DAVID HOMEOMORPHISMS VIA CARLESON BOXES

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Abstract. We construct a family of examples of increasing homeomorphisms of the real line whose local quasi-symmetric distortion blows up almost everywhere, which nevertheless can be realized as the boundary values of David homeomorphisms of the upper half-plane. The construction of such David extensions uses Carleson boxes.

1. Introduction

This paper answers a question that was posed by Zakeri, in conversation with the author, some time ago. The question concerns David homeomorphisms, which are very useful generalizations of quasi-conformal mappings introduced by David in [D] (see below for the definition). Such homeomorphisms were used for the first time in complex dynamics by Haïssinsky [H] to perform parabolic surgery on rational maps, and were later used by Petersen and Zakeri [PZ] in their study of quadratic polynomials with Siegel disks.

The question asked by Zakeri can be roughly stated as follows. Suppose a given homeomorphism of the real line is the boundary map of a David homeomorphism of the upper half-plane. How bad can its pointwise distortion be? Before we can explain the meaning and give the precise statement of Zakeri’s question, we need to recall some standard notions.

1.1. Quasi-conformality. Let \( f: U \to V \) be an orientation-preserving homeomorphism between open subsets of the plane. If \( f \) is differentiable at \( z \in U \) with \( \partial f(z) \neq 0 \), its Beltrami coefficient at \( z \) is \( \mu_f(z) = \overline{\partial f(z)}/\partial f(z) \). The homeomorphism \( f = u + iv \) is said to be absolutely continuous on lines (ACL) if the restrictions of \( u \) and \( v \) (the real and imaginary parts of \( f \), respectively) to Lebesgue almost every horizontal and Lebesgue almost every vertical lines are absolutely continuous functions. The ACL condition implies differentiability almost everywhere. We say that \( f \) is quasi-conformal if \( f \) is ACL and its Beltrami coefficient \( \mu_f(z) \) (defined almost everywhere) satisfies \( |\mu_f(z)| \leq k < 1 \) a.e., for some constant \( 0 \leq k < 1 \). The dilatation or conformal distortion of \( f \) at \( z \) is

\[
K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.
\]


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The supremum of $K_f(z)$ over all $z$ is called the maximal dilatation, denoted $K_f$. Note that $K_f < \infty$ whenever $f$ is quasi-conformal. The composition $f \circ g$ of quasi-conformal homeomorphisms $f, g$ is quasi-conformal, and $K_{f \circ g} \leq K_f K_g$. See [A] or [G] for these facts on qc-mappings, and much more.

1.2. David homeomorphisms. The concept of David homeomorphism, introduced by David in [D], is a generalization of quasi-conformality. One allows the dilatation of $f$ to degenerate—in other words, one allows $K_f(z) \to \infty$ at certain places—but in a controlled fashion. More precisely, a homeomorphism $f : U \to V$ between open sets of the plane is David if $f$ is orientation-preserving, ACL, and there exist constants $C > 0$ and $\alpha > 0$ such that, for each $\lambda > 0$ we have

$$\text{Area}_s \{ z \in U : K_f(z) > \lambda \} < C e^{-\alpha \lambda}.$$ 

Here $\text{Area}_s$ denotes the spherical area, namely

$$\text{Area}_s(E) = \iint_E \frac{dx \, dy}{(1 + |z|^2)^2},$$

for every Borel measurable set $E \subseteq \mathbb{C}$. When $U$ is bounded, we may safely replace the spherical area by the usual Euclidean area (two-dimensional Lebesgue measure) in the definition.

1.3. Quasi-symmetric distortion. Given an increasing homeomorphism of the real line $\phi : \mathbb{R} \to \mathbb{R}$ we define its quasi-symmetric local distortion function to be

$$\sigma_\phi(x) = \limsup_{t \downarrow 0} \max \left\{ \frac{\phi(x + t) - \phi(x)}{\phi(x) - \phi(x - t)}, \frac{\phi(x + t) - \phi(x)}{-\phi(x) - \phi(x - t)} \right\},$$

We say that $\phi$ is quasi-symmetric if there exists $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{\phi(x + t) - \phi(x)}{\phi(x) - \phi(x - t)} \leq M,$$

for all $x \in \mathbb{R}$ and all $t > 0$. In particular, if $\phi$ is quasi-symmetric with quasi-symmetry constant $M$, then $\sigma_\phi(x) \leq M$ for all $x$. If $\phi$ is quasi-symmetric, then it can be extended to a quasi-conformal homeomorphism of the upper half-plane (and the dilatation of the extension can be bounded in terms of the quasi-symmetry constant of $\phi$ only). Conversely, the boundary values of every quasi-conformal homeomorphism of the upper half-plane are given by a quasi-symmetric homeomorphism of the real line. These facts are well-known, and their proofs can be found in [A].

Following [Z], we define the pointwise distortion of $\phi$ at $x \in \mathbb{R}$, denoted by $\lambda_\phi(x)$, to be the analogue of (1), with ‘sup$_{t>0}$’ replacing ‘lim sup$_{t \downarrow 0}$’. Thus, $\sigma_\phi(x) \leq \lambda_\phi(x)$ for all $x$. It is proved in [Z, Thm. A] that, if $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(x + 1) = \phi(x)$ for all $x$ and $\lambda_\phi \in L^p[0, 1]$ for some $p > 0$, then $\phi$ extends to a David homeomorphism $f$ of the upper half-plane (still satisfying $f(z + 1) = f(z)$ for all $z$). This integrability condition is sufficient, but not necessary, for David extendibility. It implies, of course, that $\lambda_\phi$ is almost everywhere finite. The question arises: Is it possible for the pointwise distortion $\lambda_\phi$ to blow up in a large subset of the real line and yet $\phi$ still be extendible to a David homeomorphism? This is the question asked by Zakeri, and our answer is given by the following result.

**Theorem 1.** There exists a family of increasing homeomorphisms $\phi : \mathbb{R} \to \mathbb{R}$ having the following properties.
(a) $\sigma_\phi(x) = \infty$ for every $x$ in a full-measure subset of $\mathbb{R}$;
(b) $\phi$ has modulus of continuity $\rho(t) = t \log (1/t)$;
(c) $\phi = f \upharpoonright \mathbb{R}$, where $f$ is a David homeomorphism of the (closed) upper half-plane.

Note that, although the pointwise distortions of the examples produced by this theorem blow up almost everywhere, these homeomorphisms still exhibit some regularity, in the sense that they are $\alpha$-Hölder continuous for all $0 < \alpha < 1$ (as implied by (b)). In order to prove this theorem, we will first show that, given an increasing homeomorphism $\phi$ of the real line satisfying some suitable conditions, a David extension of $\phi$ to the upper half-plane can be constructed using a system of Carleson boxes. We will then verify that these conditions are sufficiently mild to allow for a $\phi$ with properties (a) and (b) above. Apart from an unavoidable limit, the construction of such $\phi$’s follows a rather explicit discrete, piecewise-linear scheme. The proof that (a) holds in our examples will involve a simple Borel–Cantelli argument.

Remark 1. Each example $\phi$ to be constructed in the proof of theorem 1 will be the lift of a circle homeomorphism (i.e., it will satisfy $\phi(x + 1) = \phi(x) + 1$ for all $x$), and its David extension $f$ will satisfy $f(z + 1) = f(z) + 1$ for all $z$ in the upper half-plane. Therefore everything can be quotiented down to the unit disk to produce examples of David homeomorphisms of $D$ with pointwise distortion blowing up almost everywhere on $S^1 = \partial \mathbb{D}$.

2. Dyadic approximations

Let $\phi : [0, 1] \to [0, 1]$ be an increasing homeomorphism. We associate to $\phi$ a sequence $\phi_n : [0, 1] \to [0, 1]$ of dyadic approximations defined as follows. Denote by $\mathcal{P}_n$ the $n$-th dyadic partition of $[0, 1]$ (modulo endpoints), namely

$$\mathcal{P}_n = \left\{ \left[ \frac{j}{2^n}, \frac{j + 1}{2^n} \right] : 0 \leq j \leq 2^n - 1 \right\}.$$ 

Its elements are called atoms, and the endpoints of atoms are called vertices. Each atom of $\mathcal{P}_n$ is the union of exactly two atoms of $\mathcal{P}_{n+1}$, for all $n \geq 0$. We define $\phi_n : [0, 1] \to [0, 1]$ so that $\phi_n$ is affine on each atom of $\mathcal{P}_n$ and so that it agrees with $\phi$ in the vertices of $\mathcal{P}_n$ (i.e., the points $j2^{-n}$, $0 \leq j \leq 2^n - 1$). Note in particular that $\phi_0 = \text{id}$ and that each $\phi_n$ is quasi-symmetric. Moreover, $\phi_n(\Delta) = \phi(\Delta)$ for each $\Delta \in \mathcal{P}_n$. More importantly, the sequence of piecewise linear homeomorphisms $\phi_n$ converges uniformly to $\phi$. For later use, let us define a sequence of positive numbers $M_1, M_2, \ldots$ as follows:

$$M_n = \sup \frac{|\phi_n(\Delta)|}{|\phi_n(\Delta^*)|},$$

where the supremum is taken over all pairs of atoms $\Delta, \Delta^* \in \mathcal{P}_n$ lying in a single atom of $\mathcal{P}_{n-1}$ (i.e., such that $\Delta \cup \Delta^* \in \mathcal{P}_{n-1}$). Note that $M_n \geq 1$; in fact $M_n > 1$ unless $\phi_n = \phi_{n-1}$. We define the cumulative distortion function of $\phi$ to be the function $\kappa_\phi : \mathbb{N} \to \mathbb{R}^+$ given by

$$\kappa_\phi(n) = \prod_{j=1}^{n} \frac{1 + M_j}{2}.$$
Let us also remark that the slope \( s_\Delta \) of \( \phi_n \) on each atom \( \Delta \in \mathcal{P}_n \) satisfies

\[
\prod_{j=1}^{n} \frac{2}{1 + M_j} \leq s_\Delta \leq \prod_{j=1}^{n} \frac{2M_j}{1 + M_j}.
\]

The following result is an easy consequence of the definitions given so far, and will be used in §3.

**Lemma 1.** For each \( n \geq 1 \), the piecewise affine homeomorphism \( \phi_n \times \text{id} : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R} \) has quasi-conformal distortion bounded by \( \kappa_\phi(n) \).

**Proof.** If \( h(x) = ax + b \) is an affine homeomorphism of the line with \( a > 0 \), then \( H = h \times \text{id} \) is quasi-conformal and \( K_H \leq \max\{a, a^{-1}\} \), as the reader can easily check. The lemma follows from this fact and the estimate (3), taking into account that

\[
\max\{s_\Delta, s_\Delta^{-1}\} \leq \prod_{j=1}^{n} \frac{1 + M_j}{2} = \kappa_\phi(n).
\]

\( \square \)

3. Carleson boxes extensions

Assume we are given \( \phi \) as in the previous section, and its associated sequence \( (\phi_n) \) of dyadic approximations. Let us define an extension of \( \phi \) to a homeomorphism \( f : [0,1]^2 \to [0,1]^2 \), using a Carleson boxes construction in order to bound its local quasi-conformal distortion. We will define \( f \) as a limit

\[
f = \lim_{n \to \infty} f_n \circ f_{n-1} \circ \cdots \circ f_1
\]

in the \( C^0 \)-topology, where each \( f_n \) is a piecewise affine homeomorphism of the unit square built from \( \phi_n \).

![Figure 1. Images of Carleson boxes under \( f^{(n-1)} \).](image)
Here is the detailed description. Let us fix a sequence $1 = a_0 > a_1 > a_2 > \cdots > a_n > \cdots$ of positive numbers converging to zero. We define $f_n : [0, 1]^2 \to [0, 1]^2$ to be equal to the identity map on the set $\{(x, y) \in [0, 1]^2 : y \geq a_n\}$, to be equal to $(\phi_n \circ \phi_n^{-1}) \times \text{id}$ on the set $\{(x, y) \in [0, 1]^2 : 0 \leq y < a_{n+1}\}$, and interpolating in a piecewise linear fashion between these two on the strip $[0, 1] \times [a_{n+1}, a_n]$. This interpolating map $g_n : [0, 1] \times [a_{n+1}, a_n] \to \text{id}$ in turn is defined as follows. Divide each rectangle of the form $\phi_{n-1}(\Delta) \times [a_{n+1}, a_n]$, with $\Delta \in \mathcal{P}_{n-1}$, into 3 triangles, by joining the two upper-most vertices of the rectangle to the midpoint of its bottom side, in the domain of $g_n$ (see Figure 1). Similarly, in the target, divide each rectangle $\phi_{n-1}(\Delta) \times [a_{n+1}, a_n]$ into 3 triangles by joining the upper-most vertices to the point on the bottom side that lies vertically above the image of the midpoint of $\phi_{n-1}(\Delta)$ under $\phi_n \circ \phi_n^{-1}$. Let $g_n$ map each triangle in the domain affinely onto the corresponding triangle in the target, in the obvious way (see Figure 1). Summarizing, the piecewise affine homeomorphism $f_n$ has been defined so that

$$f_n(x, y) = \begin{cases} (x, y) & \text{if } y \geq a_n, \\ g_n(x, y) & \text{if } a_{n+1} < y < a_n, \\ (\phi_n \circ \phi_n^{-1}(x), y) & \text{if } 0 \leq y \leq a_{n+1}. \end{cases}$$

Note that the composition $f^{(n)} = f_n \circ f_{n-1} \circ \cdots \circ f_1$ agrees with $\phi_n \times \text{id}$ on the strip $[0, 1] \times [0, a_{n+1}]$. Moreover, for each $(x, y) \in [0, 1]^2$ with $y > 0$, the sequence $f^{(n)}(x, y)$ is eventually constant. The limiting homeomorphism $f = \lim f^{(n)}$ agrees with $\phi = \lim \phi_n$ on $[0, 1] \times \{0\}$. We have also $f(x, y) = f^{(n)}(x, y)$ for all $(x, y) \in [0, 1] \times [a_{n+1}, a_n]$.

**Definition 1.** The homeomorphism $f$ constructed above is called a Carleson boxes extension of $\phi$.

Note that the restriction of $f$ to each strip of the form $[0, 1] \times [a_{n+1}, a_n]$ is piecewise affine, hence quasi-conformal. We wish to bound the local dilatation of $f$ in each strip. For this purpose, we decompose the unit square (in the domain) into Carleson boxes

$$B_{n,j} = \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right] \times [a_{n+1}, a_n], \quad n \geq 0, \quad 0 \leq j \leq 2^n - 1.$$ 

The first thing to notice is that the qc-distortion of $f^{(n)}$ on each box $B_{m,j}$ with $m > n$ is bounded by $\kappa_{\phi}(n)$, the cumulative distortion function introduced in (2). This is immediate from Lemma 1. In order to bound the qc-distortion of $f$, we therefore still need an estimate on $K_{f^{(n)}}(z)$ for all $z \in B_{j,n}$. Here we have to remember that $f^{(n)} = f_n \circ f^{(n-1)}$, and that $f_n$ agrees with $g_n$ on the strip $[0, 1] \times [a_{n+1}, a_n]$. This strip has been decomposed into triangles of two types: upside-down triangles, such as the one shaded in Figure 1, and those with one side supported in the bottom edge of the strip. In the latter triangles, $g_n$ agrees with $(\phi_n \circ \phi_n^{-1}) \times \text{id}$. Hence in those triangles $f^{(n)}$ agrees with $g_n \circ f^{(n-1)} = \phi_n \times \text{id}$, and therefore (again by Lemma 1) its qc-distortion there is bounded by $\kappa_{\phi}(n)$. In the upside-down triangles, $g_n$ is a linear shearing map. Hence we need the following lemma.
Lemma 2. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear shearing map given by $T(x, y) = (x + by, y)$, where $b > 0$. Then its qc-dilatation is everywhere constant, equal to

$$K_T = \left(\frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}\right)^2. \tag{5}$$

Proof. Writing $T$ in complex coordinates, we get

$$T(z, \bar{z}) = \left(1 + \frac{b}{2i}\right)z - \frac{b}{2i} \bar{z}.$$  

Hence

$$\partial T = 1 + \frac{b}{2i}, \quad \overline{\partial T} = -\frac{b}{2i},$$

and therefore

$$k = \left|\frac{\overline{\partial T}}{\partial T}\right| = \frac{b/2}{\sqrt{1 + b^2/4}}.$$

This gives us

$$K_T = \frac{1 + k}{1 - k} = \frac{b/2 + \sqrt{1 + b^2/4}}{\sqrt{1 + b^2/4} - b/2}$$

from which (5) follows. \qed

Remark 2. Note that the same result holds true if $T$ is post-composed with a translation. Note also that for $b$ large ($b > 1$) the right-hand side of (5) is $O(b^2)$, whereas for $b$ small ($b < 1$), it is simply $O(1)$.

To apply lemma 2 in our situation, each rectangle

$$R_{n, \Delta} = \phi_{n-1}(\Delta) \times [a_{n+1}, a_n]$$

should be normalized to unit size by a suitable affine map. We note en passant that $R_{n, \Delta} = f^{(n-1)}(B_{n-1, j})$ for some $j$. Let us denote by $U_{n, \Delta} \subset R_{n, \Delta}$ the upside-down triangle determined by the top side of $R_{n, \Delta}$ and the midpoint of its bottom side (see Figure 2). We want to bound the qc-distortion of $g_n$ on $U_{n, \Delta}$, and the result we need is lemma 3 below. The rough idea is that such distortion will be bounded provided the rectangles $R_{n, \Delta}$ don’t become too skinny. As we shall see, if we choose the sequence $a_n$ so that $\delta_n = a_n - a_{n+1}$ decreases to zero at a slower rate than $2^{-n}$, then the map $g_n$ restricted to the upside-down triangle $U_{n, \Delta}$ will be a linear shearing with small $b$, whose qc-distortion is $O(1)$ by lemma 2 and the above remark.

Let us agree to say that a non-decreasing sequence $\tau: \mathbb{N} \to \mathbb{R}^+$ has sub-exponential growth if

$$\lim_{n \to \infty} \frac{\log \tau(n)}{n} = 0.$$  

For instance, if $\tau(n)$ grows at most polynomially with $n$, then $\tau$ has sub-exponential growth. This is tantamount to saying that $\tau(n)\rho^{-n} \to 0$ for all $\rho > 1$. We have the following lemma.

Lemma 3. Suppose $\rho = \liminf (2^n \delta_n) > 1$, and assume also that $\kappa_\phi$ has sub-exponential growth. Then $K_{g_n|_{U_{n, \Delta}}} = O(1)$, i.e., the qc-distortion of $g_n$ in each upside-down triangle $U_{n, \Delta}$ is bounded, and uniformly so in $n$. 


Proof. We denote by $Q$ the unit square $[0,1]^2$. Let $\Lambda: R_{n,\Delta} \to Q$ be the affine orientation-preserving map sending vertices to vertices (and preserving horizontals). Since the horizontal sides of $R_{n,\Delta}$ have length $\ell_n = |\phi_{n-1}(\Delta)|$ and its vertical sides have length $\delta_n$, the map $\Lambda$ has the form

$$\Lambda(x,y) = (\ell_n^{-1}x + \alpha, \delta_n^{-1}y + \beta).$$

Here the constants $\alpha, \beta$ depend on $n, \Delta$, of course, but their actual values are irrelevant to us. Since $g_n(R_{n,\Delta}) = R_{n,\Delta}$, we can write $g_n|_{U_{n,\Delta}} = \Lambda^{-1} \circ T \circ \Lambda$, where $T$ is a linear shearing map of the form $T(x,y) = (x + by + c, y)$, where $b = O(1)$ (in fact, one can easily check that $b \leq 1/2$; see Figure 2). A simple calculation yields, for each $(x,y) \in U_{n,\Delta}$,

$$g_n(x,y) = (x + b\ell_n\delta_n^{-1}y + A, y + B),$$

where $A, B$ are constants. Now, taking into account that $|\Delta| = 2^{-n+1}$, and using (3) and (4) of lemma 1, we see that

$$\ell_n = |\phi_{n-1}(\Delta)| = s_{\Delta}|\Delta| \leq \kappa_\phi(n)2^{-n+1}.$$

This shows that

$$b\ell_n\delta_n^{-1} \leq 2b\kappa_\phi(n)(2^n\delta_n)^{-1} \leq 2b\kappa_\phi(n)\rho^{-n} \to 0$$

as $n \to \infty$, because $\kappa_\phi$ has sub-exponential growth. Taking this information back to (6) and applying lemma 2, we get the desired result. \qed

Combining all of the above facts, we arrive at the following lemma.

**Lemma 4.** Suppose $\lim \inf (2^n\delta_n) > 1$, and assume also that $\kappa_\phi(n)$ has sub-exponential growth. Then the maximal dilatation $K_{n,j}$ of $f$ in each Carleson box $B_{n,j}$ satisfies the inequality $K_{n,j} \leq c\kappa_\phi(n)$, for some uniform constant $c > 0$.

Now we have the following theorem, which is the main result of this section.

**Theorem 2.** Let $\phi: [0,1] \to [0,1]$ be an increasing homeomorphism whose cumulative distortion function $\kappa_\phi(n)$ satisfies $\kappa_\phi(n) \leq Cn$ for some $C > 0$. Then $\phi$ extends to a David homeomorphism of the unit square.
Proof. Consider the Carleson boxes extension $f$ constructed above. We assume that the sequence $\delta_n = a_n - a_{n+1}$ is chosen to be of the form $\delta_n = c_0 \delta^n$ for some $\frac{1}{2} < \delta < 1$, so that the hypothesis of lemma 4 is satisfied. Let us estimate the area of the set of points where $K_f$ is greater than $\lambda$ for any given $\lambda > 0$. Using lemma 4, we have

$$\text{Area } \{ z : K_f(z) > \lambda \} \leq \text{Area } \left\{ \bigcup_{K_n : \delta > \lambda} B_{n,j} \right\} \leq \sum_{\kappa_n > c^{-1} \lambda} \text{Area}(B_{n,j}) = \sum_{\kappa_n > c^{-1} \lambda} \delta_n.$$  

Now, suppose we know that, for all $n \geq 1$ and constants $C_1, \alpha > 0$,

$$\sum_{n \geq m} \delta_n \leq C_1 e^{-\alpha \kappa(m)}.$$  

Then, letting $m = \inf \{ n : \kappa_n(n) > c^{-1} \lambda \}$, we see from (7) that

$$\text{Area } \{ z : K_f(z) > \lambda \} \leq C_1 e^{-\alpha \kappa(m)} < C_1 e^{-\alpha \lambda/c},$$

and therefore $f$ is indeed a David homeomorphism.

Hence, all we need is that (8) be true. But on the one hand, since $\delta_n = c_0 \delta^n$, the left-hand side of (8) is

$$\sum_{n \geq m} \delta_n = \frac{c_0 \delta^m}{1 - \delta} = \frac{c_0}{1 - \delta} e^{-m \log \frac{1}{\delta}}.$$  

On the other hand, since $\kappa_n(n) < Cn$, the right-hand side of (8) is bounded from below by $e^{-\alpha Cn}$. Therefore (8) will be true provided we take $\alpha = C^{-1} \log \frac{1}{\delta}$ and $C_1 = (1 - \delta)/c_0$. \qed

4. Divergent quasi-symmetric distortion

Let us now define a special family of homeomorphisms of the unit interval, whose members have divergent local quasi-symmetric distortion almost everywhere. Each member of the family will be determined by a given sequence of quasi-symmetric weights $\mathcal{M} = (M_1, M_2, \ldots, M_n, \ldots)$, where $M_n > 1$ for each $n \geq 1$, and will accordingly be denoted by $\phi_\mathcal{M}$. We define $\phi_\mathcal{M}$ as the limit of a sequence of dyadic piecewise linear homeomorphisms $\phi_n$, inductively constructed as follows. We take $\phi_0 = \text{id}$. Suppose $\phi_n$ has been defined so that it is linear on each atom of $\mathcal{P}_n$. Given an atom $\Delta \in \mathcal{P}_n$, write $\Delta = \Delta^- \cup \Delta^+$ where $\Delta^-, \Delta^+ \in \mathcal{P}_{n+1}$ and $\Delta^-$ lies to the left of $\Delta^+$. Define $\phi_{n+1}$ on $\Delta$ so that it agrees with $\phi_n$ at the endpoints of $\Delta$, is linear with slope $s^-$ on $\Delta^-$, linear with slope $s^+$ on $\Delta^+$, and $s^- = M_{n+1} s^+$ (see Figure 3). Note that the arithmetic mean of the slopes $s^-, s^+$ is equal to the slope $s_\Delta$ of $\phi_n$ on $\Delta$, so that

$$s^- = \frac{2M_{n+1}}{1 + M_{n+1}} s_\Delta \quad \text{and} \quad s^+ = \frac{2}{1 + M_{n+1}} s_\Delta.$$  

Note that, inductively defined in this way, the sequence $(\phi_n)$ satisfies $\phi_n \leq \phi_{n+1}$ for all $n$. We let $\phi_\mathcal{M} = \lim \phi_n$. This limit is uniform, i.e., it holds in the $C^0$ topology. Under suitable conditions on $\mathcal{M}$, the limiting map is better than continuous, as the following result shows.
Lemma 5. Let $\mathcal{M} = (M_n)_{n \geq 1}$ be a sequence of quasi-symmetric weights such that $\prod_{j=1}^{n} M_j \leq cn$ for all $n$ (and some constant $c > 0$). Then $\phi_{\mathcal{M}}$ has modulus of continuity $\rho(t) = t \log (1/t)$. In particular, it is $\alpha$-Hölder for every $0 < \alpha < 1$.

Proof. Let us write $\phi = \phi_{\mathcal{M}}$ in this proof. Given $x, y \in [0, 1]$, say with $x < y$, let $n \geq 0$ be the smallest integer such that $[x, y] \supseteq \Delta_{x,y}$ for some $\Delta_{x,y} \in \mathcal{P}_n$. Then $[x, y]$ is contained in the union of at most three atoms of $\mathcal{P}_n$, and so $|\Delta_{x,y}| \leq |x - y| \leq 3|\Delta_{x,y}|$. Since $|\Delta_{x,y}| = 2^{-n}$, this can be rewritten as

$$c_1^{-1} n \leq \log \frac{1}{|x - y|} \leq c_1 n$$

for some $c_1 > 1$. Note that for each atom $\Delta \in \mathcal{P}_n$ we have $\phi(\Delta) = \phi_n(\Delta)$, and $|\phi_n(\Delta)| = s_\Delta |\Delta|$, where $s_\Delta$ is the slope of $\phi_n|_{\Delta}$, But clearly (see (9))

$$s_\Delta \leq \prod_{j=1}^{n} \frac{2M_j}{1 + M_j}.$$ 

Combining these facts, we see that

$$|\phi(x) - \phi(y)| \leq 3 \prod_{j=1}^{n} \frac{2M_j}{1 + M_j} |\Delta_{x,y}| = 3 \prod_{j=1}^{n} \frac{M_j}{1 + M_j}$$

$$= 3 \left( \prod_{j=1}^{n} \frac{1}{1 + M_j} \right) \left( \prod_{j=1}^{n} M_j \right) \leq 3c \frac{n}{2^n}.$$
But then, using (10), we deduce that
\[ |\phi(x) - \phi(y)| \leq C|x - y| \log \frac{1}{|x - y|}, \]
for some \( C > 0 \), and so the lemma is proved. \( \square \)

Let us now convince ourselves that, under a simple condition on \( M \), the local quasi-symmetric distortion of \( \phi_M \) blows up at every dyadic rational. In what follows, we denote by \( D^+ h(x) \) and \( D^- h(x) \) the right and left derivatives of \( h \) at \( x \) respectively. These Dini derivatives certainly exist at all points when \( h \) is piecewise linear.

**Lemma 6.** Let \((\phi_n)\) be the defining sequence of dyadic approximations of \( \phi = \phi_M \). If \( \xi \in [0,1] \) has the form \( \xi = p2^{-n} \) with \( p \) odd, then for all \( k \geq 1 \) we have
\begin{equation}
\frac{D^+ \phi_{n+k}(\xi)}{D^- \phi_{n+k}(\xi)} = \frac{1}{M_n} \prod_{j=1}^{k} M_{n+j}. \tag{11}
\end{equation}

In particular, if \( \prod_{j=1}^{\infty} M_j = \infty \), then \( \sigma_{\phi}(\xi) = \infty \).

**Proof.** This follows at once from the following facts which are immediate from the construction of \( \phi \). First, we have on the one hand
\[ \frac{D^+ \phi_n(\xi)}{D^- \phi_n(\xi)} = \frac{1}{M_n}. \]

On the other hand, when we construct \( \phi_{j+1} \) from \( \phi_j \), we replace each linear piece by two new ones; the slope of the new one on the left is multiplied by \( 2M_j/(1 + M_j) \), while the slope of the new piece on the right is multiplied by \( 2/(1 + M_j) \). Thus, working inductively we see that for each \( k \geq 1 \)
\begin{equation}
D^+ \phi_{n+k}(\xi) = \prod_{j=1}^{k} \frac{2M_{n+j}}{1 + M_{n+j}} D^+ \phi_n(\xi). \tag{12}
\end{equation}

Likewise, we have
\begin{equation}
D^- \phi_{n+k}(\xi) = \prod_{j=1}^{k} \frac{2}{1 + M_{n+j}} D^+ \phi_n(\xi). \tag{13}
\end{equation}

Dividing (12) by (13) we get (11) as desired. Finally, for each \( k \geq 1 \) let \( \Delta_k \) and \( \Delta^*_k \) be the two atoms of \( \mathscr{P}_{n+k} \) that have \( \xi \) as their common vertex, with \( \Delta^*_k \) on the left of \( \Delta_k \). Then
\[ |\phi(\Delta_k)| = |\phi_{n+k}(\Delta_k)| = D^+ \phi_{n+k}(\xi)|\Delta_k|, \]
and similarly
\[ |\phi(\Delta^*_k)| = |\phi_{n+k}(\Delta^*_k)| = D^- \phi_{n+k}(\xi)|\Delta_k|, \]
where we have used that \( |\Delta^*_k| = |\Delta_k| \). Therefore, by (11),
\[ \frac{|\phi(\Delta_k)|}{|\phi(\Delta^*_k)|} = \frac{1}{M_n} \prod_{j=1}^{k} M_{n+j}. \]

Letting \( k \to \infty \), we deduce that \( \sigma_{\phi}(\xi) = \infty \). \( \square \)
Before we proceed, we need to introduce a special class of sequences of quasi-symmetric weights which, for lack of a better name, we call \textit{sparsely-jumping sequences}. The ingredients in the construction of such a sequence are two other auxiliary sequences of positive numbers, \((u_n)\) and \((\theta_n)\). They must satisfy the following conditions.

(i) We have \(1 < u_n \leq 2\) for all \(n\) and \(C_0 = \prod_{n=1}^{\infty} u_n < \infty\) (for example, we can take \(u_n = 1 + n^{-2}\)).

(ii) There exists a constant \(C_1 > 1\) such that \(C_1^{-1} \log n \leq \theta_n \leq C_1 \log n\) for all \(n\); in particular \(\prod_{n=1}^{\infty} \theta_n = \infty\).

Given the above ingredients satisfying these two conditions, let us write \(r_n = \theta_1 \theta_2 \cdots \theta_n\) and \(k_n = \left\lceil r_n \right\rceil\). By discarding the first few terms if necessary, we may assume that \(k_1 < k_2 < \ldots\). Note that \(k_{n+1}/k_n \to \infty\) as \(n \to \infty\).

Now define \(M_n\) as follows

\[
M_n = \begin{cases} 
  u_n & \text{if } n \notin \{k_1, k_2, \ldots\} \\
  \theta_m u_{k_m} & \text{if } n = k_m
\end{cases}
\]

\textbf{Definition 2.} A sequence \(\mathcal{M} = (M_n)\) constructed by the procedure just described is said to be a \textit{sparsely-jumping sequence}.

There are two very easy but important properties enjoyed by a sparsely-jumping sequence which we state as a lemma.

\textbf{Lemma 7.} Every sparsely-jumping sequence \((M_n)\) as above satisfies the following two properties:

(i) \(\prod_{j=1}^{n} M_j \leq Cn\) for some \(C > 0\); 

(ii) \(M_{k_n} \to \infty\) as \(n \to \infty\).

\textbf{Proof.} Let \(C = \prod_{j=1}^{\infty} u_n\). Since we have

\[
\prod_{j=1}^{n} M_j = \left(\prod_{j=1}^{n} u_j\right) \left(\prod_{j=1}^{m} \theta_j\right),
\]

where \(m\) is largest with the property that \(k_m \leq n\), it follows that

\[
\prod_{j=1}^{n} M_j \leq Cr_m \leq Ck_m \leq Cn.
\]

At the same time, we have \(M_{k_n} = \theta_n u_{k_n} \to \infty\) as \(n \to \infty\). \hfill \square

\textbf{Remark 3.} Let \(\mathcal{M} = (M_n)\) be a sparsely-jumping sequence of quasi-symmetric weights. Then property (i) in the above lemma implies that the cumulative distortion function of \(\phi = \phi_{\mathcal{M}}\) satisfies the inequality \(\kappa_{\phi}(n) \leq Cn\) (this is simply because \(\frac{1}{2}(1 + M_j) < M_j\) for all \(j\)).

\textbf{Theorem 3.} Let \(\mathcal{M}\) be a sparsely-jumping sequence of quasi-symmetric weights. Then the associated homeomorphism \(\phi_{\mathcal{M}} : [0, 1] \to [0, 1]\) has modulus of continuity \(\rho(t) = t \log (1/t)\), and \(\sigma_{\phi_{\mathcal{M}}}(x) = \infty\) for Lebesgue almost every \(x\) in the unit interval.

\textbf{Proof.} Let us write \(\phi = \phi_{\mathcal{M}}\) in this proof. The first assertion, concerning the modulus of continuity of \(\phi\), follows from Lemma 5. Hence we focus on the second assertion. We start by identifying a set of points \(A_{\infty} \subseteq [0, 1]\) where \(\sigma_{\phi}\) blows up.
Recall that any $x \in [0, 1]$ can be written in binary as

$$x = \sum_{j=1}^{\infty} \frac{x_j}{2^j},$$

where $x_j \in \{0, 1\}$ for each $j$, and the expansion is unique if we throw away a countable set of points. For each $n \geq 1$, let

$$A_n = \{x \in (0, 1) : x_{k_n-1} = 1; x_{k_n} = x_{k_n+1} = \cdots = x_{k_n+q_n} = 0\}.$$

Here, $(q_n)$ is the non-decreasing sequence of positive integers given by $q_n = 1 + \lceil \log_2 n \rceil$. Note that $q_n \to \infty$ but $\sum_{n=1}^{\infty} 2^{-q_n} = \infty$. Since we have $\mu(A_n) = 2^{-q_n-2}$ for all $n$ (where $\mu$ denotes one-dimensional Lebesgue measure), it follows that

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{2q_n+2} = \infty.$$

Moreover, the $A_n$’s are independent events, as long as $q_n < k_{n+1} - 1$ for all $n$, which is not a problem to assume since $k_{n+1}/k_n \to \infty$ (see the proof of Lemma 7). Therefore, by the Borel–Cantelli lemma, the set

$$A_\infty = \limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$$

has full Lebesgue measure. We will show that $\sigma_\phi(x) = \infty$ for all $x \in A_\infty$.

![Figure 4. Forcing large quasi-symmetric distortion at $x \in A_\infty$.](image)
Given $x \in A_\infty$, let $n_i \to \infty$ be a sequence such that $x \in A_{n_i}$ for all $i$. Write

$$\xi_i = \sum_{j=1}^{n_i-1} \frac{x_j}{2^j},$$

so that

$$|x - \xi_i| < \frac{1}{2^{n_i+q_i}}.$$

Consider the atom of $\mathcal{P}_{k_i} n_i$ containing $x$, namely $\Delta_i = [\xi_i, \eta_i]$, where $\eta_i = \xi_i + 2^{-k_i+1}$. For each $j \geq 0$, let $\Delta_i^{(j)} = [\xi_i, \xi_i + 2^{-k_i-j+1}]$, so that

$$\Delta_i = \Delta_i^{(0)} \supset \Delta_i^{(1)} \supset \Delta_i^{(2)} \supset \cdots \supset \Delta_i^{(q_i)} \supset [\xi_i, x].$$

We refer the reader to Figure 4. Our goal is to estimate $\phi(x)$ from below, where $\eta_i^*$ is the point symmetric to $\eta_i$ with respect to $x$, i.e., $\eta_i^* = 2x - \eta_i$. To do this, we first estimate $\phi(x) - \phi(\xi_i)$ from above. Since $[\xi_i, x] \subset \Delta_i^{(q_i)}$, we see using (12) that

$$|\phi(x) - \phi(\xi_i)| \leq \left| \phi(\Delta_i^{(q_i)}) \right| \leq \prod_{j=k_i+1}^{n_i} \frac{2M_j}{1 + M_j} \left| \Delta_i^{(q_i)} \right| s_i^+.$$

Here $s_i^+ = D^+ \phi_{k_i}(\xi_i)$. Since $\left| \Delta_i^{(q_i)} \right| = 2^{-q_i} |\Delta_i|$, it follows that

$$|\phi(x) - \phi(\xi_i)| \leq \prod_{j=k_i+1}^{n_i} \frac{M_j}{1 + M_j} |\Delta_i| s_i^+.$$

But by construction of $M$, we have

$$\prod_{j=k_i+1}^{n_i} \frac{M_j}{1 + M_j} \leq \prod_{j=k_i+1}^{n_i} \frac{u_j}{1 + u_j} \leq \left( \frac{2}{3} \right)^{q_i}.$$

This yields the estimate

$$|\phi(x) - \phi(\xi_i)| \leq \beta_i |\Delta_i| s_i^+, \tag{15}$$

where $\beta_i = \left( \frac{2}{3} \right)^{q_i}$.

Next, we estimate $\phi(\xi_i) - \phi(\eta_i^*)$, also from above. Let $\Delta_i^*$ the atom of $\mathcal{P}_{k_i} n_i$ adjacent to $\Delta_i$ on the left. Then $[\eta_i^*, \xi_i] \subset \Delta_i^*$ and so

$$|\phi(\xi_i) - \phi(\eta_i^*)| \leq |\phi(\Delta_i^*)| = |\Delta_i| s_i^-,$$

where $s_i^- = D^- \phi_{k_i}(\xi_i)$. Combining (15) with (16) we get

$$|\phi(x) - \phi(\eta_i)| \leq (s_i^- + \beta_i s_i^+)|\Delta_i|, \tag{17}$$

The next step is to estimate $|\phi(\eta_i) - \phi(x)|$ from below. But this is easy from (15), since

$$|\phi(\eta_i) - \phi(x)| = |\phi(\Delta_i)| - |\phi(x) - \phi(\xi_i)| \geq (1 - \beta_i)|\Delta_i| s_i^+.$$

$$|\phi(\eta_i) - \phi(x)| = |\phi(\Delta_i)| - |\phi(x) - \phi(\xi_i)| \geq (1 - \beta_i)|\Delta_i| s_i^+. \tag{18}$$
Here we have used that $|\phi(\Delta_i)| = s_i^+|\Delta_i|$. Putting (17) and (18) together, we deduce that
\[
\frac{\phi(\eta_i) - \phi(x)}{\phi(x) - \phi(\eta_i^*)} \geq \frac{(1 - \beta_i)s_i^+}{s_i^- + \beta_is_i^+} = \frac{(1 - \beta_i)(s_i^+/s_i^-)}{1 + \beta_i(s_i^+/s_i^-)}.
\]
But from (11) and the construction of $(M_n)$, we know that
\[
\frac{s_i^+}{s_i^-} = \frac{M_{k_n}}{M_{k_n-1}} \in \left[\frac{\theta_{n_i}}{2}, 2\theta_{n_i}\right].
\]
Hence we arrive at
\[
\frac{\phi(\eta_i) - \phi(x)}{\phi(x) - \phi(\eta_i^*)} \geq \frac{1}{2}(1 - \beta_i)\theta_{n_i}.
\]
Finally, we have by construction
\[
\theta_{n_i} \leq C_1 \log n_i \leq C_2 q_{n_i},
\]
and therefore
\[
\beta_i\theta_{n_i} \leq C_2 q_{n_i} \left(\frac{2}{3}\right)^{q_{n_i}} \to 0 \text{ as } i \to \infty.
\]
This shows that the right hand side of (19) diverges to $\infty$. Hence $\sigma_\phi(x) = \infty$ for all $x \in A_\infty$, and the theorem is proved.

\section{5. Proof of Theorem 1}

The proof of theorem 1 is obtained by putting together the results of Theorems 2 and 3. Given any $\phi$ as in theorem 3, we know that its cumulative distortion function $\kappa_\phi$ satisfies the inequality $\kappa_\phi(n) \leq Cn$ for some constant $C > 0$. Hence, by Theorem 2, the Carleson boxes extension $f$ of $\phi$ to the unit square is a David homomorphism. Note that by construction $f$ agrees with the identity on the three sides of the unit square lying in the upper-half plane. We can extend $f$ to the infinite half-strip $[0, 1] \times [0, \infty)$ by setting it to be the identity on $\{(x, y): 0 \leq x \leq 1, \ y \geq 1\}$. Now extend $\phi$ to the whole real line taking $\phi(x+n) = \phi(x)+n$ for all $x \in [0, 1]$ and all $n \in \mathbb{Z}$. Likewise, extend $f$ to the whole half-plane $\mathbb{H}$ putting $f(z+n) = f(z)+n$ for all $z \in \mathbb{H}$ and all $n \in \mathbb{Z}$. Then the pointwise distortion of $\phi$: $\mathbb{R} \rightarrow \mathbb{R}$ is infinite almost everywhere, so that $\phi$ satisfies (a) of theorem 1. And it satisfies (b) of that theorem too, by lemma 5. Finally, its extension $f$: $\mathbb{H} \rightarrow \mathbb{H}$ is a David homeomorphism. This finishes the proof of Theorem 1.

\textbf{Remark 4.} In $[Z]$, Zakeri defines the \textit{scalewise distortion} of $\phi$ to be
\[
\rho_\phi(t) = \sup_{x \in \mathbb{R}} \max \left\{ \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \right\}.
\]
He proves that if
\[
(20) \quad \rho_\phi(t) = O \left( \log \frac{1}{t} \right)
\]
then $\phi$ extends to a David homeomorphism of the upper half-plane. Using this result, it is possible to prove Theorem 1 without having to invoke Theorem 2. All one needs to do is to first prove a lemma showing that the examples we constructed in §4 satisfy Zakeri’s condition (20). Conversely, it is also possible to use Theorem 2 to produce an
alternative proof of Zakeri’s result (Zakeri’s proof uses the Beurling–Ahlfors extension of φ).

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