A MEAN-VALUE THEOREM FOR SOME EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR ON THE UPPER-HALF SPACE

Sirkka-Liisa Eriksson and Heikki Orelma

Tampere University of Technology, Department of Mathematics
P.O. Box 553, 33101 Tampere, Finland; sirkka-liisa.eriksson@tut.fi
Tampere University of Technology, Department of Mathematics
P.O. Box 553, 33101 Tampere, Finland; heikki.orelma@tut.fi

Abstract. In this paper we study a mean-value property for solutions of the eigenvalue equation of the Laplace–Beltrami operator

$$\Delta_{lb}h = -(n-1)h$$

with respect to the volume and the surface integrals on the Poincaré upper-half space $\mathbb{R}^{n+1}_+ = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$ with the Riemannian metric $ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}$.

1. Preliminaries

In this section we recall the Laplace–Beltrami operator in the Poincaré upper-half space and formulate its connections with the so-called hypermonogenic functions. Let us denote $\mathbb{R}^{n+1}_+ = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$. The Poincaré half-space is the Riemannian manifold $(\mathbb{R}^{n+1}_+, ds^2)$, where the Riemannian metric is

$$ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

The Laplace–Beltrami operator on the Poincaré upper-half space is the operator (details are available for example in [7])

$$\Delta_{lb}f = x_n^2 \Delta f - (n-1)x_n \frac{\partial f}{\partial x_n},$$

where $f : \Omega \to \mathbb{R}$ is a smooth enough function defined on an open subset $\Omega$ of $\mathbb{R}^{n+1}_+$ and $\Delta = \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. The solutions of the Laplace–Beltrami equation $\Delta_{lb}f = 0$ are called hyperbolic harmonic functions.

The Clifford algebra $\mathcal{C}\ell_{0,n}$ is the free associative algebra with unit generated by the symbols $e_1, \ldots, e_n$ together with the defining relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

for $i, j = 1, \ldots, n$. As a vector space the dimension of the Clifford algebra $\mathcal{C}\ell_{0,n}$ is $2^n$. A canonical basis is given by $e_A = e_{a_1} \cdots e_{a_k}$, where $A = \{a_1, \ldots, a_k\} \subset \{1, \ldots, n\}$ and $1 \leq a_1 < \ldots < a_k \leq n$. In particular, we denote $e_\emptyset = e_0 = 1$ and $e_{\{j\}} = e_j$. The


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(n + 1)-dimensional Euclidean space $\mathbb{R}^{n+1}$ is a subspace of $\mathcal{C}_0^{n+1}$ under the canonical embedding

$$(x_0, x_1, \ldots, x_n) \mapsto \sum_{j=0}^{n} x_j e_j$$

and thus we may assume that $\mathbb{R}^{n+1} \subset \mathcal{C}_0^{n+1}$. An element $a \in \mathcal{C}_0^{n+1}$ is called a Clifford number and often the algebra $\mathcal{C}_0^{n+1}$ is called the algebra of Clifford numbers. Elements $x = \sum_{j=1}^{n} x_j e_j \in \mathcal{C}_0^{n+1}$ are called vectors. Thus we see that an element $x$ of $\mathbb{R}^{n+1}$ may be written as

$$x = x_0 + x$$

with $x = x_1 e_1 + \cdots + x_n e_n$ and it is called a paravector.

The conjugation is the algebra anti-automorphism on the Clifford algebra defined by $\overline{x} = x_0 - x$, that is, if $a, b \in \mathcal{C}_0^{n+1}$, then $\overline{ab} = \overline{b}\overline{a}$. Also, $x^2 = xx = -x_1^2 - \cdots - x_n^2$.

Thus we may compute

$$x\overline{x} = (x_0 + x)(x_0 - x) = x_0^2 + x_1^2 + \cdots + x_n^2$$

for $x \in \mathbb{R}^{n+1}$. The Euclidean norm is then $|x|^2 = x\overline{x} = \overline{x}x$. The main-involution is the algebra automorphism denoted and defined by $x' = x_0 - x$, that is, if $a, b \in \mathcal{C}_0^{n+1}$, then $(ab)' = a'b'$.

Let us consider the Clifford algebra valued functions $f : \Omega \to \mathcal{C}_0^{n+1}$, where $\Omega \subset \mathbb{R}^{n+1}$ is an open subset. Since the Clifford algebra $\mathcal{C}_0^{n+1}$ is generated by the symbols $e_1, \ldots, e_n$, we obtain that then the Clifford algebra $\mathcal{C}_0^{n+1}$ is generated by the symbols $e_1, \ldots, e_{n-1}$. Hence each $a \in \mathcal{C}_0^{n+1}$ may be represented in the form

$$a = b + ce_n,$$

where $b, c \in \mathcal{C}_0^{n+1}$. We abbreviate $Pa = b$ and $Qa = c$ and $Q'a = (Qa)'$ and $P'a = (Pa)'$. Then we define the modified Dirac operator by

$$Mf = Df + \frac{n-1}{x_n} Q' f,$$

where $D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$ is the Dirac operator on $\mathbb{R}^{n+1}$. The theory of null-solutions of the modified Dirac operator is called hyperbolic function theory, see, e.g., [4].

The function $f : \Omega \to \mathcal{C}_0^{n+1}$ is called a hypermonogenic on $\Omega$ if $Mf(x) = 0$ for each $x \in \Omega$. Hypermonogenic functions have many nice function theoretic properties, for example, they have Cauchy-type integral formulas. Also, the function $x \mapsto x^k$, where $k \in \mathbb{Z}$, is hypermonogenic. Many properties and more references can be found from the survey article [4].

The conjugate of the modified Dirac operator is defined by

$$\overline{M}f = \overline{D}f - \frac{n-1}{x_n} Q' f,$$

where $\overline{D} = \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - \cdots - e_n \frac{\partial}{\partial x_n}$.

In the hyperbolic function theory we define hyperbolic harmonic functions $f : \Omega \to \mathcal{C}_0^{n+1}$ as solutions of the equation

$$\overline{M}Mf(x) = 0$$

for $x \in \Omega$. 
The next theorem gives us a connection between hypermonogenic functions and hyperbolic harmonic functions. Also, we see that the equation in above is really a good generalization for real-valued hyperbolic harmonic functions.

**Theorem 1.1.** [2] Let \( \Omega \subset \mathbb{R}^{n+1}_+ \) be an open subset and let \( f : \Omega \to \mathcal{C}^{0,n}_0 \) be a twice differentiable function. Then

\[
P(\overline{M}Mf) = \Delta f - \frac{n-1}{x_n} \frac{\partial f}{\partial x_n}
\]

and

\[
Q(\overline{M}Mf) = \Delta Qf - \frac{n-1}{x_n} \frac{\partial Qf}{\partial x_n} + (n-1) \frac{Qf}{x^2_n}.
\]

If \( f \) is hypermonogenic, then \( Pf \) satisfies the equation

\[
\Delta Pf - \frac{n-1}{x_n} \frac{\partial Pf}{\partial x_n} = 0
\]

and \( Qf \) satisfies the equation

\[
\Delta Qf - \frac{n-1}{x_n} \frac{\partial Qf}{\partial x_n} + (n-1) \frac{Qf}{x^2_n} = 0.
\]

Thus we see that the \( Q \)-part of a hypermonogenic function is a solution of the following eigenvalue equation

\[
\Delta_\mathbb{H} h = -(n-1)h.
\]

In the next section we shall study more detailed what is the structure of the above eigenfunctions.

Also, we see that the \( P \)-part of a hypermonogenic function is a direct generalization of a real-valued hyperbolic harmonic function. For a \( \mathcal{C}^{0,n-1} \)-valued function, especially for the \( P \)-part of a hypermonogenic function, we obtained the following structure theorem.

**Theorem 1.2.** [5] Let \( \Omega \subset \mathbb{R}^{n+1}_+ \) be open and \( g : \Omega \to \mathcal{C}^{0,n-1}_0 \) be a differentiable function. The following properties are equivalent.

(a) \( g \) is a solution of the equation

\[
\Delta g - \frac{n-1}{x_n} \frac{\partial g}{\partial x_n} = 0.
\]

(b) \( g \) is smooth and

\[
g(a) = \frac{1}{\omega_{n+1} \sinh^n R_h} \int_{\partial B_h(a,R_h)} g(x) \, d\sigma_h(x)
\]

for all \( \overline{B}_h(a,R_h) \subset \Omega \). In the formula \( \omega_{n+1} \) denotes the surface area of the \( n \)-dimensional unit sphere.

(c) \( g \) is smooth and

\[
g(a) = \frac{1}{V(B_h(a,R_h))} \int_{B_h(a,R_h)} g(x) \, dx_h(x)
\]

for all \( \overline{B}_h(a,R_h) \subset \Omega \), where \( V(B_h(a,R_h)) = \sigma_n \int_0^{R_h} \sinh^n t \, dt \) is the volume of the ball \( B_h(a,R_h) \).
In the previous theorem $B_h(a, R_h)$ is the hyperbolic ball with the center $a$ and the radius $R_h$. In the next section we shall give more detailed description for it. Since $R$ is a canonical subset of $C_0$, we obtain the following obvious corollary.

**Corollary 1.3.** The preceding theorem is true also for real valued functions.

In the next section we shall state and prove a similar theorem for the preceding eigenfunctions.

2. A mean-value theorem for some eigenfunctions of the Laplace–Beltrami operator

Our aim is to give a detailed proof for the following structure theorem of the eigenfunctions represented in the previous section. First we recall a few basic facts from the hyperbolic geometry. A more detailed survey to the topic is available in [6]. In [6] it is shown that the hyperbolic ball with the radius $R_h$ and the center $a$ is the Euclidean ball with the center $\tau(a, R_h)$ and the radius $R_e(a, R_h)$,

$$B_h(a, R_h) = \{ x \in \mathbb{R}^{n+1} : |x - \tau(a, R_h)| < R_e(a, R_h) \},$$

where

$$\tau(a, R_h) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n \cosh R_h$$

and

$$R_e(a, R_h) = a_n \sinh R_h.$$

The $n$-form

$$d\sigma = \sum_{j=0}^{n} (-1)^j e_j d\tilde{x}_j$$

is often very useful vector valued differential form on $\mathbb{R}^{n+1}$, where $d\tilde{x}_j = dx_0 \cdots dx_{j-1} \cdot dx_{j+1} \cdots dx_n$.

Let $K$ be an $(n + 1)$-dimensional manifold-with-boundary. On the boundary $\partial K$ the form $d\sigma$ admits the representation $d\sigma = \nu dS$, where $\nu$ is the outer unit normal vector field and $dS$ a scalar $n$-form. The corresponding surface form on the hyperbolic space is $d\sigma_h = \frac{d\sigma}{\omega_n}$ and if $dx$ is the volume form on the Euclidean space then the corresponding hyperbolic form is $dx_h = \frac{dx}{\omega_n}$. More detailed introduction to integration and certain differential forms in the Poincaré upper-half space can be found from [6].

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open subset and let $h : \Omega \to C_0, n-1$ be a smooth function. The following properties are equivalent:

(i) $h$ is an eigenfunction of the Laplace–Beltrami operator with the eigenvalue $-(n - 1)$, i.e, is a solution of

$$\Delta_l h(x) = -(n - 1) h(x)$$

for $x \in \Omega$.

(ii)

$$h(a) = \frac{1}{\omega_{n+1} \psi(R_h)} \int_{\partial B_h(a, R_h)} h(x) d\sigma_h(x),$$

where

$$\psi(R_h) = \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt$$
whenever $B(a, R_h) \subset \Omega$.

(iii)

$$h(a) = \frac{n - 1}{\omega_{n+1}} \int_{B_h(a, R_h)} h(x) \, dx_h,$$

where $\omega_{n+1}$ is the surface area of the $(n + 1)$-unit sphere and

$$\phi(R_h) = (n - 1) \cosh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt - \sinh^{n-1} R_h$$

whenever $B(a, R_h) \subset \Omega$.

The corresponding result in the case $n = 2$ is already known. Leutwiler proved the theorem in his paper [8] using Green’s functions which are simple in the case $n = 2$. Authors wishes to emphasize that the methods of Leutwiler are available only in the special case $n = 2$ since the Green’s functions have much more complicated form in higher dimensions.

The first consequence is the following remark.

**Corollary 2.2.** The preceding theorem is true also for functions $h : \Omega \rightarrow \mathbb{R}$.

The proof of the theorem is based on a sequence of lemmata. First we recall the Cauchy’s formula for the $Q$-part of a hypermonogenic function and other useful results.

**Proposition 2.3.** [1] If $f$ is a hypermonogenic function on $\Omega$ and $K \subset \Omega$ is an oriented $(n + 1)$-dimensional manifold-with-boundary, then for each $a \in K$ we have

$$Qf(a) = \frac{2^n a^{n-1}}{\omega_{n+1}} \int_{\partial K} Q(q(x, a) \nu(x) f(x)) \, dS(x),$$

where $dS$ is the scalar surface element, $\nu$ is the outer unit normal vector field, and

$$q(x, a) = -\frac{1}{2(n - 1)} |x - a|^{n-1} |x - \hat{a}|^{n-1} = \frac{1}{2} |x - a|^{n-1} |x - \hat{a}|^{n-1}.$$

The kernel in the above integral admits the following expression.

**Theorem 2.4.** [6]

$$q(x, a) = \frac{(x - \tau(a, x)) \cosh d_h(x, a) - a_n \sinh^2 d_h(x, a) e_n}{(2a_n x_n)^n \sinh^{n+1} d_h(x, a)},$$

where

$$\tau(a, x) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n \cosh d_h(x, a) e_n$$

and $d_h$ is the distance function with respect to the hyperbolic metric.

Also we need the following integration result. We define a generalized version of the modified Dirac operator by

$$M_n f = Df - \frac{n}{x_n} Q' f.$$

**Theorem 2.5.** [3] Let $\Omega$ be an open subset of $\mathbb{R}^{n+1}_+$. If $K \subset \Omega$ is an oriented $(n+1)$-dimensional manifold-with-boundary and $g$ is a smooth Clifford algebra-valued function on $\Omega$, then

$$\int_{\partial K} P(\nu(x) g(x)) \frac{dS(x)}{x_n} = \int_K P(M_n g(x)) \frac{dx}{x_n}.$$
Using the preceding result we are able to prove the following lemma.

**Lemma 2.6.** Assume that $f$ is hypermonogenic on $\Omega$ and $\overline{B_h(a,R_h)} \subset \Omega$. Then

$$\int_{\partial B_h(a,R_h)} Q\left(\frac{e_n\nu(x)f(x)}{x_n^m}\right) dS(x) = \int_{B_h(a,R_h)} Qf(x) dx_h.$$

**Proof.** It is easy to see that $Q(e_n\nu(x)f(x)) = P'(\nu(x)f(x))$. Using Theorem 2.5 we have

$$\int_{\partial B_h(a,R_h)} Q\left(\frac{e_n\nu(x)f(x)}{x_n^m}\right) dS(x) = \int_{B_h(a,R_h)} P'(M_n f(x)) \frac{dx}{x_n^m}.$$

Since

$$M f(x) = D f(x) + \frac{n-1}{x_n} Q' f(x),$$

we obtain

$$M_n f(x) = M f(x) + \frac{Q' f(x)}{x_n} = \frac{Q' f(x)}{x_n}.$$  

The proof is complete. \qed

Also we shall need the following result.

**Lemma 2.7.** [5] If $f$ is a twice continuously differentiable function from $\Omega \subset \mathbb{R}^{n+1}_+$ into $C_\ell_{0,n}$, and $B_h(e_n, R_h) \subset \Omega$, we obtain

$$\frac{d}{dR_h} \left( \frac{1}{\sinh^n R_h} \int_{\partial B_h(a,R_h)} f \, d\sigma_h \right) = \frac{1}{\sinh^n R_h} \int_{B_h(a,R_h)} \Delta_h f \, dx_h.$$

Next we deduce that any eigenfunction of the Laplace–Beltrami operator is a $Q$-part of some hypermonogenic function. The theorem is formulated only for a ball but similar theorem holds also for more general star-shaped domains (cf. [2]).

**Theorem 2.8.** [2] Let $h : B_h(a, R) \to C_\ell_{0,n}$ be a solution of the equation

$$\Delta_h h(x) = -(n-1) h(x) \text{ on } \overline{B_h(a, R)}.$$

There exists a hypermonogenic function $f : B_h(a, R) \to C_\ell_{0,n}$ satisfying $h = Q f$ on $B_h(a, R)$.

Now we may start to give the proof for the Theorem 2.1. First we show that the statement (1) implies (2).

**Lemma 2.9.** Let $h : \Omega \to C_\ell_{0,n}$ be a solution of

$$\Delta_h h(x) = -(n-1) h(x) \text{ on } \Omega \text{ and let } \overline{B_h(a, R_h)} \subset \Omega.$$

Then

$$h(a) = \frac{1}{(n-1) \omega_n+1 \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt} \int_{\partial B_h(a,R_h)} h(x) \, d\sigma_h(x).$$

**Proof.** Let $f$ be a hypermonogenic function satisfying $Q f = h$ on $B_h(a, R_h)$. Applying Proposition 2.3 and Theorem 2.4 we obtain

$$\omega_{n+1} Q f(a) = \int_{\partial B_h(a,R_h)} Q\left(\frac{x - \tau(a,x)}{a_n x_n^m} \sinh d_h(x,a) - a_n \sinh^2 d_h(x,a) e_n \nu(x)f(x)\right) dS(x).$$
Since on the ball $B_h(a, R_h)$ the unit normal field is given by $\nu(x) = \frac{x - \tau(a, x)}{R_e(a, R_h)}$, we infer

$$\omega_{n+1} Qf(a) = \frac{R_e(a, R_h) \cosh R_h}{a_n \sinh^{n+1} R_h} \int_{\partial B_h(a, R_h)} Qf(x) d\sigma_h$$

$$- \frac{1}{\sinh^{n-1} R_h} \int_{\partial B_h(a, R_h)} Q\left( \frac{\epsilon_n \nu(x) f(x)}{x_n} \right) dS(x).$$

Since $\nu(x)\nu(x) = 1$, we obtain

$$\omega_{n+1} Qf(a) = \frac{R_e(a, R_h) \cosh R_h}{a_n \sinh^{n+1} R_h} \int_{\partial B_h(a, R_h)} Qf(x) d\sigma_h$$

$$- \frac{1}{\sinh^{n-1} R_h} \int_{B_h(a, R_h)} Qf(x) dR_h.$$

Using Lemma 2.7 and the assumption we have

$$\omega_{n+1} Qf(a) = \frac{\cosh R_h}{\sinh^n R_h} \int_{\partial B_h(a, R_h)} Qf(x) d\sigma_h$$

$$- \frac{\sinh R_h}{n-1} \int_{\partial B_h(a, R_h)} \frac{1}{\sinh^n R_h} Qf(x) d\sigma_h.$$

The equation in the above gives us the differential equation

$$\sinh(R_h) g'(R_h) + (n-1) \cosh(R_h) g(R_h) = C,$$

where $C = (n-1) Qf(a)$ and

$$g(R_h) = \frac{1}{\omega_{n+1} \sinh^n R_h} \int_{\partial B_h(a, R_h)} Qf(x) d\sigma_h.$$

The general solution of this equation is

$$g(R_h) = C \int_0^{R_h} \sinh^{n-2}(t) dt + C_0 \frac{\sinh^{n-1}(R_h)}{\sinh^{n-2}(R_h)}.$$

Since $g$ is a continuous function, we have

$$\lim_{R_h \to 0^+} g(R_h) = Qf(a)$$

and then $C_0 = 0$. The proof is complete. \qed

We show next that the statement (2) implies (3).

**Lemma 2.10.** Assume

$$h(a) = \frac{1}{\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) dt} \int_{\partial B_h(a, R_h)} h(x) d\sigma_h(x).$$

Then

$$h(a) = \frac{n-1}{\omega_{n+1} \phi(R_h)} \int_{B_h(a, R_h)} Qf(x) dx_h,$$

where

$$\phi(R_h) = (n-1) \cosh R_h \int_0^{R_h} \sinh^{n-2}(t) dt - \sinh^{n-1} R_h.$$
Proof. Using Lemma 2.7 we have
\[ -\frac{n-1}{\sinh^n R_h} \int_{B_{e_n}(R_h)} h(x) \, dx_h = \frac{d}{dR_h} \left( \frac{1}{\sinh^n R_h} \int_{\partial B_{e_n}(R_h)} h(x) \, d\sigma_h \right). \]

By the assumptions
\[ -\frac{n-1}{\omega_{n+1} \sinh^n R_h} \int_{B_{e_n}(R_h)} h(x) \, dx_h \]
\[ = \left( \frac{1}{\sinh R_h} - (n-1) \frac{\cosh R_h \int_0^{R_h} \sinh^{n-2} t \, dt}{\sinh^n R_h} \right) h(a). \]

Then
\[ h(a) = \frac{n-1}{\omega_{n+1} (n-1) \cosh R_h \int_0^{R_h} \sinh^{n-2} (t) \, dt - \sinh^{n-1} R_h} \int_{B_{e_n}(R_h)} h(x) \, dx_h, \]
and the proof is complete. \(\Box\)

We show next that (3) implies (2). First we need the following lemma.

**Lemma 2.11.** Let \( T : B_{e_n}(R_h) \to B_{e_n}(R_h) \) be the mapping \( T(x) = a_n x + Pa \), where \( a \in \mathbb{R}^{n+1}_+ \). Then \( T \) is diffeomorphism, and the following transformation rules hold:

(a) \( \int_{\partial B_{e_n}(R_h)} f(y) \, d\sigma_h(y) = \int_{\partial B_{e_n}(e_n R_h)} f \circ T^{-1}(x) \, d\sigma_h(x) \),
(b) \( \int_{B_{e_n}(R_h)} f \circ T(x) \, d\sigma_h(x) = \int_{B_{e_n}(e_n R_h)} f(y) \, d\sigma_h(y) \),
(c) \( \int_{B_{e_n}(R_h)} h(y) \, dy_h = \int_{B_{e_n}(e_n R_h)} h \circ T^{-1}(x) \, dx_h \),
(d) \( \int_{B_{e_n}(R_h)} h \circ T(x) \, dx_h = \int_{B_{e_n}(e_n R_h)} h(y) \, dy_h \).

That allows us to prove the following lemma.

**Lemma 2.12.** Assume
\[ h(a) = \frac{n-1}{\omega_{n+1} \phi(R_h)} \int_{B_{e_n}(R_h)} h(x) \, dx_h. \]

Then
\[ h(a) = \frac{1}{\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2} (t) \, dt} \int_{\partial B_{e_n}(R_h)} h(x) \, d\sigma_h(x). \]

Proof. Using the previous proposition we infer
\[ h(a) = \int_{B_{e_n}(e_n R_h)} h \circ T^{-1}(x) \, dx_h. \]

Using the polar coordinates we have
\[ h(a) = \frac{n-1}{\omega_{n+1} \phi(R_h)} \int_0^{R_h} \int_{\partial B_{e_n}(e_n t)} h \circ T^{-1}(x) \, d\sigma_h(x) \, dt. \]

Then
\[ \omega_{n+1} \phi(R_h) h(a) = (n-1) \int_0^{R_h} \int_{\partial B_{e_n}(e_n t)} h \circ T^{-1}(x) \, d\sigma_h(x) \, dt. \]
Then for \( x \) using the (a)-part of the preceding proposition we have

\[
\phi'(R_h) = (n - 1) \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt,
\]

using Lemma 2.7 we have

\[
h(a) = \frac{1}{\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt} \int_{\partial B_h(a,R_h)} h \circ T^{-1}(x) \, d\sigma_h(x).
\]

Then using the (a)-part of the preceding proposition we have

\[
h(a) = \frac{1}{\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt} \int_{\partial B_h(a,R_h)} h(y) \, d\sigma_h(y).
\]

The proof is complete. 

Lastly we deduce that (2) implies (1).

**Lemma 2.13.** Assume

\[
h(a) = \frac{1}{\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt} \int_{\partial B_h(a,R_h)} h(x) \, d\sigma_h(x).
\]

Then

\[
\Delta h(x) = -(n - 1)h(x)
\]

for \( x \in B_h(a,R_h) \).

**Proof.** Since

\[
\frac{d}{dR_h} \int_0^{R_h} \sinh^{n-2}(t) \, dt = \frac{\sinh^{n-2} R_h \phi(R_h)}{\left( \int_0^{R_h} \sinh^{n-2}(t) \, dt \right)^2},
\]

using Lemma 2.7 we obtain

\[
0 = \frac{\sinh^{n-2} R_h \phi(R_h)}{\left( \int_0^{R_h} \sinh^{n-2}(t) \, dt \right)^2} \frac{1}{\sinh^n R_h} \int_{\partial B_h(a,R_h)} h(x) \, d\sigma_h(x)
\]

\[
+ \frac{\sinh^{n-1} R_h}{\int_0^{R_h} \sinh^{n-2}(t) \, dt} \frac{1}{\sinh^n R_h} \int_{B_h(a,R_h)} \Delta h(x) \, dx_h.
\]

Since (2) and (3) are equivalent, we obtain the formula

\[
\int_{\partial B_h(a,R_h)} h(x) \, d\sigma_h(x) = (n - 1) \frac{\sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt}{\phi(R_h)} \int_{B_h(a,R_h)} h(x) \, dx_h.
\]

Then

\[
0 = \frac{(n - 1) \phi(R_h)}{\left( \int_0^{R_h} \sinh^{n-2}(t) \, dt \right)^2} \frac{1}{\sinh^2 R_h} \int_{B_h(a,R_h)} h(x) \, dx_h
\]

\[
+ \frac{1}{\int_0^{R_h} \sinh^{n-2}(t) \, dt} \frac{1}{\sinh R_h} \int_{B_h(a,R_h)} \Delta h(x) \, dx_h,
\]

that is,

\[
\frac{1}{\sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt} \int_{B_h(a,R_h)} (\Delta h(x) + (n - 1)h(x)) \, dx_h = 0.
\]
Since $R_h$ is arbitrary, we obtain that
\[ \Delta_0 h(a) + (n - 1) h(a) = 0. \]
The proof is complete. \qed

References


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