UNIONS OF JOHN DOMAINS AND UNIFORM DOMAINS IN REAL NORMED VECTOR SPACES

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Abstract. Let $E$ be real normed vector spaces with the dimension at least 2. In this paper we study the following questions: When is the union of two John domains in $E$ a John domain and when is the union of two uniform domains in $E$ a uniform domain?

1. Introduction and main results

Throughout the paper, we always assume that $E$ denotes a real normed vector space with $\dim E \geq 2$ and that $D$ is a proper subdomain in $E$. The norm of a vector $z$ in $E$ is written as $|z|$, and for any two points $z_1, z_2$ in $E$, the distance between them is denoted by $|z_1 - z_2|$, and the closed line segment with endpoints $z_1$ and $z_2$ by $[z_1, z_2]$. For $x \in E$ and $r > 0$, we let $B(x, r)$ denote the open ball in $E$ with center $x$ and radius $r$. For real numbers $r$ and $s$, we use the notation: $r \wedge s = \min\{r, s\}$.

John domains in Euclidean spaces $\mathbb{R}^n$ were introduced by John [1] in connection with his work on elasticity. The term is due to Martio and Sarvas [3]. Roughly speaking, a domain is a John domain if it is possible to travel from one point of the domain to another without going too close to the boundary. The precise definition is as follows.

Definition 1.1. $D$ is called a $c$-John domain if for every pair of points $x_1, x_2 \in D$ there is a rectifiable arc $\gamma$ joining them with

$$\ell(\gamma[x_1, x]) \wedge \ell(\gamma[x_2, x]) \leq c \, d(x)$$

for all $x \in \gamma$, where $c$ is a positive constant, $\gamma[x_j, x]$ denotes the closed subarc of $\gamma$ with endpoints $x_j$ and $x$ ($j = 1, 2$), $\ell(\gamma[x_j, x])$ the arclength of $\gamma[x_j, x]$. $\gamma$ is called a $c$-John arc joining $x_1$ and $x_2$.

See [4] for several characterizations of John domains. In the study of John domains, the following question is natural:

Question 1.1. Is the union of two John domains in $E$ still a John domain when their intersection is not empty?

Väisälä considered this question when $E = \mathbb{R}^n$. In [7], Väisälä constructed an example to show that, in general, the answer to this question is negative. Note
that the definition of John domains used in [7] is based on diameter cigar, which is quantitatively equivalent to Definition 1.1 when $E = \mathbb{R}^n$. In the same paper, Väisälä proved that if the intersection of two John domains is not too thin then their union is a John domain as the following result shows.

**Theorem A.** [7, Theorem 3.1] Suppose that $D_1$ and $D_2$ are $c$-John domains in $\mathbb{R}^n$. If there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $B(z_0, r) \subset D_1 \cap D_2$ and $d(D_1) \wedge d(D_2) \leq c_0 r$, where $d(D_i)$ denotes the diameter of $D_i$ $(i = 1, 2)$, then $D_1 \cup D_2$ is a $c'$-John domain with $c' = 2c_0 + 1$.

As the first aim of this paper, we study Question 1.1 further. Our result is as follows.

**Theorem 1.1.** Suppose that both $D_1$ and $D_2$ are $c$-John domains in $E$, and that there are $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $B(z_0, r) \subset D_1 \cap D_2$. If there exists some $r_1 > 0$ such that $r_1 \leq c_0 r$ and $D_1 \subset B(z_0, r_1)$, where $c_0 > 1$ is a constant, then $D_1 \cup D_2$ is a $c'$-John domain with $c' = c(4c_0 + 1)$.

The proof of Theorem 1.1 will be presented in Section 2. Our proof method is different from that in [7]. Hence when $E = \mathbb{R}^n$, we also give a different proof for Theorem A.

We remark that the assumption “$d(D_1) \wedge d(D_2) \leq c_0 r$” in Theorem A is equivalent to the statement “at least one of $D_1$ and $D_2$ is bounded”, and the assumption “there exists some $r_1 > 0$ such that $r_1 \leq c_0 r$ and $D_1 \subset B(z_0, r_1)$” in Theorem 1.1 is equivalent to the statement “$D_1$ is bounded”. The following example shows that the requirement that “at least one of $D_1$ and $D_2$ must be bounded” in Theorem 1.1 is necessary.

**Example 1.1.** Let $D_1 = \{(x, y) \in \mathbb{R}^2; x < 0\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2; |y| < x + 1\}$. Then both $D_1$ and $D_2$ are John domains, but $D = D_1 \cup D_2$ is not a John domain.

The proof of Example 1.1 will be given in Section 3.

**Definition 1.2.** $D$ is called $c$-uniform in the norm metric in $E$ provided there exists a positive constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ satisfying (cf. [5, Section 6.3])

1. $\ell(\gamma[z_1, z_2]) \leq c d(z)$ for all $z \in \gamma$, and
2. $\ell(\gamma) \leq c |z_1 - z_2|$.

$D$ is called uniform if it is $c$-uniform for some $c > 0$, and $\gamma$ is called a $c$-uniform arc if it satisfies (1) and (2) (cf. [6, Section 2.16]). See [2, 9] for the generalization of this definition.

As the second aim of this paper, we consider the following question:

**Question 1.2.** Does Theorem 1.1 hold for uniform domains in $E$?

The following example shows that even when both $D_1$ and $D_2$ are bounded uniform domains their union may not be uniform.

**Example 1.2.** Let $D_1 = \{(x, y) \in \mathbb{R}^2; -2 < x < 1, 0 < y < 2\}$ and $D_2 = D_3 \cup D_4$, where $D_3 = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, -1 < y < 1\}$ and $D_4 = \{(x, y) \in \mathbb{R}^2; -2 < x < 1, -1 < y < 0\}$. Then both $D_1$ and $D_2$ are uniform domains, but $D = D_1 \cup D_2$ is not a uniform domain.
The proof of Example 1.2 will be given in Section 3. For convex and bounded domains in $E$, the following result due to Väisälä, which is from [6].

**Theorem B.** [6, Theorem 2.19] Suppose that $G$ is a convex domain and that $B(x_0, r') \subset G \subset B(x, R')$. Then $G$ is $c_1$-uniform with $c_1 = 2 \frac{r'}{r}$.

In the following we consider Question 1.2 for convex domains and we get

**Theorem 1.2.** Suppose that $D_1$ and $D_2$ are convex domains in $E$, where $D_1$ is bounded and $D_2$ is $c$-uniform for some $c > 1$, and that there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $B(z_0, r) \subset D_1 \cap D_2$. If there exist constants $R_1 > 0$ and $c_0 > 1$ such that $R_1 \leq c_0 r$ and $D_1 \subset B(z_0, R_1)$, then $D_1 \cup D_2$ is a $c'$-uniform domain with $c' = (c + 1)(2c_0 + 1) + c$.

The proof of Theorem 1.2 will be presented in Section 4.

**Remark 1.1.** Example 1.2 shows that the hypothesis “$D_2$ being convex” in Theorem 1.2 is necessary.

### 2. The proof of Theorem 1.1

We show that the theorem holds with $c' = c(4c_0 + 1)$. Set $D = D_1 \cup D_2$. Let $x \in D \setminus D_2$, $y \in D \setminus D_1$, $d_j(x) = d(x, \partial D_j)$ for $j = 1, 2$. Then there are John arcs $\alpha \subset D_1$ from $x$ to $z_0$ and $\beta \subset D_2$ from $z_0$ to $y$, and an arc $\gamma \subset \alpha \cup \beta$ from $x$ and $y$. To prove that $\gamma$ is a $c'$-John arc in $D$ it suffices to show that

1. $\ell(\alpha[x, w]) \leq c'd(w)$ for all $w \in \alpha$,
2. $\zeta(z) := (\ell(\alpha) + \ell(\beta[z_0, z])) \wedge \ell(\beta[y, y]) \leq c'd(z)$ for all $z \in \beta$.

We let $x_0 \in \alpha$ be the point bisecting the length of $\alpha$ and choose $x_1$ such that $\ell(\alpha[x_1, z_0]) = \frac{r_1}{2}$. For any $w \in \alpha$, if $w \in \alpha[x, x_0]$, then we have

$$\ell(\alpha[x, w]) = \ell(\alpha[x, w_0]) \wedge \ell(\alpha[w, z_0]) \leq cd_1(w) \leq cd(w),$$

and (1) is proved.

If $w \in \alpha[x_1, z_0]$, then

$$d(w) \geq \frac{r}{2} \geq \frac{r_1}{2c_0'},$$

and

$$\ell(\alpha) = 2\ell(\alpha[x, x_0]) \leq 2cd_1(x_0) \leq 2cr_1,$$

which show that

$$\ell(\alpha[x, w]) \leq \ell(\alpha) \leq 2cr_1 \leq 4c_0d(w),$$

and we obtain (1).

Let $w \in \alpha[x_0, x_1]$. Obviously, $\ell(\alpha[x, w]) \wedge \ell(\alpha[z_0, w]) = \ell(\alpha[z_0, w]) \geq \frac{r_1}{2}$, which together with (2.1) show that

$$\ell(\alpha[x, w]) \leq 2cr_1 \leq 4c_0\frac{r_1}{r}\ell(\alpha[z_0, w]) \leq 4c^2c_0d(w),$$

which is (1).

The proof of (1) is complete. In the following, we come to prove (2). We let $y_0 \in \beta$ be the point bisecting the length of $\beta$ and choose $y_1$ such that $\ell(\alpha[z_0, y_1]) = \frac{r}{2}$. For any $z \in \beta$, if $z \in \beta[y, y_0]$, then (2) easily follows because $\beta$ is $c$-John in $D_2$.

If $z \in \beta[y_1, z_0]$, then (2.1) implies that

$$\zeta(z) \leq \ell(\alpha) + \ell(\beta[z_0, z]) \leq 2cr_1 + cd_2(z) \leq 4c_0d(z) + cd(z) = c(4c_0 + 1)d(z),$$
since \( r_1 \leq c_0 r \leq 2c_0 d(z) \). If \( z \in \beta[y_0, y_1] \), then we have
\[
\ell(\beta[y, z]) \wedge \ell(\beta[z, z_0]) = \ell(\beta[z, z_0]) \geq \frac{r}{2} \geq \frac{r_1}{2c_0}
\]
and
\[
\ell(\beta[y, z]) \wedge \ell(\beta[z, z_0]) \leq c d_2(z) \leq c d(z),
\]
which together with (2.1) imply
\[
\zeta(z) \leq \ell(\alpha) + \ell(\beta[z_0, z]) \leq 2cr_1 + c d(z) \leq 4c^2 c_0 d(z) + c d(z) = c(4cc_0 + 1)d(z).
\]
The arbitrariness of \( x \) and \( y \) shows that \( D = D_1 \cup D_2 \) is a \( c' \)-John domain with \( c' = c(4cc_0 + 1) \).

3. The proof of Examples 1.1 and 1.2

3.1. Proof of Example 1.1. The proof of both \( D_1 \) and \( D_2 \) being John domains easily follows from the fact that an \( L \)-bilipschitz image of a \( c \)-John domain is \( c' \)-John with \( c' = L^2 c \). Obviously, \( z_0 = (-\frac{1}{2}, 0) \in D_1 \cap D_2 \) and \( \mathbf{B}(z_0, \frac{1}{2}) \subset D_1 \cap D_2 \). Let \( D = D_1 \cup D_2 \). Then for any positive integer \( n \), \( w_n = (-n, 0) \) and \( z_n = (n, 0) \in D \). For any \( \gamma_n \) joining \( w_n \) and \( z_n \), there must exist a point \( u_n \in \gamma_n \cap (D_1 \cap D_2) \) such that \( d(u_n) < 2 \) and \( \ell(\gamma_n[u_n, w_n]) \wedge \ell(\gamma_n[z_n, u_n]) \to \infty \) as \( n \to \infty \). This implies that \( D_1 \cup D_2 \) is not a John domain.

In order to prove Example 1.2, we introduce the following definition.

**Definition 3.1.** A domain \( D \subset E \) is \( c \)-quasiconvex if each pair of points \( a, b \in D \) can be joined with an arc \( \gamma \subset D \) with
\[
\ell(\gamma) \leq c |a - b|,
\]
where \( c > 1 \) is a constant.

**Proposition 3.1.** If \( D \subset E \) is uniform, then it must be quasiconvex.

3.2. Proof of Example 1.2. Theorem B implies that all domains \( D_1, D_3 \) and \( D_4 \) are uniform, and Theorem 1.2 shows that \( D_2 \) is also uniform. Obviously, \( D = D_1 \cup D_2 \) is not quasiconvex, hence by Proposition 3.1, \( D \) is not uniform.

4. The proof of Theorem 1.2

Before the proof of Theorem 1.2, we introduce two lemmas.

**Lemma C.** [8, Lemma 3.4] Suppose that \( D \subset E \) is a convex domain. The function \( d: D \to \mathbb{R} \) is concave, that is,
\[
d(ta + (1 - t)b) \geq td(a) + (1 - t)d(b)
\]
whenever \( a, b \in D \) and \( t \in [0, 1] \).

**Lemma 4.1.** Suppose that \( X \) is a vector space with \( \dim X \geq 2 \), the vectors \( x_i, y_i, z_i \in X \) are linearly independent for each \( i \in \{1, 2\} \), and that
\[
x_1 - y_1 = \lambda(x_2 - y_2), \quad y_1 - z_1 = \mu(y_2 - z_2) \quad \text{and} \quad z_1 - x_1 = \tau(z_2 - x_2)
\]
for constants \( \lambda, \mu, \tau \). Then \( \lambda = \mu = \tau \).
Proof. Since
\[ z_1 - x_1 = (z_1 - y_1) + (y_1 - x_1) = \mu(z_2 - y_2) + \lambda(y_2 - x_2), \]
we know
\[ (\tau - \lambda)x_2 + (\lambda - \mu)y_2 + (\mu - \tau)z_2 = 0. \]
By the linear independence of \( \{x_2, y_2, z_2\} \) we get \( \lambda = \mu = \tau. \)

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We show that the theorem holds with \( c' = (c + 1)(2c_0 + 1) + c. \) Set \( D = D_1 \cup D_2 \) and \( D_0 = D_1 \cap D_2. \) Let \( a \in D \setminus D_2, b \in D \setminus D_1, z_1 = \frac{a + b}{2} \)
and \( s = |a - b|. \)

Case 4.1. \( |z_1 - z_0| \leq 2c_0s. \) There is a \( c \)-uniform arc \( \beta \subset D_2 \) from \( z_0 \) to \( b \) and an arc \( \gamma \subset \{a - z_0\} \cup \beta \) from \( a \) to \( b. \) To prove that \( \gamma \) is \( c' \)-uniform in \( D \) it suffices to show that

1. \( |a - z_0| + \ell(\beta) \leq c'|a - b|, \)
2. \( |a - x| \leq c_0d(x) \) for all \( x \in [a, z_0], \)
3. \( \zeta(y) := \left( |a - z_0| + \ell(\beta[z_0, y]) \right) \wedge \ell(\beta[b, y]) \leq c'd(y) \) for all \( y \in \beta. \)

Since \( |a - z_0| \leq |a - z_1| + |z_1 - z_0| \leq s + 2c_0s \) and similarly \( |b - z_0| \leq s + 2c_0s, \) we have

\[ |a - z_0| + \ell(\beta) \leq (2c_0 + 1)s + c|b - z_0| \leq (c + 1)(2c_0 + 1)s \leq c'|a - b|, \]
and (1) is proved.

If \( x \in [a, z_0], \) then \( x = (1 - t)a + tz_0 \) for some \( t \in [0, 1], \) and we have \( |a - x| = t|a - z_0| \leq tc_0r. \) Lemma C implies that

\[ d(x) \geq d_1(x) \geq (1 - t)d_1(a) + td_1(z_0) \geq td_1(z_0) \geq tr. \]

As \( |a - x| = t|a - z_0| \leq tc_0r, \) this yields (2).

Let \( y \in \beta \) and let \( y_0 \in \beta \) be the point bisecting the length of \( \beta. \) If \( y \in \beta[y_0, b], \)
then the \( c \)-uniformity of \( \beta \) gives \( \zeta(y) = \ell(\beta[y, b]) \leq cd_2(y). \) If \( y \in \beta[z_0, y_0], \)
then

\[ \zeta(y) \leq |a - z_0| + \ell(\beta[z_0, y]) \leq c_0r + cd_2(y) \leq c_0d(z_0) + cd(y). \]

Here \( d(z_0) \leq d(y) + |z_0 - y| \leq d(y) + \ell(\beta[z_0, y]) \leq (1 + c)d(y), \) and we obtain (3).

Case 4.2. \( |z_1 - z_0| > 2c_0s. \) Set \( e = (b - a)/|b - a| \) and \( a_0 = z_0 + re. \) As \( a \notin D_2 \) and \( b \notin D_1, \) these points do not lie on the line through \( z_0 \) and \( z_1. \) Hence there is a unique point \( w \in [z_0, z_1] \cap [a_0, a]. \) Applying Lemma 4.1 to the triples \( (w, z_0, a_0) \) and \( (w, z_1, a) \)
we get \( |w - z_1| = s|w - z_0|/r. \) Replacing \( a \) and \( a_0 \) by \( b \) and \( b_0 = z_0 - re, \) respectively, we see that \( w \in [z_0, z_1] \cap [b_0, b]. \) Hence \( w \in D_0, \) which implies that \( [z_0, w] \subset D_0. \)
Since \( w \in D_1 \subset B(z_0, c_0r), \) we have \( |w - z_0| \leq c_0r, \) whence \( |w - z_1| \leq c_0s. \)

Set \( u = (z_1 - z_0)/|z_1 - z_0| \) and \( y_1 = w - c_0su. \) Then \( |y_1 - z_1| = |y_1 - w| + |w - z_1| \leq 2c_0s, \) whence \( y_1 \in [z_0, w]. \) There is a \( c \)-uniform arc \( \beta_1 \subset D_2 \) from \( y_1 \) to \( b \) and an arc \( \gamma \subset [a, y_1] \cup \beta_1 \) from \( a \) to \( b. \) To prove that \( \gamma \) is \( c' \)-uniform in \( D \) it suffices to show that

1. \( |a - y_1| + \ell(\beta_1) \leq c'|a - b|, \)
2. \( |a - x| \leq (2c_0 + 1)d(x) \) for all \( x \in [a, y_1], \)
3. \( \zeta_1(y) := (|a - y_1| + \ell(\beta_1[y_1, y])) \wedge \ell(\beta_1[b, y]) \leq c'd(y) \) for all \( y \in \beta_1. \)
We have
\[ |a - y_1| \leq |a - z_1| + |z_1 - y_1| \leq s + 2c_0 s, \quad \ell(\beta_1) \leq c|b - y_1| \leq c(2c_0 + 1)s, \]
and (1) follows.

If \( x \in [a, y_1] \), then \( x = (1 - t)a + ty_1 \) for some \( t \in [0, 1] \). It follows from Lemma C that
\[ d(x) \geq d_1(x) \geq (1 - t)d_1(a) + td_1(y_1) \geq td_1(y_1). \]
As \( |w - z_0| \leq c_0 r \), we similarly obtain
\[ d_1(y_1) \geq \frac{c_0 s}{|w - z_0|} d_1(z_0) \geq \frac{c_0 s r}{|w - z_0|} \geq s. \]
Hence
\[ |a - x| = t|a - y_1| \leq t(2c_0 + 1)s \leq (2c_0 + 1)d(x), \]
which is (2).

Let \( y \in \beta_1 \) and \( y_0 \in \beta_1 \) be the point bisecting the length of \( \beta_1 \). If \( y \in \beta_1[y_0, b] \), then (3) follows from the \( c \)-uniformity of \( \beta_1 \) in \( D_2 \). Let \( y \in \beta_1[y_1, y_0] \). Now (2) and the \( c \)-uniformity of \( \beta_1 \) imply that
\[ \zeta_1(y) \leq |a - y_1| + \ell(\beta_1[y_1, y]) \leq (2c_0 + 1)d_1(y_1) + cd_2(y). \]
Here \( d(y_1) \leq d(y) + |y - y_1| \leq d(y) + \ell(\beta_1[y_1, y]) \leq (c + 1)d(y) \), and we obtain (3).

The proof of Theorem 1.2 is complete. \( \square \)

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