \\
+ \sum_{k=2}^{m-1} \cos \alpha_1 \cos \alpha'_1 \cos \alpha_k \cos \alpha'_k \cos \alpha_k \sin \alpha_k' \sin \alpha'_k
\]
\[
= 1 - \sum_{k=1}^{m-1} \sigma_k (1 - \cos(\alpha_k - \alpha'_k)) \leq 1 - \sigma_j (1 - \cos(\alpha_j - \alpha'_j)),
\]
where $\sigma_k = \cos \alpha_1 \cos \alpha'_1 \cdots \cos \alpha_k \cos \alpha'_k$. Note that $\sigma_1 = 1$ and $4^{-m+2} \leq \sigma_j < 1$. Thus, since $\pi/(6N) \leq |\alpha_j - \alpha'_j| \leq \pi/6$,
\[
d(\zeta, \xi) \geq (1 - |\langle \zeta, \xi \rangle|)^{1/2} \geq \sqrt{2} \sigma_j^{1/2} \sin \frac{|\alpha_j - \alpha'_j|}{2} \geq \sqrt{2} \sigma_j^{1/2} \sin \frac{\pi}{12N} \geq \frac{2^{-m}}{N}.
\]
(ii) $n_k = n_k'$ for $k = 1, \ldots, m - 1$ and there exists a $j$ such that $1 \leq j \leq m - 1$ and $l_j \neq l'_j$. Let $\sigma_k = \beta_k - \beta'_k$ for $k = 1, \ldots, m - 1$, and let $A_1 = \sin^2 \alpha_1, A_k = \cos^2 \alpha_1 \cdots \cos^2 \alpha_k \sin^2 \alpha_k$ for $k = 2, \ldots, m - 1$, and $A_m = \cos^2 \alpha_1 \cdots \cos^2 \alpha_{m-1}$. Then,
\[
|\langle \zeta, \xi \rangle| = A_1 + \sum_{k=2}^{m} e^{i\sigma_{k-1}} A_k,
\]
and
\[
|\langle \zeta, \xi \rangle|^2 = \left( A_1 + \sum_{k=2}^{m} A_k \cos \sigma_{k-1} \right)^2 + \left( \sum_{k=2}^{m} A_k \sin \sigma_{k-1} \right)^2
\]
\[
= \sum_{k=1}^{m} A_k^2 + 2A_1 \sum_{k=2}^{m} A_k \cos \sigma_{k-1} + 2 \sum_{2 \leq k < l \leq m} A_k A_l \cos(\sigma_{k-1} - \sigma_{l-1})
\]
\[
\leq (A_1 + \cdots + A_m)^2 - 2A_1 A_{j+1} (1 - \cos(\beta_j - \beta'_j)).
\]
Since $A_1 + \cdots + A_m = 1$, $A_1 \geq 1/4$ and $A_{j+1} \geq 1/4^{-m-1}$, we have
\[
|\langle \zeta, \xi \rangle|^2 \leq 1 - 4^{-(m-1)} \sin^2 \frac{|\beta_j - \beta'_j|}{2},
\]
and
\[
d(\zeta, \xi) \geq 2^{-(m-1)} \sin \frac{\pi}{4N} \geq \frac{2^{-m}}{N}.
\]
The lemma is proved. □

By Lemma 2, $\Gamma_N$ is $d$-separated by $2^{-m}/(2N)$. Thus, using Lemma 1, we have

**Lemma 3.** If $n$ is a positive integer, then
\[
\sum_{\zeta \in \Gamma_N} |\langle \xi, \zeta \rangle|^n \leq 1 + \sum_{k=1}^{\infty} (k+2)^{2m-2} e^{-k^24^{-m-3/2}n^2/N^2} \quad \text{for } \xi \in S.
\]

The following lemma is a direct consequence of Lemma 3.

**Lemma 4.** If $n$ is a positive integer, then
\[
\sum_{\zeta \in \Gamma_N, \zeta \neq \xi} |\langle \xi, \zeta \rangle|^n \leq \sum_{k=1}^{\infty} (k+2)^{2m-2} e^{-k^24^{-m-3/2}n^2/N^2} \quad \text{for } \xi \in \Gamma_N.
\]
It is easy to see that there exists a positive integer $n_0$, depending on $m$ only, such that

\begin{equation}
\sum_{k=1}^{\infty} (k + 2)^{2m-2} e^{-4^{-m-3/2}n_0 k^2} < \frac{1}{2}.
\end{equation}

For an integer $n \geq n_0$, let $N_n$ be the largest positive integer such that $n/N_n^2 \geq n_0$.

Note that

\begin{equation}
\sum_{\substack{\zeta, \xi \in \Gamma_{N_n}, \xi \neq \zeta}} |\langle \zeta, \xi \rangle|^n < \frac{1}{2} \quad \text{for} \quad \zeta \in \Gamma_{N_n},
\end{equation}

and

\begin{equation}
\sum_{\substack{\zeta, \xi \in \Gamma_{N_n}} \langle \zeta, \xi \rangle^n \geq \sum_{\zeta \in \Gamma_{N_n}} 1 - \sum_{\zeta \in \Gamma_{N_n}} \sum_{\xi \in \Gamma_{N_n}, \xi \neq \zeta} |\langle \zeta, \xi \rangle|^n
\quad > \sum_{\zeta \in \Gamma_{N_n}} \frac{1}{2} = \frac{1}{2} N_{n}^{2(m-1)} \geq 2^{1-2m} n_0^{1-m} n^{m-1}.
\end{equation}

Now, for $n \geq n_0$, define

\begin{equation}
f_n(z) = \sum_{\zeta \in \Gamma_{N_n}} \langle z, \zeta \rangle^n.
\end{equation}

There is an estimate for $|f_n(z)|$. In fact, by Lemma 3 and the definition of $N_n$,

\begin{equation}
|f_n(z)| \leq |z|^n \sum_{\zeta \in \Gamma_{N_n}} |\langle z/|z|, \zeta \rangle|^n \leq |z|^n \left( 1 + \sum_{k=1}^{\infty} (k + 2)^{2m-2} e^{-k^2 4^{-m-3/2}N_{n}^2} \right)
\quad \leq |z|^n \left( 1 + \sum_{k=1}^{\infty} (k + 2)^{2m-2} e^{-k^2 4^{-m-3/2}n_0} \right) = c|z|^n.
\end{equation}

Let $\Lambda_n \subset S$ for $n = n_0, n_0 + 1, \cdots$. The sequence of homogeneous polynomials

\begin{equation}
f_n(z) = \sum_{\zeta \in \Lambda_n} \langle z, \zeta \rangle^n
\end{equation}

is called a normal sequence if it possesses the following property: there exists a positive constant $C$ such that

(i) $|f_n(z)| \leq C|z|^n$ for $z \in B^m$ and $n = n_0, n_0 + 1, \cdots$,

(ii) \( \sum_{\zeta, \xi \in \Lambda_n} \langle \zeta, \xi \rangle^n \geq C^{-1} n^{m-1} \) for $n = n_0, n_0 + 1, \cdots$.

Because of (2.3) and (2.5), the sequence $f_n(z)$ defined by (2.4) is a normal sequence. In what following, we will consider all lacunary series defined by normal sequences of homogeneous polynomials.
3. A necessary condition for a lacunary series to be in $Q_K$ spaces

In this section we prove a necessary condition for a lacunary series defined by a normal sequence to belong to a $Q_K$ space on the unit ball.

**Theorem 1.** Let $f_n(z), n = n_0, n_0 + 1, \ldots$, be a normal sequence and

$$f(z) = \sum_{k=1}^{\infty} a_k f_{n_k}(z),$$

where $n_0 \leq n_1 < n_2 < \cdots < n_k < \cdots$ is a sequence of positive integers such that $\liminf_{k \to \infty} n_{k+1}/n_k > 1$. If $f \in Q_K$, then

$$(3.1) \quad \sum_{k=1}^{\infty} n_k^m K(n_k^{-m})|a_k|^2 < \infty.$$

**Proof.** Let $f \in Q_K$. We have, by (1.1),

$$\left| \tilde{\nabla} f(z) \right|^2 = (1 - |z|^2) \left( \sum_{k=1}^{\infty} n_k a_k \sum_{\xi \in \Lambda_{n_k}} \langle z, \xi \rangle^{n_k-1} \xi \right)^2 - \left( \sum_{k=1}^{\infty} n_k a_k \sum_{\xi \in \Lambda_{n_k}} \langle z, \xi \rangle^{n_k} \right)^2,$$

and

$$\int_{B^n} \left| \tilde{\nabla} f(z) \right|^2 |K(g(|z|))| d\tau(z) = 2m \int_0^1 r^{2m-1} \left( \int_S |A|^2 - |B|^2 \right) d\sigma(z) K(g(r)) dr,$$

where

$$A = \sum_{k=1}^{\infty} n_k a_k r^{n_k-1} \sum_{\xi \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k-1} \xi, \quad B = \sum_{k=1}^{\infty} n_k a_k r^{n_k} \sum_{\xi \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k}.$$

If $k \neq k'$, integrating by slices gives

$$\int_S \sum_{\xi \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k} \sum_{\xi' \in \Lambda_{n_{k'}}} \overline{\langle \zeta, \xi' \rangle}^{n_{k'}-1} d\sigma(z) = 0$$

and

$$\int_S \sum_{\xi \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k-1} \sum_{\xi' \in \Lambda_{n_{k'}}} \overline{\langle \zeta, \xi' \rangle}^{n_{k'}-1} d\sigma(z) = 0.$$

Thus,

$$(3.2) \quad \int_{B^n} \left| \tilde{\nabla} f(z) \right|^2 |K(g(|z|))| d\tau(z) = \sum_{k=1}^{\infty} 2mn_k^2 |a_k|^2 \int_0^1 r^{2n_k+2m-3} \left( \int_S A_k(z) d\sigma(z) - r^2 \int_S B_k(z) d\sigma(z) \right) K(g(r)) dr,$$

where

$$A_k(z) = \sum_{\xi, \xi' \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k-1} \overline{\langle \xi, \xi' \rangle}^{n_{k'}-1} \langle \xi, \xi' \rangle,$$

$$B_k(z) = \sum_{\xi, \xi' \in \Lambda_{n_k}} \langle \zeta, \xi \rangle^{n_k} \overline{\langle \xi, \xi' \rangle}^{n_{k'}}. $$
For $\xi, \xi' \in \Lambda_{n_k}$, let $T$ be a unitary transformation such that $T\xi = (1, 0, \ldots, 0)$. Denote $\xi'' = T\xi' = (\xi''_1, \ldots, \xi''_m)$. Then, $\xi''_1 = \langle T\xi', T\xi \rangle = \langle \xi', \xi \rangle = \langle \xi, \xi' \rangle$, and

$$
\langle \xi, \xi' \rangle \int_S \langle \zeta, \zeta \rangle^{n_k-1} \langle \zeta, \zeta \rangle^{n_k-1} d\sigma(\zeta) = \int_S \langle \zeta, T\xi \rangle^{n_k-1} \langle \zeta, T\xi \rangle^{n_k-1} d\sigma(\zeta)
$$

$$
= (\xi, \xi') \int \zeta^{n_k-1} \left( \zeta_1 \langle \xi, \xi' \rangle + \zeta_2 \xi''_1 + \cdots + \zeta_m \xi''_m \right)^{n_k-1} d\sigma(\zeta).
$$

Integrating by slices, we obtain

$$
\langle \xi, \xi' \rangle \int_S \langle \zeta, \zeta \rangle^{n_k-1} \langle \zeta, \zeta \rangle^{n_k-1} d\sigma(\zeta) = (\xi, \xi')^{n_k} \int_S |\zeta_1|^{2(n_k-1)} d\sigma(\zeta)
$$

$$
= (\xi, \xi')^{n_k} \cdot \frac{(m-1)!}{(m-1 + n_k - 1)!} (n_k - 1)! \frac{(m-1)!}{(m-1 + n_k - 1)!},
$$

where the known formula [13]

$$
\int_S |\zeta_1|^{\beta_1} \cdots |\zeta_m|^{\beta_m} d\sigma(\zeta) = \frac{(m-1)!\beta_1! \cdots \beta_m!}{(m-1 + \beta_1 + \cdots + \beta_m)!}
$$

is used. By the same reason,

$$
\int_S \langle \zeta, \zeta \rangle^{n_k} \langle \zeta, \zeta \rangle^{n_k} d\sigma(\zeta) = (\xi, \xi')^{n_k} \cdot \frac{0}{(m-1 + n_k - 1)!} (n_k - 1)! \frac{0}{(m-1 + n_k - 1)!}
$$

Thus, by condition (ii) of the normal sequence, (3.3), (3.4) and (3.5),

$$
\int_S A_k(\zeta) d\sigma(\zeta) = (m-1)!(n_k - 1)! (m-1 + n_k - 1)! \frac{\xi, \xi' \in \Lambda_{n_k}}{\xi, \xi' \in \Lambda_{n_k}}
$$

$$
= \frac{\beta_1! \cdots \beta_m!}{(m-1 + \beta_1 + \cdots + \beta_m)!}
$$

$$
= \int_S B_k(\zeta) d\sigma(\zeta)
$$

$$
= (m-1)! (n_k - 1)! (m-1 + n_k - 1)! \frac{\xi, \xi' \in \Lambda_{n_k}}{\xi, \xi' \in \Lambda_{n_k}}
$$

$$
\stackrel{\partial}{\geq} C^{-1} (m-1)! n_k^{-2k} n_k! (m-1 + n_k - 1)! (m-1 + n_k - 1)!,
$$

Now, it follows from (3.2) and (3.6) that

$$
\int_S |\nabla f(z)|^2 |K(g(|z|))| d\sigma(z) \geq c^{-1} C^{-1} \sum_{k=1}^{\infty} |a_k|^2 n_k \int_0^1 \frac{r^{2n_k + 2m - 3}}{(1 - r^2)^m} \cdot K(g(r)) dr,
$$

where $c$ is a positive number depending on $m$ only. By (1.2),

$$
K(g(r)) \geq K(c_2^{-1}(1 - r)^m) \quad \text{for} \quad 1/2 \leq r \leq 1.
$$

Consequently,

$$
\int_0^1 \frac{r^{2n_k + 2m - 3}}{(1 - r^2)^m} \cdot K(g(r)) dr \geq c_1^{-1} \int_0^1 \frac{r^{2n_k - 1}}{(1 - r^m)^m} \cdot K(c_2^{-1}(1 - r)^m) dr
$$

$$
\geq c_1^{-1} \int_0^{\log 2} t^{-m} e^{-2nt} K(c_2^{-1} t^m) dt \geq c_1^{-1} K(n_k^{-m}) \int_0^{\log 2} t^{-m} e^{-2nt} dt
$$

$$
= c_1^{-1} n_k^{-m} K(n_k^{-m}) \int_0^{\log 2} t^{-m} e^{-2t} dt.
$$
Let $k'$ be sufficiently large such that $n_{k'} \log 2 \geq c_3^{1/m} + 1$. Then, for $k \geq k'$,
\[
\int_0^1 r^{2m-3} (1 - r^2)^m \cdot K(g(r)) \, dr \geq C^{-1} c^{-1} n_k^{-m} K(n_k^{-m}),
\]
and, by (3.7),
\[
\infty > \int_{B^m} |\nabla f(z)|^2 |K(g(|z|))| \, d\tau(z) \geq C^{-1} c^{-1} \sum_{k=k'}^\infty n_k^m K(n_k^{-m}) |a_k|^2.
\]
This shows (3.1) and the theorem is proved. \qed

It can be seen from the proof that the condition (i) for the normal sequence $f_\nu(z)$ and the lacunary condition for the sequence $n_k$ are not necessary for Theorem 1. As an implication of Theorem 1, we prove that (1.5) is also necessary for $Q_K = \mathcal{B}$ on the unit ball $B^m$.

**Theorem 2.** If $Q_K = \mathcal{B}$ on the unit ball $B^m$, then (1.5) holds.

**Proof.** Assume that $Q_K = \mathcal{B}$. Among lacunary series defined by normal sequences, we consider
\[
f(z) = \sum_{k=k_0}^{\infty} f_{2k}(z),
\]
where $f_{2k}(z)$ are constructed by (2.4) and $2k_0 \geq n_0$. By (2.5), $|f_{2k}(z)| \leq c |z|^{2k}$ for $k \geq k_0$ and $z \in B^m$. Thus,
\[
(1 - |z|^2)|\mathcal{R}f(z)| \leq (1 - |z|^2) \sum_{k=k_0}^{\infty} |\mathcal{R}f_{2k}(z)| \leq (1 - |z|^2) \sum_{k=k_0}^{\infty} 2^k |f_{2k}(z)|
\]
\[
\leq c(1 - |z|^2) \sum_{k=k_0}^{\infty} 2^k |z|^{2k} \leq 4c(1 - |z|) \sum_{n=1}^{\infty} |z|^n \leq 4c.
\]

By (1.4), this shows that $f \in \mathcal{B}$ and, consequently, $f \in Q_K$ since $Q_K = \mathcal{B}$. Using Theorem 1 gives
\[
\sum_{k=1}^{\infty} 2^{mk} K(2^{-mk}) < \infty.
\]

By (1.2), we have
\[
\int_{1/2}^1 r^{2m-1} (1 - r^2)^m \cdot K(g(r)) \, dr \leq \int_{1/2}^1 K(c(1 - r)^m) \frac{d}{(1 - r)^{m+1}} \leq c_1 \int_0^{c_1/m \log 2} t^{-(m+1)} K(t^m) \, dt.
\]
On the other hand,
\[
\int_0^{1/2} t^{-(m+1)} K(t^m) \, dt = \sum_{k=1}^{\infty} \int_0^{2^{-k}} t^{-(m+1)} K(t^m) \, dt
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-(k+1)} 2^{(m+1)(k+1)} K(2^{-mk})
\]
\[
= 2^m \sum_{k=1}^{\infty} 2^{mk} K(2^{-mk}),
\]

since $K$ is non-decreasing. Thus,
\[ \int_{1/2}^{1} r^{2m-1} (1 - r^2)^{m+1} K(g(r)) \, dr < \infty. \]
Combining this with (1.3), we obtain (1.5). The theorem is proved. \qed

4. The necessary and sufficient condition for a lacunary series to be in $Q_p$ spaces

The condition (3.1) in Theorem 1 is not sufficient if one does not put any extra restriction on $K$. We don’t know what is a better restriction. Now, we can prove the sufficiency only for $Q_p$ spaces, $(m - 1)/m < p \leq 1$.

In [10], the following equivalent characterization for $Q_p$ spaces on the unit ball $B^m$ with $m > 1$ was proved.

**Lemma 5.** Let $(m - 1)/m < p \leq 1$. Then, for $f \in H(B^m)$, $f \in Q_p$ if and only if
\[
\sup_{a \in B^m} \int_{B^m} (1 - |z|^2)^2 |\mathcal{R} f(z)|^2 (1 - |\phi_a(z)|^2)^{mp} \, d\tau(z) < \infty.
\]

The following lemma can be found in [13].

**Lemma 6.** If $\lambda > 0$ and $z \in B^m$, then
\[
\int_{S} \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z \rangle|^{m+\lambda}} \leq \frac{c}{(1 - |z|^2)^\lambda}.
\]

**Theorem 3.** Let $(m - 1)/m < p \leq 1$ and $f$ be defined as in Theorem 1. Then, $f \in Q_p$ if and only if
\[
\sum_{k=1}^{\infty} |a_k|^{2m(1-p)} < \infty.
\]

**Proof.** The necessity of the condition (4.3) follows from Theorem 1. Now, assume that (4.3) holds. By condition (i) of the normal sequence, for $z \in B^m$,
\[ |\mathcal{R} f(z)| \leq \sum_{k=1}^{\infty} |a_k| n_k f_{n_k}(z) \leq C \sum_{k=1}^{\infty} |a_k| n_k |z|^{n_k}. \]
Let $0 < \eta < 1$. For $0 < r < 1$, using Schwarz’s inequality gives
\[ \left( \sum_{k=1}^{\infty} |a_k| n_k r^{n_k} \right)^2 \leq \left( \sum_{k=1}^{\infty} n_k^{\eta} r^{n_k} \right) \left( \sum_{k=1}^{\infty} n_k^{2-\eta} |a_k|^2 r^{n_k} \right). \]
There exists an $M > 1$ such that $n_k/(n_k - n_{k-1}) \leq M$ for $k = 1, 2, \ldots$, since $\liminf_{k \to \infty} n_{k+1}/n_k > 1$. Let $\mu_0 = 0$ and $n_k |\log r| = \mu_k$ for $k = 1, 2, \ldots$. Then,
\[ \sum_{k=1}^{\infty} n_k^{\eta} r^{n_k} = |\log r|^{-\eta} \sum_{k=1}^{\infty} \mu_k^{\eta} e^{-\mu_k} \]
\[ = |\log r|^{-\eta} \sum_{k=1}^{\infty} \frac{n_k}{n_k - n_{k-1}} (\mu_k - \mu_{k-1}) \mu_k^{\eta-1} e^{-\mu_k} \]
\[ \leq M |\log r|^{-\eta} \Gamma(\eta). \]
By (1.2), the concavity of \( t^p \) and (4.2), for \( a \in B^m \), we have

\[
\int_S (1 - |\phi_a(r\zeta)|^2)^{mp} \, d\sigma(\zeta) = (1 - |a|^2)^{mp}(1 - r^2)^{mp} \int_S \frac{d\sigma(\zeta)}{1 - \langle ra, \zeta \rangle^{2mp}} \\
\leq (1 - |a|^2)^{mp}(1 - r^2)^{mp} \left( \int_S \frac{d\sigma(\zeta)}{1 - \langle ra, \zeta \rangle^{2m}} \right)^p \\
\leq c \cdot \frac{(1 - |a|^2)^{mp}(1 - r^2)^{mp}}{1 - |a|^2 r^2} \leq c(1 - r^2)^{mp}.
\]

Combining the above estimates, we obtain

\[
I(a) = \int_{B^m} (1 - |z|^2)^2 |\mathcal{H} f(z)|^2 (1 - |\phi_a(z)|^2)^{mp} \, d\tau(z) \\
= \int_{B^m} (1 - |z|^2)^{m+1} |\mathcal{H} f(z)|^2 (1 - |\phi_a(z)|^2)^{mp} \, d\nu(z) \\
\leq c_1 M \Gamma(\eta) \int_0^1 \frac{r^{2m-1} \log r - \eta}{(1 - r^2)^{m-1}} \sum_{k=1}^{\infty} |a_k|^2 n_k^{2-\eta} r^{nk} \, dr \int_S (1 - |\phi_a(r\zeta)|^2)^{mp} \, d\sigma(\zeta) \\
\leq c_2 M \Gamma(\eta) \sum_{k=1}^{\infty} |a_k|^2 n_k^{2-\eta} \int_0^1 r^{2m-1+nk} \log r - \eta (1 - r^2)^{1-m(1-p)} \, dr.
\]

It is easy to calculate that

\[
\int_0^1 r^{2m-1+nk} \log r - \eta (1 - r^2)^{1-m(1-p)} \, dr \leq 2 \int_0^1 r^{nk} \log r - \eta (1 - r)^{1-m(1-p)} \, dr \\
= 2 \int_0^\infty t^{\eta} (1 - e^{-t})^{1-m(1-p)} e^{-nt} \, dt \\
\leq 2 \int_0^\infty t^{1-\eta-m(1-p)} e^{-nt} \, dt \\
= 2n^{\eta+m(1-p)-2} \Gamma(2 - \eta - m(1-p)).
\]

Thus,

\[
I(a) \leq c M \Gamma(\eta) \Gamma(2 - \eta - m(1-p)) \sum_{k=1}^{\infty} |a_k|^2 n_k^{m(1-p)}
\]

holds for \( a \in B^m \). Because of (4.3), the right side of the above inequality is a finite number. Using Lemma 5, we see that \( f \in Q_p \). The theorem is proved. \( \Box \)

References


Lacunary series and $Q_K$ spaces on the unit ball


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