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INTERPOLATION OF A MEASURE OF WEAK NON-COMPACTNESS

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Dedicated to Sylwia, my beloved wife.

Abstract. We study the measure of weak non-compactness of operators between abstract interpolation spaces. We prove an estimate of this measure, depending on the fundamental function of the space. We specialize our results and show applications to weak compactness of operators.

1. Introduction

The behavior of compact linear operators under interpolation has been studied since the 1960s. First results were established by Krasnosel'skiĭ [21], who proved that under the hypothesis of the Riesz–Thorin interpolation theorem, that is $T\colon L_{p_i} \to L_{q_i}$ is bounded for $i = 0, 1$ where $1 \le p_i, q_i \le \infty$, and the additional assumption that $T\colon L_{p_0} \to L_{q_0}$ is compact, $q_0 < \infty$, it follows that $T\colon L_p \to L_q$ is also compact, where $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $0 < \theta < 1$.

His results lead to the question whether similar results hold in the abstract interpolation case, when (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1}) is replaced by a Banach pair (A_0, A_1) and (B_0, B_1), respectively. The complete answer is still unknown.

The first results for the real interpolation method were obtained in 1964 by Lions and Peetre [24] for the case when $A_0 = A_1$ or $B_0 = B_1$, and by Persson [28] for the general case $A_0 \neq A_1$ and $B_0 \neq B_1$ with an approximation condition on the couple (B_0, B_1). In these cases, they showed that the operator $T\colon \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}$ is compact for $0 < \theta < 1$, $1 \le q \le \infty$. We refer to the monograph [5] for a more detailed history of research.

In 1969 Hayakawa [18] gave the result for the real interpolation method without approximation hypothesis but with the assumption that $T\colon (A_0, A_1) \to (B_0, B_1)$ and the restrictions $T\colon A_0 \to B_0$, $T\colon A_1 \to B_1$ are both compact operators and $1 \le q < \infty$. New approaches to Hayakawa's result can be found in the paper by Cobos and Peetre [14] and the references given therein. Finally, in 1992 Cwikel [15] and Cobos, Kühn and Schonbek [10] showed that the theorem holds whenever $T\colon A_0 \to B_0$ is compact and $T\colon A_1 \to B_1$ is bounded.

A similar question for weak non-compactness was raised as well. In 1978 Beauzamy proved in [4] that if the inclusion map

$$A_0 \cap A_1 \hookrightarrow A_0 + A_1 \text{ is a weakly compact operator,}$$

then $\overline{A}_{\theta,q}$ is a reflexive Banach space, for $1 < q < \infty$, $0 < \theta < 1$.

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Next, Heinrich [19] extended this result to closed operator ideals. Generalizations of Beauzamy's result for the real method are due to Aizenstein and Brudnyi [6], Maligranda and Quevedo [25], and Mastyło [26]. The result is the following: if

$$T\colon A_0 \cap A_1 \to B_0 + B_1 \text{ is a weakly compact operator,}$$

then so is

$$T\colon \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}, \text{ where } 1 < q < \infty,\ 0 < \theta < 1.$$

It is perhaps worth remarking that applications to reflexivity of the interpolation spaces were given by Brudnyĭ and Krugljak [6], Mastyło [26] and Cobos, Fernández-Cabrera, Manzano and Martínez [7].

We recall that the classical interpolation Riesz–Thorin theorem for operators between L_p-spaces states that

$$\|T\|_{L_p \to L_q} \le C \|T\|^{1-\theta}_{L_{p_0} \to L_{q_0}} \|T\|^{\theta}_{L_{p_1} \to L_{q_1}}.$$

The similar result holds for the real interpolation spaces $\overline{A}_{\theta,q}$, $\overline{B}_{\theta,q}$, with a corresponding estimate

$$\|T\|_{\overline{A}_{\theta,q} \to \overline{B}_{\theta,q}} \le C \|T\|^{1-\theta}_{A_0 \to B_0} \|T\|^{\theta}_{A_1 \to B_1}.$$

Note that in both theorems the logarithmically convex function $s^{1-\theta}t^{\theta}$ appears. In case of abstract real interpolation spaces a general variant of a function was introduced in [30]. The natural version of above theorems takes place, namely

$$\|T\|_{\overline{A}_E \to \overline{B}_E} \le C\psi_E(\|T\|_{A_0 \to B_0}, \|T\|_{A_1 \to B_1}),$$

where E is an appropriate Banach lattice, $\overline{A}_E$, $\overline{B}_E$ are abstract real interpolation spaces and ψ_E is a corresponding function.

Nowadays we search for quantitative versions of interpolating compactness and weakly compactness results, using measures of non-compactness. For instance let us mention β and separation measures. In 1999, Cobos, Fernández-Martínez and Martínez in their remarkable paper [9] obtained a logarithmic type inequality, namely

$$\beta(T\colon \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}) \le C\beta(T\colon A_0 \to B_0)^{1-\theta}\beta(T\colon A_1 \to B_1)^{\theta}.$$

In 2006, the above inequality was extended to the abstract real interpolation case in [30]

$$\beta\left(T\colon \overline{A}_E \to \overline{B}_E\right) \le C\psi_E\left(\beta\left(T\colon A_0 \to B_0\right), \beta\left(T\colon A_1 \to B_1\right)\right).$$

(see also [8] where an equivalent variant of such estimate was obtained as well).

A well known measure of weak non-compactness ω introduced by De Blasi [16] can be treated as a counterpart of the Hausdorff measure of non-compactness. In 2000 Kryczka, Prus and Szczepanik (see [23]) introduced a new measure of weak non-compacness γ, which can be seen as a counterpart of the separation measure of non-compactness. In general γ and ω are not equivalent. The reason is that the measure γ appeals directly to the norm topology, while in the definition of ω the weak topology is involved.

Kryczka, Prus and Szczepanik in [23] showed the logarithmically convex type estimate for the measure of weak non-compactness γ, i.e.,

$$\gamma(T\colon \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}) \le C\gamma(T\colon A_0 \to B_0)^{1-\theta}\gamma(T\colon A_1 \to B_1)^{\theta},$$

where $1 < q < \infty$, $0 < \theta < 1$.

We also note that interpolation properties of a different type were established by Aksoy and Maligranda [2], Cobos, Manzano and Martínez [11], and Cobos and Martínez [12, 13] for the real method and Fernández-Cabrera and Martínez [17] for the abstract real method.

The main aim of this paper is to show an analogue of the above inequality for abstract real interpolation methods, namely

$$\gamma\big(T\colon \overline{A}_E \to \overline{B}_E\big) \le C\psi_E\big(\gamma(T\colon A_0 \to B_0), \gamma(T\colon A_1 \to B_1)\big).$$

2. Preliminaries and notation

We use the standard notations from the interpolation theory (see [5, 6] for more details). Let $\overline{A} = (A_0, A_1)$ be a Banach couple. As usual we let $\Delta(\overline{A}) := A_0 \cap A_1$ and $\Sigma(\overline{A}) := A_0 + A_1$. For all $t > 0$ the K-functional is defined by

$$K(t, a; \overline{A}) := \inf_{a=a_0+a_1} \left\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}\right\} \quad \text{for all } \ a \in \Sigma(\overline{A})$$

and the J-functional by

$$J(t, a; \overline{A}) := \max\left\{\|a\|_{A_0}, t\,\|a\|_{A_1}\right\} \quad \text{for all } \ a \in \Delta(\overline{A}).$$

A real sequence $w = \{w_n\}_{n\in\mathbf{Z}}$ is called a weight sequence if each w_n is positive. If E is a Banach sequence lattice modelled on $\mathbf{Z}$ and $w = \{w_n\}$ is a weight sequence, we define the weighted Banach sequence lattice $E(w) := \{\{x_n\} : \{x_n w_n\} \in E\}$. The space $E(w)$ is equipped with the norm $\|x\|_{E(w)} := \|\{x_n w_n\}\|_E$.

Following the terminology from [27], the space E is said to be K-non-trivial (resp., J-non-trivial) when $\ell_\infty \cap \ell_\infty(2^{-n}) \subset E$ (resp., $E \subset \ell_1 + \ell_1(2^{-n})$).

For a J-non-trivial Banach sequence lattice E, the J-method space $\overline{A}_{E;J} := (A_0, A_1)_{E;J}$ consists of all elements $a \in \Sigma(\overline{A})$ which can be represented in the form

$$a = \sum_{n=-\infty}^{\infty} a_n \quad (\text{convergence in } \Sigma(\overline{A}))$$

such that $\left\{J(2^n, a_n; \overline{A})\right\}_{n\in\mathbf{Z}} \in E$ with the associated norm

$$\|a\| := \inf\left\{\left\|\left\{J(2^n, a_n; \overline{A})\right\}\right\|_E : a = \sum_{n=-\infty}^{\infty} a_n\right\}.$$

For a K-non-trivial Banach sequence lattice E, we define the K-method space $\overline{A}_{E;K} := (A_0, A_1)_{E;K}$ which contains all elements $a \in \Sigma(\overline{A})$ with the property that $\left\{K(2^n, a; \overline{A})\right\} \in E$, equipped with the norm

$$\|a\| := \left\|\left\{K(2^n, a; \overline{A})\right\}\right\|_E.$$

It is well known from [27] that if E is a parameter of the real method (i.e., $\ell_\infty \cap \ell_\infty(2^{-n}) \subset E \subset \ell_1 + \ell_1(2^{-n})$) then for any Banach couple $\overline{A}$

$$\overline{A}_{E;K} \hookrightarrow \overline{A}_{E;J}$$

and the norm of the inclusion map is less than 4. If the Calderón operator Ω defined on $\ell_1 + \ell_1(2^{-n})$ by

$$\Omega\left\{\xi_n\right\} := \left\{\sum_{k=-\infty}^{\infty} \min\left\{1, 2^{n-k}\right\}|\xi_k|\right\}_n, \quad \{\xi_n\} \in \ell_1 + \ell_1(2^{-n})$$

is bounded on E, then $\overline{A}_{E;J} = \overline{A}_{E;K}$ with equivalence of the norms. In this case we put $\overline{A}_E$ instead of $\overline{A}_{E;J}$ or $\overline{A}_{E;K}$.

Let us note that from the point of view of applications the classical interpolation spaces play an important role. For this recall that if ρ is a *function parameter*, i.e., $\rho\colon (0,\infty) \to (0,\infty)$ is a quasi-concave function ($t \mapsto \rho(t)$ increases and $t \mapsto \rho(t)/t$ decreases) and

$$s_\rho(t) = o(1) \text{ as } t \to 0 \quad\text{and}\quad s_\rho(t) = o(t) \text{ as } t \to \infty,$$

where $s_\rho(t) = \sup\{\rho(tu)/\rho(u)\colon u > 0\}$ for every $t > 0$. Now, if we take $E = \ell_q(1/\rho(2^m))$ with $1 \le q \le \infty$, then we have

$$(A_0, A_1)_{\ell_q(1/\rho(2^m));K} = (A_0, A_1)_{\ell_q(1/\rho(2^m));J} = (A_0, A_1)_{\rho,q}.$$

If $\rho(t) = t^\theta$, $\theta \in (0,1)$, we get the classical real interpolation spaces $(A_0, A_1)_{\theta,q}$ (see, e.g., [5, 6]).

Let $\omega(\mathbf{Z})$ be the space of all real sequences on $\mathbf{Z}$. For all $\nu \in \mathbf{Z}$, the shift operator $\tau_\nu\colon \omega(\mathbf{Z}) \to \omega(\mathbf{Z})$ is defined by $\tau_\nu\{\xi_m\} := \{\xi_{m+\nu}\}$.

Throughout the rest of the paper we consider Banach lattices E modelled on $\mathbf{Z}$ such that the shift operator τ_ν is bounded in E for all $\nu \in \mathbf{Z}$. For such E we define a function $\varphi_E\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ by

$$\varphi_E(2^m, 2^n) := 2^m \|\tau_{n-m}\|_{E\to E} \quad\text{for all}\quad m, n \in \mathbf{Z}$$

and

$$\varphi_E(s,t) := \varphi_E\left(2^{[\log_2 s]}, 2^{[\log_2 t]}\right) \quad\text{for all}\quad s, t > 0,$$

where $[\cdot]$ denotes the greatest integer function.

Further, we will need to work with a function $\psi_E\colon [0,\infty) \times [0,\infty) \to [0,\infty]$ which is an extension of φ_E. The function ψ_E is defined by the following:

$$\begin{aligned} \psi_E(0,0) := 0,\ \ \psi_E(s,0) &:= \liminf_{v\to 0+} \varphi_E(s,v) \quad\text{for all}\quad s > 0, \\ \psi_E(0,t) &:= \liminf_{u\to 0+} \varphi_E(u,t) \quad\text{for all}\quad t > 0, \\ \text{and}\quad \psi_E(s,t) &:= \varphi_E(s,t) \quad\text{if}\quad s, t > 0. \end{aligned}$$

The following technical lemma shows some fundamental properties of ψ_E.

Lemma 2.1. *Let E be a Banach sequence lattice on $\mathbf{Z}$ such that the shift operator τ_n is bounded in E for all $n \in \mathbf{N}$. Then the function ψ_E has the following properties:*

(i) *$\psi_E(2^m s, 2^n t) \le \psi_E(2^m, 2^n)\psi_E(s,t)$ for all $m, n \in \mathbf{Z}$ and $s, t \ge 0$.*

(ii) *There exists a constant $C_1 = C_1(E) > 0$ such that*

$$\psi_E(su, tv) \le C_1 \psi_E(s,t)\psi_E(u,v)$$

for every $s, t, u, v > 0$.

(iii) *If $\lim_{(s,t)\to(0,0)} \psi_E(s,t) < \infty$, then there exists $C_2 = C_2(E) > 0$ such that*

$$\psi_E(s,t) \le C_2 \psi_E(u,v)$$

for all $s \le u$, $t \le v$.

(iv) *$\psi_E(s,s) \le s \le 2\psi_E(s,s)$ for every $s \ge 0$*

Proof. Properties (i)–(iii) are proved in [30, Lemma 2.3]. (iv) follows immediately from inequalities $2^{[\log_2 s]} \le s < 2 \cdot 2^{[\log_2 s]}$ for every $s > 0$. □

Let us note that condition (iii) from the previous lemma, can be reformulated in the following way:

Lemma 2.2. *Let E be a Banach sequence lattice on $\mathbf{Z}$ such that the shift operator τ_n is bounded in E for all $n \in \mathbf{N}$. The following conditions are equivalent:*

(i) $\sup_{s,t\in(0,1]} \psi_E(s,t) < \infty$.

(ii) $\sup_{s\in(0,1]} \psi_E(s,1) < \infty$ *and* $\sup_{t\in(0,1]} \psi_E(1,t) < \infty$.

(iii) *There exists a constant $C > 0$ such that*

$$\psi_E(s,t) \le C \max\{s,t\} \quad \textit{for every} \ \ s,t \ge 0.$$

Proof. Implications (iii)⇒(i)⇒(ii) are straightforward. Assume that condition (ii) holds. Then by Lemma 2.1 there exists a constant $C_1 > 0$ such that

$$\psi_E(s,t) \le C_1 \psi_E(s,1)\psi_E(1,t)$$

for every $s,t \ge 0$. Therefore (i) holds. Suppose that (i) is satisfied. It follows by Lemma 2.1 that there exists a constant $C_2 > 0$ such that

$$\psi_E(s,t) \le C_2 \psi_E(\max\{s,t\}, \max\{s,t\}) \le C_2 \max\{s,t\}$$

for every $s,t \ge 0$, which gives (iii). □

The significance of the function ψ_E to interpolation is due to the following. Let T be a bounded operator from the Banach couple $\overline{A} = (A_0, A_1)$ to the Banach couple $\overline{B} = (B_0, B_1)$. If E is K-non-trivial, then

$$\|T\|_{\overline{A}_{E;K} \to \overline{B}_{E;K}} \le 2\psi_E(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}).$$

Similarly, if E is J-non-trivial, then

$$\|T\|_{\overline{A}_{E;J} \to \overline{B}_{E;J}} \le 2\psi_E(\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}).$$

(see [30, Lemma 2.4]). To use these estimates we need to impose some growth conditions on ψ_E; specifically, we will say that a Banach sequence lattice E on $\mathbf{Z}$ is *admissible* if it satisfies condition (iii), hence also conditions (i) and (ii), of Lemma 2.2. In what follows, we will only be interested in admissible Banach lattices.

Proposition 2.3. *If E is an interpolation Banach lattice with respect to a couple $(\ell_p, \ell_p(2^{-n}))$ for some $1 \le p \le \infty$, then E is admissible.*

Proof. Let E be an interpolation space between ℓ_p and $\ell_p(2^{-n})$, with a constant $C > 0$. Since τ_ν has bounded norms $\|\tau_\nu\|_{\ell_p\to\ell_p} = 1$ and $\|\tau_\nu\|_{\ell_p(2^{-n})\to\ell_p(2^{-n})} = 2^\nu$, we have $\|\tau_\nu\|_{E\to E} \le C\max\{1, 2^\nu\}$, for each $\nu \in \mathbf{Z}$. Therefore

$$\begin{aligned}\varphi_E(s,t) &= 2^{[\log_2 s]} \left\|\tau_{[\log_2 t]-[\log_2 s]}\right\|_{E\to E} \le C 2^{[\log_2 s]} \max\left\{1, 2^{[\log_2 t]-[\log_2 s]}\right\} \\ &= C\max\left\{2^{[\log_2 s]}, 2^{[\log_2 t]}\right\} \le C\max\{s,t\},\end{aligned}$$

for all $s,t > 0$. We thus get $\psi_E(s,t) \le C\max\{s,t\}$ for all $s,t \ge 0$. □

The following result is an obvious observation (cf. [8, Example 4.5] or [30, Corollary 3.6]).

Lemma 2.4. *Let ρ be a function parameter, $1 \le q \le \infty$ and $E = \ell_q\left(\frac{1}{\rho(2^m)}\right)$. Then $\|\tau_n\|_{E\to E} \le s_\rho(2^n)$ for all $n \in \mathbf{Z}$.*

For the sake of completeness, we also include a short proof.

Proof. Since $1/\rho(2^m) \leq \rho(2^n)/\rho(2^{m+n})$ for all $m, n \in \mathbf{N}$, we have

$$\begin{aligned}\|\tau_n \{\xi_m\}\|_E &= \left(\sum_{m\in\mathbf{Z}} \left(|\xi_{m+n}| \frac{1}{\rho(2^m)}\right)^q\right)^{1/q} \leq s_\rho(2^n) \left(\sum_{m\in\mathbf{Z}} \left(|\xi_{m+n}| \frac{1}{\rho(2^{m+n})}\right)^q\right)^{1/q} \\ &= s_\rho(2^n) \|\{\xi_m\}\|_E\end{aligned}$$

for every $\{\xi_m\} \in E$. □

Lemma 2.5. *Let ρ be a function parameter, $1 \leq q \leq \infty$ and $E = \ell_q(\frac{1}{\rho(2^m)})$. Then there exists a constant $C > 0$ such that*

$$\psi_E(s,t) \leq C \max\{s,t\} \quad \text{for every} \quad s,t \geq 0.$$

Moreover,

$$\psi_E(s,0) = \psi_E(0,t) = 0 \quad \text{for every} \quad s,t \geq 0.$$

Proof. It follows from Lemma 2.2 that it suffices to show that

$$\lim_{s\to 0+} \psi_E(s,1) = 0 \quad \text{and} \quad \lim_{t\to 0+} \psi_E(1,t) = 0. \tag{2.1}$$

By the definition of a function parameter ρ we get

$$\lim_{m\to-\infty} s_\rho(2^m) = 0 \quad \text{and} \quad \lim_{m\to\infty} s_\rho(2^m)/2^m = 0.$$

Thus Lemma 2.4 gives

$$\lim_{m\to-\infty} \psi_E(2^m,1) = 0 \quad \text{and} \quad \lim_{m\to-\infty} \psi_E(1,2^m) = 0,$$

and the proof is complete by Lemma 2.1. □

The equivalence of

$$\lim_{n\to\infty} 2^n \|\tau_n\|_{E\to E} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\tau_{-n}\|_{E\to E} = 0,$$

and (2.1) follows by the definition of ψ. In particular this implies that E is admissible. The above condition was considered in [8, 17, 30].

3. Norms of Lions–Peetre type

In this section we will look more closely at another equivalent norms in abstract real interpolation spaces. Some of them were defined in the well-known paper by Lions and Peetre [24]. In particular, we derive Theorem 3.3, which is useful for calculation of $\|\cdot\|_{A_{E;K}}$ and $\|\cdot\|_{A_{E;J}}$ norms.

Let E be a Banach lattice and X be a Banach space. Denote by $E(X)$ the Köthe–Bochner space

$$E(X) := \left\{\{x_n\}_{n\in\mathbf{Z}} \in \prod_{n=-\infty}^{\infty} X \colon \{\|x_n\|_X\}_{n\in\mathbf{Z}} \in E\right\},$$

equipped with the norm $\|\{x_n\}\| := \|\{\|x_n\|_X\}\|_E$.

In spaces $\overline{A}_{E;K}$ and $\overline{A}_{E;J}$ we define the following norms:

(i) For a K-non-trivial Banach lattice E

$$|||a|||_{\overline{A}_{E;K}} := \inf_{\{a\}=\{a_n^0\}+\{a_n^1\}} \max\left\{\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right\},$$

where the infimum is taken over all sequences $\{a_n^0\} \subset A_0$ and $\{a_n^1\} \subset A_1$ such that $a = a_n^0 + a_n^1$ for each $n \in \mathbf{Z}$.

(ii) For a J-non-trivial Banach lattice E

$$|||a|||_{\overline{A}_{E;J}} := \inf_{a=\sum a_n} \max\left\{\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right\},$$

where the infimum is taken over all sequences $\{a_n\} \subset A_0 \cap A_1$ such that the series $\sum_{n\in\mathbf{Z}} a_n$ converges to a in $A_0 + A_1$.

Lemma 3.1. *Let E be a Banach lattice.*

(i) *If E is K-non-trivial, then $|||\cdot|||_{\overline{A}_{E;K}} \sim \|\cdot\|_{\overline{A}_{E;K}}$.*

(ii) *If E is J-non-trivial, then $|||\cdot|||_{\overline{A}_{E;J}} \sim \|\cdot\|_{\overline{A}_{E;J}}$.*

Proof. (i) It is easy to check that $|||a|||_{\overline{A}_{E;K}} \le \|a\|_{\overline{A}_{E;K}} \le 2\,|||a|||_{\overline{A}_{E;K}}$.

(ii) Similarly, it follows easily that $|||a|||_{\overline{A}_{E;J}} \le 2\,\|a\|_{\overline{A}_{E;J}} \le 4\,|||a|||_{\overline{A}_{E;J}}$. □

Theorem 3.2. *Let E be a Banach lattice.*

(i) *If E is K-non-trivial, then*

$$|||a|||_{\overline{A}_{E;K}} \le 2 \inf_{\{a\}=\{a_n^0\}+\{a_n^1\}} \psi_E\left(\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right),$$

where the infimum is taken over all sequences $\{a_n^0\} \subset A_0$ and $\{a_n^1\} \subset A_1$ such that $a = a_n^0 + a_n^1$ for each $n \in \mathbf{Z}$.

(ii) *If E is J-non-trivial, then*

$$|||a|||_{\overline{A}_{E;J}} \le 2 \inf_{a=\sum a_n} \psi_E\left(\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right),$$

where the infimum is taken over all sequences $\{a_n\} \subset A_0 \cap A_1$ such that the series $\sum_{n\in\mathbf{Z}} a_n$ converges to a in $A_0 + A_1$.

Proof. (i) Fix $a \in \overline{A}_{E;K}$ and choose sequences $\{a_n^0\} \in E(A_0)$ and $\{a_n^1\} \in E(A_1)$ satisfying decompositions $a = a_n^1 + a_n^0$ for each $n \in \mathbf{Z}$. Fix $\varepsilon_j \ge 0$ for $j = 0, 1$, where $\varepsilon_j > 0$ if and only if $\|\{2^{jn} a_n^j\}\|_{E(A_j)} = 0$. Let $2^{k_j - 1} \le \|\{2^{jn} a_n^j\}\|_{E(A_j)} + \varepsilon_j < 2^{k_j}$, $j = 0, 1$ and let $\nu = k_1 - k_0$. Hence

$$\begin{aligned}\left\|\{2^{jn} a_{n+\nu}^j\}\right\|_{E(A_j)} &= \left\|\left\{2^{jn}\left\|a_{n+\nu}^j\right\|_{A_j}\right\}\right\|_E = 2^{-j\nu}\left\|\left\{2^{j(n+\nu)}\left\|a_{n+\nu}^j\right\|_{A_j}\right\}\right\|_E \\ &= 2^{-j\nu}\left\|\tau_\nu\left\{2^{jn}\left\|a_n^j\right\|_{A_j}\right\}\right\|_E \le 2^{-j\nu}\left\|\{2^{jn} a_n^j\}\right\|_{E(A_j)} \|\tau_\nu\|_{E\to E},\end{aligned}$$

for $j = 0, 1$. Thus

$$\begin{aligned}|||a||| &\le \max\left\{\left\|\{a_{n+\nu}^0\}\right\|_{E(A_0)}, \left\|\{2^n a_{n+\nu}^1\}\right\|_{E(A_1)}\right\} \le 2^{k_0}\|\tau_\nu\|_{E\to E} \\ &= 2\cdot 2^{k_0-1}\|\tau_{k_1-1-k_0+1}\|_{E\to E} = 2\psi_E\left(\left\|\{a_n^0\}\right\|_{E(A_0)} + \varepsilon_0, \left\|\{2^n a_n^1\}\right\|_{E(A_1)} + \varepsilon_1\right).\end{aligned}$$

In consequence

$$|||a||| \le 2\psi_E\left(\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right),$$

which completes the proof.

(ii) It is obvious that (ii) holds for $a = 0$. Choose $0 \neq a \in \overline{A}_{E;J}$ and a sequence $\{a_n\} \subset A_0 \cap A_1$ such that $\{a_n\} \in E(A_0)$ and $\{2^n a_n\} \in E(A_1)$ for each $n \in \mathbf{Z}$, and $a = \sum_{n\in\mathbf{Z}} a_n$ is convergent in $A_0 + A_1$. Clearly $\|\{a_n\}\|_{E(A_0)} > 0$ and $\|\{2^n a_n\}\|_{E(A_1)} > 0$. Let $2^{k_j - 1} \leq \|\{2^{jn} a_n\}\|_{E(A_j)} < 2^{k_j}$, $j = 0, 1$ and let $\nu = k_1 - k_0$. Hence

$$\begin{aligned}\left\|\{2^{jn} a_{n+\nu}\}\right\|_{E(A_j)} &= \left\|\left\{2^{jn} \|a_{n+\nu}\|_{A_j}\right\}\right\|_E = 2^{-j\nu} \left\|\left\{2^{j(n+\nu)} \|a_{n+\nu}\|_{A_j}\right\}\right\|_E \\ &= 2^{-j\nu} \left\|\tau_\nu \left\{2^{jn} \|a_n\|_{A_j}\right\}\right\|_E \leq 2^{-j\nu} \left\|\{2^{jn} a_n\}\right\|_{E(A_j)} \|\tau_\nu\|_{E\to E},\end{aligned}$$

for $j = 0, 1$. We thus get

$$\begin{aligned}\|a\| &\leq \max\left\{\|\{a_{n+\nu}\}\|_{E(A_0)}, \|\{2^n a_{n+\nu}\}\|_{E(A_1)}\right\} \leq 2^{k_0} \|\tau_\nu\|_{E\to E} \\ &= 2 \cdot 2^{k_0 - 1} \|\tau_{k_1 - 1 - k_0 + 1}\|_{E\to E} = 2\psi_E\left(\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right),\end{aligned}$$

and the theorem follows. □

Theorem 3.3. *Let E be an admissible Banach lattice.*

(i) *If E is J-non-trivial, then*

$$\|a\|_{\overline{A}_{E;K}} \sim \inf_{\{a\}=\{a_n^0\}+\{a_n^1\}} \psi_E\left(\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right),$$

where the infimum is taken over all sequences $\{a_n^0\} \subset A_0$ and $\{a_n^1\} \subset A_1$ such that $a = a_n^0 + a_n^1$ for each $n \in \mathbf{Z}$.

(ii) *If E is K-non-trivial, then*

$$\|a\|_{\overline{A}_{E;J}} \sim \inf_{a=\sum a_n} \psi_E\left(\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right),$$

where the infimum is taken over all sequences $\{a_n\} \subset A_0 \cap A_1$ such that the series $\sum_{n\in\mathbf{Z}} a_n$ converges to a in $A_0 + A_1$.

Proof. By the definition of $\|\cdot\|_{\overline{A}_{E;K}}$ we get

$$\begin{aligned}&\inf_{\{a\}=\{a_n^0\}+\{a_n^1\}} \psi_E\left(\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right) \\ &\quad\leq C \inf_{\{a\}=\{a_n^0\}+\{a_n^1\}} \max\left\{\left\|\{a_n^0\}\right\|_{E(A_0)}, \left\|\{2^n a_n^1\}\right\|_{E(A_1)}\right\} = C\,\|a\|_{\overline{A}_{E;K}},\end{aligned}$$

where infimum is taken over all sequences $\{a_n^0\} \subset A_0$ and $\{a_n^1\} \subset A_1$ such that $a = a_n^0 + a_n^1$ for each $n \in \mathbf{Z}$, which establishes the formula.

Similarly we obtain

$$\begin{aligned}&\inf_{a=\sum a_n} \psi_E\left(\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right) \\ &\quad\leq C \inf_{a=\sum a_n} \max\left\{\|\{a_n\}\|_{E(A_0)}, \|\{2^n a_n\}\|_{E(A_1)}\right\} = \|a\|_{\overline{A}_{E;J}},\end{aligned}$$

where the infimum is taken over all sequences $\{a_n\} \subset A_0 \cap A_1$ such that the series $\sum_{n\in\mathbf{Z}} a_n$ converges to a in $A_0 + A_1$. The opposite inequality in (i) and (ii) follows from Theorem 3.2. This completes the proof. □

4. Measure of weak non-compactness

Throughout the rest of the paper we consider Banach spaces over the real numbers. We say that a Banach lattice E has an order continuous norm, if $\|x_n\|_E \to 0$ for every $\{x_n\} \subset E$ such that $0 \le x_n \downarrow 0$. The Köthe dual space E' of the Banach lattice E is defined to be the space of all $\{y_n\}$ such that $\{x_n y_n\} \in \ell_1(\mathbf{Z})$ for every $\{x_n\} \in E$, equipped with the norm

$$\|\{y_n\}\|_{E'} := \sup \left\{ \sum_{n\in\mathbf{Z}} |x_n y_n| : \|\{x_n\}\|_E \le 1 \right\}.$$

We recall that a Banach lattice E is reflexive if and only if both E and E' have order continuous norms and $E = E''$ (see [20]). It is well-known (see [22]) that if a Banach lattice E has an order continuous norm, then the operator $x^* = \{x_n^*\} \mapsto \phi_{x^*}$ defined by

$$\phi_{x^*}(x) := \sum_{n\in\mathbf{Z}} x_n^*(x_n) \ \text{ for every } \ x = \{x_n\} \in E(X),$$

is an isometric isomorphism acting from the Köthe–Bochner space $E'(X^*)$ onto $(E(X))^*$. Therefore $E(X)^* \simeq E'(X^*)$.

For any $n \in \mathbf{N}$, we denote by P_n, Q_n^+, Q_n^- the operators on the Banach sequence lattice E, defined as follows

$$\begin{aligned} P_n \{u_m\} &= \{\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots\}, \\ Q_n^+ \{u_m\} &= \{\ldots, 0, 0, u_{n+1}, u_{n+2}, \ldots\}, \\ Q_n^- \{u_m\} &= \{\ldots, u_{-n-2}, u_{-n-1}, 0, 0, \ldots\}. \end{aligned}$$

The following properties of the above operators are obvious:

- $I = P_n + Q_n^+ + Q_n^-$, where I denotes the identity operator on E.
- Operators P_n, Q_n^+, Q_n^-, $Q_n^+ + Q_n^-$ have on E the norm ≤ 1.

In a similar way, we define on the Köthe dual E' of E the operators R_n, S_n^+, S_n^-.

We follow the notation used in [23]. Let $\{x_n\}$ be a sequence in a Banach space X. We say that $\{y_n\}$ is a sequence of *successive convex combinations* (in short *scc*) if there exists a sequence $\{p_n\}_{n=1}^\infty \subset \mathbf{Z}$ such that

$$0 = p_1 < p_2 < p_3 < \ldots \quad \text{and} \quad y_n \in \operatorname{conv}\{x_i\}_{i=p_n+1}^{p_{n+1}} \ \text{ for all } \ n \in \mathbf{N}.$$

Vectors u_1, u_2 are said to be a pair of *scc* for $\{x_n\}$ if

$$u_1 \in \operatorname{conv}\{x_i\}_{i=1}^{p} \quad \text{and} \quad u_2 \in \operatorname{conv}\{x_i\}_{i=p+1}^{\infty} \ \text{ for some } \ p \in \mathbf{N}.$$

In the sequel the following useful theorem from [23] will be used.

Theorem 4.1. *Let $\{x_n\}$ be a bounded sequence in a Banach space X. For every $\varepsilon > 0$ there exists a sequence $\{y_n\}$ of scc for x_n such that if u_1, u_2 and v_1, v_2 are any pairs of scc for $\{y_n\}$, then $\big|\, \|u_1 - u_2\| - \|v_1 - v_2\| \,\big| < \varepsilon$.*

Following [3], we recall an axiomatic approach to the notion of a measure of weak non-compactness. Let μ be a real-valued function defined on the family of all bounded and nonempty subsets of a Banach space X. We call μ a *measure of weak non-compactness*, if the following conditions are satisfied for any subsets $A, B \subset X$ and $c \in \mathbf{R}$:

(1) $\mu(A) = 0 \iff A$ is a relatively weakly compact set.

(2) if $A \subset B$, then $\mu(A) \leq \mu(B)$.
(3) $\mu(\text{conv}A) = \mu(A)$.
(4) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(5) $\mu(A + B) \leq \mu(A) + \mu(B)$.
(6) $\mu(cA) = |c|\,\mu(A)$.

Definition 4.2. Let X be a Banach space and let $\emptyset \neq A \subset X$ be a bounded set.

$$\gamma(A) := \sup\{\text{csep}\{x_n\} : \{x_n\} \subset \text{conv}A\},$$

where

$$\text{csep}\{x_n\} := \inf\{\|y_1 - y_2\| : y_1, y_2 \text{ is a pair of } scc \text{ for } \{x_n\}\}.$$

γ is a measure of weak non-compactness (see [23, Theorem 2.3] for more details). Moreover, an alternative formula for the measure γ was established in [23].

Theorem 4.3. *Let A be nonempty and bounded subset of a Banach space X. Then*

$$\gamma(A) = \sup \text{dist}(x^{**}, \text{conv}\{x_n\}),$$

where the supremum is taken over all sequences $\{x_n\} \subset \text{conv}A$ and all ω^*-cluster points $x^{**} \in X^{**}$ of a sequence $\{x_n\}$.

Throughout the rest of the paper for given Banach spaces X and Y and every operator $T\colon X \to Y$, we define the measure of weak non-compactness of T by

$$\gamma(T) := \gamma(T(B_X)).$$

To deal with the measure γ, the ultrafilter notion will be used (for more details concerning filters we refer the reader to [1, 29]). We recall the following useful fact.

Lemma 4.4. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$ and $M \in \mathcal{U}$. If $f\colon M \to \mathbf{N}$ is the bijection given by $f(n_k) = k$, then $\mathcal{U}_M = \{f(M \cap A)\colon A \in \mathcal{U}\}$ is a free ultrafilter on $\mathbf{N}$. Moreover, if*

$$x = \lim_{\mathcal{U}} x_n, \quad \textit{then} \;\; x = \lim_{\mathcal{U}_M} x_{n_k}.$$

Proof. Since an ultrafilter $\mathcal{U}$ is free if and only if $\mathcal{U}$ does not contain a finite set, the proof is straightforward. □

The following lemma is a fairly straightforward generalization of [23, Lemma 3.4]

Lemma 4.5. *Let Y be a Banach space and let E be a reflexive Banach lattice modelled on $\mathbf{Z}$. If $y = \{y_n\}$, $y_n = \{y_{n,m}\}_{m\in\mathbf{Z}} \in E(Y^{**})$ for all $n \in \mathbf{N}$ and*

$$y = \omega^* - \lim_{\mathcal{U}} y_n$$

over some free ultrafilter $\mathcal{U}$ on $\mathbf{N}$, then

$$y_m = \omega^* - \lim_{\mathcal{U}} y_{n,m} \;\; \textit{for each} \;\; m \in \mathbf{N}.$$

Proof. Let $m \in \mathbf{Z}$ and take $v = \{0, \ldots, 0, v_m, 0, \ldots, 0\} \in E'(Y^*)$. From representation of the dual of $E'(Y^*)$, the functional given by

$$f_v(z) = z(v) = z_m(v_m) \;\; \text{for} \;\; z \in E(Y^{**})$$

is ω^*-continuous. It follows that $f_v(y_n) = y_{n,m}(v_m)$ and

$$y_m(v_m) = f_v(y) = \omega^* - \lim_{\mathcal{U}} y_{n,m}(v_m)$$

for each $v_m \in Y^*$, hence that $y_m = \omega^* - \lim_{\mathcal{U}} y_{n,m}$. $\square$

Let X, Y be Banach spaces, $T \in L(X,Y)$ and E be a Banach lattice. The operator $\widetilde{T}\colon E(X) \to E(Y)$ is defined by $\widetilde{T}\{x_n\} := \{Tx_n\}$ for $x = \{x_n\} \in E(X)$.

Theorem 4.6. *Let X, Y be Banach spaces and let E be a reflexive Banach lattice modelled on $\mathbf{Z}$. Then for every bounded operator $T\colon X \to Y$*

$$\gamma(\widetilde{T}\colon E(X) \to E(Y)) = \gamma(T\colon X \to Y).$$

Proof. Since $\widetilde{T}\{\ldots,0,0,x,0,0\ldots\} = \{\ldots,0,0,Tx,0,0\ldots\}$, we have $\gamma(\widetilde{T}) \geq \gamma(T)$. If $\gamma(\widetilde{T}) = 0$, we have $\gamma(T) = 0$. Let $\gamma(\widetilde{T}) > 0$. Choose $0 < \varepsilon < \gamma(\widetilde{T})$. By Theorem 4.3, there exists $\{x_n\} \subset B_{E(X)}$ such that for $y_n = Tx_n$ we obtain

$$0 < \gamma(\widetilde{T}) - \varepsilon \leq \operatorname{dist}\{y, \operatorname{conv}\{y_n\}\} \leq \operatorname{dist}\{y, \{y_n\}\},$$

where $y \in E''(Y^{**})$ is a ω^*-cluster point of $\{y_n\}$, and hence $y = \omega^* - \lim_{\mathcal{U}} y_n$ over some free ultrafilter $\mathcal{U}$ in $\mathbf{N}$. By the separation theorem there exists

$$\phi = \{\phi_m\} \in E'(Y^{***}) \quad \text{with} \quad \|\phi\|_{E'(Y^{***})} \leq 1$$

such that for all $z \in y - \operatorname{conv}\{y_n\}$, we have $\phi(z) \geq \gamma(\widetilde{T}) - \varepsilon$. Since E' has an order continuous norm, there exists $M \in \mathbf{N}$ satisfying

$$\left\|\left(S_M^+ + S_M^-\right)\{\|\phi_m\|_{Y^{***}}\}\right\|_{E'} < \varepsilon.$$

The boundedness of $y - \operatorname{conv}\{y_n\}$ gives constant $c > 0$ such that

$$\begin{aligned}
\gamma(\widetilde{T}) - \varepsilon \leq & \sum_{m=-M}^{M} \phi_m(y_m - y_{n,m}) \\
& + \left\|\left(S_M^+ + S_M^-\right)\{\|\phi_m\|_{Y^{***}}\}\right\|_{E'} \left\|\left(Q_M^+ + Q_M^-\right)\left\{\|y_m - y_{n,m}\|_{Y^{**}}\right\}\right\|_E \\
\leq & \sum_{i=-M}^{M} \phi_m(y_m - y_{n,m}) + \varepsilon c.
\end{aligned}$$

for all $n \in \mathbf{N}$, where by Lemma 4.5 $y_m = \omega^* - \lim_{\mathcal{U}} y_{n,m}$ for all $m \in \mathbf{Z}$. Taking $I = \{m \in \mathbf{Z}\colon -M \leq m \leq M \text{ and } \phi_m \neq 0\}$ and

$$\psi_m = \frac{\phi_m}{\|\phi_m\|_{Y^{***}}}, \qquad \alpha_m = \lim_{\mathcal{U}} \|x_{n,m}\|_X,$$

$$v_m = \frac{y_m}{\alpha_m + \varepsilon/\|P_M\{1\}\|_E}, \qquad v_{n,m} = \frac{y_{n,m}}{\alpha_m + \varepsilon/\|P_M\{1\}\|_E}$$

by the definition of the norm in the Köthe dual E', we get

$$\begin{aligned}
&\gamma(\widetilde{T}) - \varepsilon(1+c) \leq \sum_{m\in I} \|\phi_m\|_{Y^{***}} \left(\alpha_m + \frac{\varepsilon}{\|P_M\{1\}\|_E}\right) \psi_m(v_m - v_{n,m}) \\
&\leq \|R_M\{\|\phi_m\|_{Y^{***}}\}\|_{E'} \left\|P_M\left\{\alpha_m + \frac{\varepsilon}{\|P_M\{1\}\|_E}\right\}\right\|_E \max_{m\in I} \psi_m(v_m - v_{n,m}) \\
&\leq (1+\varepsilon) \max_{m\in I} \psi_m(v_m - v_{n,m}).
\end{aligned}$$

For each $m \in \{-M, \dots, M\}$, denote by Z_m the set of all integers n which satisfy the following inequality

$$\left(\gamma(\widetilde{T}) - \varepsilon(1+c)\right)(1+\varepsilon)^{-1} \leq \psi_m\left(\upsilon_m - \upsilon_{n,m}\right).$$

Since $\bigcup_m Z_m = \mathbf{Z}$, therefore $Z_j \in \mathcal{U}$ for some j. Denote by

$$\mathcal{M} = Z_j \cap \left\{ n \in \mathbf{Z} : \left| \|x_{n,j}\|_X - \alpha_j \right| < \frac{\varepsilon}{\|P_m\{1\}\|_E} \right\} = \{n_k\} \in \mathcal{U}.$$

Lemma 4.4 applied for the set $\mathcal{M}$, yields a new free ultrafilter $\mathcal{U}_{\mathcal{M}}$. By the above

$$\left(\gamma(\widetilde{T}) - \varepsilon(1+c)\right)(1+\varepsilon)^{-1} \leq \psi_j\left(\upsilon_j - \upsilon_{n_k,j}\right),$$

for all $k \in \mathbf{N}$. Hence

$$\left(\gamma(\widetilde{T}) - \varepsilon(1+c)\right)(1+\varepsilon)^{-1} \leq \operatorname{dist}\left\{\upsilon_j, \operatorname{conv}\left\{\upsilon_{n_k,j}\right\}\right\},$$

where $\{\upsilon_{n_k,j}\} \subset T(B_X)$ and $\upsilon_j = \omega^*\!-\!\lim_{\mathcal{U}_{\mathcal{M}}} \upsilon_{n_k,j}$. This implies

$$\left(\gamma(\widetilde{T}) - \varepsilon(1+c)\right)(1+\varepsilon)^{-1} \leq \gamma(T).$$

Since $\varepsilon > 0$ was taken arbitrarily, the required inequality $\gamma(\widetilde{T}) \leq \gamma(T)$ follows. $\square$

5. Main results

Theorem 5.1. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let E be an admissible J-non-trivial reflexive Banach lattice. Then there exists a constant $D = D(E) > 0$ such that for every bounded operator $T \colon \overline{A} \to \overline{B}$*

$$\gamma\big(T\colon \overline{A}_{E;J} \to \overline{B}_{E;J}\big) \leq D\psi_E\big(\gamma(T\colon A_0 \to B_0), \gamma(T\colon A_1 \to B_1)\big).$$

Proof. Fix $\varepsilon > 0$. Choose a sequence $\{a_n\} \subset B_{\overline{A}_{E;J}}$. For each a_n, there exists a sequence $\{a_{n,m}\} \subset A_0 \cap A_1$ such that

$$\{a_{n,m}\}_{m\in\mathbf{Z}} \in (1+\varepsilon)B_{E(A_0)} \quad \text{and} \quad \{2^m a_{n,m}\}_{m\in\mathbf{Z}} \in (1+\varepsilon)B_{E(A_1)},$$

where the series $\sum_{m\in\mathbf{Z}} a_{n,m}$ converges to a_n in $A_0 + A_1$. Set

$$b_n^0 = \{Ta_{n,m}\}_{m\in\mathbf{Z}}, \quad b_n^1 = \{2^m Ta_{n,m}\}_{m\in\mathbf{Z}}, \quad b_n = Ta_n.$$

By Theorem 4.1, there exists a sequence $\{b_k^{0\prime}\}$ of *scc* for $\{b_n^0\}$ satisfying

$$\left| \|u_1 - u_2\|_{E(B_0)} - \|v_1 - v_2\|_{E(B_0)} \right| < \varepsilon \tag{5.1}$$

for every pairs u_1, u_2 and v_1, v_2 of *scc* for $\{b_k^{0\prime}\}$. Hence

$$b_k^{0\prime} = \sum_{j=n_k+1}^{n_{k+1}} \mu_j b_j^0, \quad \text{where } \mu_j \geq 0 \text{ and } \sum_{j=n_k+1}^{n_{k+1}} \mu_j = 1.$$

Define sequence $\{b_k^{1\prime}\}$ by

$$b_k^{1\prime} = \sum_{j=n_k+1}^{n_{k+1}} \mu_j b_j^1 \quad \text{for all } k \in \mathbf{N}.$$

Similarly, as shown above, there exists a sequence $\{b_l^{1\prime\prime}\}$ of *scc* for $\{b_k^{1\prime}\}$ such that

$$\left| \|w_1 - w_2\|_{E(B_1)} - \left\|b_1^{1\prime\prime} - b_2^{1\prime\prime}\right\|_{E(B_1)} \right| < \varepsilon \tag{5.2}$$

for each pair w_1, w_2 of *scc* for $\{b_l^{1\prime\prime}\}$. It is easy to see that $\{b_l^{1\prime\prime}\}$ is also a sequence of *scc* of $\{b_l^1\}$, say

$$b_l^{1\prime\prime} = \sum_{j=m_l+1}^{m_{l+1}} \lambda_j b_j^1, \quad \text{where} \quad \lambda_j \geq 0 \quad \text{and} \quad \sum_{j=m_l+1}^{m_{l+1}} \lambda_j = 1.$$

Setting

$$b_l^{0\prime\prime} = \sum_{j=m_l+1}^{m_{l+1}} \lambda_j b_j^0 \quad \text{for all} \quad l \in \mathbf{N},$$

any pair of *scc* for $\{b_l^{0\prime\prime}\}$ is also a pair of *scc* for $\{b_k^{0\prime}\}$, thus, by (5.1),

$$\left| \|u_1 - u_2\|_{E(B_0)} - \left\|b_1^{0\prime\prime} - b_2^{0\prime\prime}\right\|_{E(B_0)} \right| < \varepsilon \tag{5.3}$$

for all *scc* pairs u_1, u_2 for $\{b_l^{0\prime\prime}\}$. From (5.2), (5.3), and the definition of $\operatorname{csep}\{\cdot\}$, we obtain

$$\left\|b_1^{i\prime\prime} - b_2^{i\prime\prime}\right\|_{E(B_i)} \leq \operatorname{csep}\left\{b_n^{i\prime\prime}\right\} + \varepsilon \quad \text{for} \quad i = 0, 1. \tag{5.4}$$

Set

$$b_1'' = \sum_{j=1}^{n_2} \lambda_j b_j \quad \text{and} \quad b_2'' = \sum_{j=n_2+1}^{n_3} \lambda_j b_j.$$

Obviously b_1'', b_2'' are a pair of *scc* for $\{b_n\}$. From this equality

$$b_1'' - b_2'' = \sum_{m \in \mathbf{Z}} T\left(\sum_{j=1}^{n_2} \lambda_j a_{j,m} - \sum_{j=n_2+1}^{n_3} \lambda_j a_{j,m}\right),$$

where the corresponding series is convergent in $B_0 + B_1$, it follows that

$$\operatorname{csep}\{b_n\} \leq 2\,\|b_1'' - b_2''\|_{\overline{B}_{E;J}} \leq 4\psi_E\left(\left\|b_1^{0\prime\prime} - b_2^{0\prime\prime}\right\|_{E(B_0)}, \left\|b_1^{1\prime\prime} - b_2^{1\prime\prime}\right\|_{E(B_1)}\right).$$

Since

$$\left\{b_n^{i\prime\prime}\right\} \in \widetilde{T}\left((1+\varepsilon)B_{E(A_i)}\right) \quad \text{for} \quad i = 0, 1,$$

the estimate (5.4) and Theorem 4.6 show that for $i = 0, 1$ we have

$$\left\|b_1^{i\prime\prime} - b_2^{i\prime\prime}\right\|_{E(B_i)} \leq \gamma\left(\widetilde{T}\left((1+\varepsilon)B_{E(A_i)}\right)\right) + \varepsilon = (1+\varepsilon)\gamma(T\colon A_i \to B_i) + \varepsilon.$$

By the properties of ψ_E (see Lemma 2.1), there exists a constant C_2 satisfying

$$\operatorname{csep}\{b_n\} \leq 4C_2\psi_E\big((1+\varepsilon)\gamma(T\colon A_0 \to B_0) + \varepsilon, (1+\varepsilon)\gamma(T\colon A_1 \to B_1) + \varepsilon\big).$$

Finally, an arbitrary choice of ε implies

$$\operatorname{csep}\{b_n\} \leq D\psi_E\left(\gamma(T\colon A_0 \to B_0), \gamma(T\colon A_1 \to B_1)\right),$$

where $D = D(E) > 0$. The assertion follows from the definition of $\gamma(T\colon \overline{A}_{E;J} \to \overline{B}_{E;J})$. □

The following result may be proved in the same way as Theorem 5.1.

Theorem 5.2. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let E be an admissible K-non-trivial reflexive Banach lattice. Then there exists a constant $D = D(E) > 0$ such that for every bounded operator $T\colon \overline{A} \to \overline{B}$*

$$\gamma\big(T\colon \overline{A}_{E;K} \to \overline{B}_{E;K}\big) \leq D\psi_E\big(\gamma(T\colon A_0 \to B_0), \gamma(T\colon A_1 \to B_1)\big).$$

Proof. Fix $\varepsilon > 0$. Choose a sequence $\{a_n\} \subset B_{\overline{A}_{E;K}}$. For each a_n, there exist sequences $\left\{a^0_{n,m}\right\}$ and $\left\{a^1_{n,m}\right\}$ such that

$$\left\{a^0_{n,m}\right\}_{m\in\mathbf{Z}} \in (1+\varepsilon)B_{E(A_0)} \quad \text{and} \quad \left\{2^m a^1_{n,m}\right\}_{m\in\mathbf{Z}} \in (1+\varepsilon)B_{E(A_1)}$$

where $a^0_{n,m} + a^1_{n,m} = a_n$ for all $m \in \mathbf{Z}$. Set

$$b^0_n = \left\{Ta^0_{n,m}\right\}_{m\in\mathbf{Z}}, \quad b^1_n = \left\{2^m Ta^1_{n,m}\right\}_{m\in\mathbf{Z}}, \quad b_n = Ta_n.$$

Analysis similar to that in the proof of Theorem 5.1 shows that there exist sequences $\{b^{0\prime\prime}_n\}$ and $\{b^{1\prime\prime}_n\}$ of *scc* for $\{b^0_n\}$ and $\{b^1_n\}$ respectively, such that we have for $i = 0, 1$

$$\left\|b^{i\prime\prime}_1 - b^{i\prime\prime}_2\right\|_{E(B_i)} \leq \operatorname{csep}\left\{b^{i\prime\prime}_n\right\} + \varepsilon. \tag{5.4}$$

In the same manner, we can see that $b''_1 = b^{0\prime\prime}_1 + b^{1\prime\prime}_1$, $b''_2 = b^{0\prime\prime}_2 + b^{1\prime\prime}_2$ are pairs of *scc* for $\{b_n\}$. Therefore

$$\operatorname{csep}\{b_n\} \leq 2\,\|b''_1 - b''_2\|_{\overline{B}_{E;K}} \leq 4\psi_E\left(\left\|b^{0\prime\prime}_1 - b^{0\prime\prime}_2\right\|_{E(B_0)}, \left\|b^{1\prime\prime}_1 - b^{1\prime\prime}_2\right\|_{E(B_1)}\right). \tag{5.5}$$

Applying Theorem 4.6 and (5.4), similarly as in the proof of Theorem 5.1, we obtain for $i = 0, 1$

$$\left\|b^{i\prime\prime}_1 - b^{i\prime\prime}_2\right\|_{E(B_i)} \leq \gamma\left(\widetilde{T}\left((1+\varepsilon)B_{E(A_i)}\right)\right) + \varepsilon = (1+\varepsilon)\gamma(T\colon A_i \to B_i) + \varepsilon.$$

By the above and (5.5) we have

$$\begin{aligned}\operatorname{csep}\{b_n\} &\leq 4\psi_E\left(\left\|b^{0\prime\prime}_1 - b^{0\prime\prime}_2\right\|_{E(B_0)}, \left\|b^{1\prime\prime}_1 - b^{1\prime\prime}_2\right\|_{E(B_1)}\right)\\ &\leq 4C_2\psi_E((1+\varepsilon)\gamma(T\colon A_0 \to B_0) + \varepsilon, (1+\varepsilon)\gamma(T\colon A_1 \to B_1) + \varepsilon).\end{aligned}$$

The rest of the proof runs as in Theorem 5.1, with $\gamma\big(T\colon \overline{A}_{E;J} \to \overline{B}_{E;J}\big)$ replaced by $\gamma\big(T\colon \overline{A}_{E;K} \to \overline{B}_{E;K}\big)$. $\square$

From Theorems 5.1 and 5.2 the following corollaries follow.

Corollary 5.3. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let E be an admissible J-non-trivial reflexive Banach lattice. If $T\colon \overline{A} \to \overline{B}$, then*

$$T\colon \overline{A}_{E;J} \to \overline{B}_{E;J} \ \text{ is a weakly compact operator}$$

provided one of the following conditions holds:

(i) *$T\colon A_0 \to B_0$ is weakly compact and $\psi_E(0,1) = 0$.*
(ii) *$T\colon A_1 \to B_1$ is weakly compact and $\psi_E(1,0) = 0$.*

In particular, if A_0 is reflexive and $\psi_E(0,1) = 0$, or A_1 is reflexive and $\psi_E(1,0) = 0$, then $\overline{A}_{E;J}$ is reflexive.

Corollary 5.4. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples. Let E be an admissible K-non-trivial reflexive Banach lattice. If $T\colon \overline{A} \to \overline{B}$, then*

$$T\colon \overline{A}_{E;K} \to \overline{B}_{E;K} \ \text{ is a weakly compact operator}$$

provided one of the following conditions holds:

(i) *$T\colon A_0 \to B_0$ is weakly compact and $\psi_E(0,1) = 0$.*
(ii) *$T\colon A_1 \to B_1$ is weakly compact and $\psi_E(1,0) = 0$.*

In particular, if A_0 is reflexive and $\psi_E(0,1) = 0$, or A_1 is reflexive and $\psi_E(1,0) = 0$, then $\overline{A}_{E;K}$ is reflexive.

It is worth remarking that Corollaries 5.3 and 5.4 do not allow to show sufficient conditions on E such that weak compactness of the interpolated operator follows from the fact that $T\colon A_0 \cap A_1 \to B_0 + B_1$ is weakly compact. An additional assumption imposed on E is required. We refer to [7, Corollary 4.4] or [17, Section 5] for such results.

Proposition 5.5. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T\colon \overline{A} \to \overline{B}$ be a bounded operator. Let ρ be a function parameter and $1 < q < \infty$.*

(i) *If $T\colon A_0 \to B_0$ and $T\colon A_1 \to B_1$ are weakly non-compact, then there exists a constant $C > 0$, such that*

$$\gamma(T\colon \overline{A}_{\rho,q} \to \overline{B}_{\rho,q}) \le C\gamma(T\colon A_0 \to B_0) s_\rho\left(\frac{\gamma(T\colon A_1 \to B_1)}{\gamma(T\colon A_0 \to B_0)}\right).$$

(ii) *If $T\colon A_0 \to B_0$ or $T\colon A_1 \to B_1$ is weakly compact, then operator*

$$T\colon \overline{A}_{\rho,q} \to \overline{B}_{\rho,q} \text{ is weakly compact.}$$

In particular, if A_0 or A_1 is reflexive then space $\overline{A}_{\rho,q}$ is reflexive.

Proof. In the light of Lemmas 2.4 and 2.5, it is sufficient to use Theorem 5.1 and Corollary 5.3. □

We note that above result for the case $\rho(t) = t^\theta$, $\theta \in (0,1)$ was proved in [23, Theorem 3.8].

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