POSITIVE HARMONIC FUNCTIONS
ON COMB-LIKE DOMAINS

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Abstract. This paper investigates positive harmonic functions on a domain which contains an infinite cylinder, and whose boundary is contained in the union of parallel hyperplanes. (In the plane its boundary consists of two sets of vertical semi-infinite lines.) It characterizes, in terms of the spacing between the hyperplanes, those domains for which there exist minimal harmonic functions with a certain exponential growth.

1. Introduction

The subtle relationship between the structure of positive harmonic functions on a domain $\Omega$ in $\mathbb{R}^N$ ($N \geq 2$) and boundary geometry has been much studied. One avenue of investigation has been to examine the effect of modifying the boundary of a familiar domain such as a half-space, cone or cylinder. Thus many authors have been led to investigate the case of Denjoy domains $\Omega$, where the complement $\mathbb{R}^N \setminus \Omega$ is contained in a hyperplane, say $\mathbb{R}^{N-1} \times \{0\}$ (see [6, 11, 14, 1, 24, 25, 8, 10, 2, 21]). For example, Benedicks [6] has established a harmonic measure criterion that describes when the cone of positive harmonic functions on $\Omega$ that vanish on the boundary $\partial \Omega$ is generated by two linearly independent minimal harmonic functions. (We recall that a positive harmonic function $h$ on a domain $\Omega$ is called minimal if any non-negative harmonic minorant of $h$ on $\Omega$ is proportional to $h$.) Benedicks’ criterion is also equivalent to the existence of a harmonic function $u$ on $\Omega$ vanishing on $\partial \Omega$ and satisfying $u(x) \geq |x_N|$ on $\Omega$, and thus describes when a Denjoy domain behaves like the union of two half-spaces from the point of view of potential theory. Related results, based on sectors, cones or cylinders, may be found in [12, 21, 18]. The purpose of this paper is to describe what happens in the case of another relative of the infinite cylinder. More precisely, let $(a_n)$ be a strictly increasing sequence of non-negative numbers such that $a_n \to +\infty$ and $a_{n+1} - a_n \to 0$ as $n \to \infty$, and let $B'$ be the unit ball in $\mathbb{R}^{N-1}$. We define

$$E = \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{N-1} \setminus B') \times \{a_n\}$$

and investigate when the domain $\Omega = \mathbb{R}^N \setminus E$ inherits the potential theoretic character of the cylinder $U = B' \times \mathbb{R}$; that is, when the set $E$ imitates $\partial U$ in terms of its effect on the asymptotic behaviour of positive harmonic functions on $\Omega$. We call

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such domains $\Omega$ *comb-like* because they are a generalization of comb domains in the plane.

Let $x = (x', x_N)$ denote a typical point of Euclidean space $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$. It is known (see [15], for example) that the cone of positive harmonic functions on $U$ that vanish on $\partial U$ is generated by two minimal harmonic functions $h_\pm(x', x_N) = e^{\pm x_N \phi(x')}$, where $\alpha$ is the square root of the first eigenvalue of the operator $-\Delta = -\sum_{j=1}^{N-1} \partial^2/\partial x_j^2$ on $B'$ and $\phi$ is the corresponding eigenfunction, normalized by $\phi(0) = 1$. We describe below when a comb-like domain admits a minimal harmonic function $u$ that vanishes on $\partial \Omega$ and satisfies $u \geq h_+$ on $U$.

**Theorem 1.1.** Let $\nu > 1$. Assume that $(a_n)$ satisfies the following condition

$$
\frac{1}{\nu} \leq \frac{a_{k+1} - a_k}{a_{j+1} - a_j} \leq \nu
$$

whenever $|a_k - a_j| < 4$. The following statements are equivalent:

(a) there exists a positive harmonic function $u$ on $\Omega$ that satisfies $u \geq h_+$ on $U$ and $u$ vanishes continuously on $E$;

(b) $\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < +\infty$.

Moreover, if (b) holds, then $u$ can be chosen to be minimal in part (a).

We will prove Theorem 1.1 by combining methods from [14], [12] and [18] with some new ideas. It is known (see [7, 9, 16]) that the behaviour of minimal harmonic functions on simply connected domains is intimately related to the classical angular derivative problem. We note that when $N = 2$, condition (b) of Theorem 1.1 is necessary and sufficient for the comb domain $\Omega$ to have an angular derivative at $+\infty$ (see [22, 23, 20]).

### 2. Notation and preliminary results

We use $\partial^\infty D$ to denote the boundary of a domain $D$ in compactified space $\mathbb{R}^N \cup \{\infty\}$. Let $B_\rho(x)$ denote the open ball in $\mathbb{R}^N$ of centre $x$ and radius $\rho > 0$. We write $B'_\rho$ (resp. $B_\rho$) for the open ball in $\mathbb{R}^{N-1}$ (resp. $\mathbb{R}^N$) of centre 0 and radius $\rho$, and $V(\rho) = \partial B'_\rho \times \mathbb{R}$. If $\rho = 1$, we write $B'$ instead of $B'_1$. For $0 < \rho_1 < \rho_2$ let $A(\rho_1, \rho_2) = (B'_\rho \setminus B'_1) \times \mathbb{R}$. We denote by $\mu^D_x$ the harmonic measure for an open set $D \subset \mathbb{R}^N$ evaluated at $x \in D$. If $f$ is a function defined on $\partial^\infty D$, we use $\overline{H}_f^D$ to denote the upper Perron–Wiener–Brelot solution to the Dirichlet problem on $D$ and $H^D_f$ for the PWB solution of the Dirichlet problem on $D$ when it exists. We denote by $P_D(\cdot, y)$ the Poisson kernel for $D$ with pole $y \in \partial D$, where $\partial D$ is smooth enough for it to be defined. If $W \subseteq D$ and $u$ is a superharmonic function on $D$, we denote by $R^W_u$ (resp. $\tilde{R}^W_u$) the reduced function (resp. the regularized reduced function) of $u$ relative to $W$ in $D$. We denote surface area measure on a given surface by $\sigma$. We use $C(a, b, \ldots)$ to denote a constant depending at most on $a, b, \ldots$, the value of which may change from line to line.

For the remainder of the paper, we fix $0 < r < 1 < R$ and for $x \in \partial U$ we define $F_x = \partial B'_r \times [x_N - 1, x_N + 1]$ and $T_x = (B'_R \setminus \overline{B'_r}) \times (x_N - 1, x_N + 1)$. 
We note that the first eigenfunction $\phi$ of $-\Delta$ in $B'$ is comparable with the distance to $\partial B'$, that is
\begin{equation}
C_1(N)(1 - |x'|) \leq \phi(x') \leq C_2(N)(1 - |x'|) \quad (x' \in B').
\end{equation}
A simple proof of (2.1) can be found in [17, pp. 419–420]. The following estimate for the Poisson kernel (see [18, Section 2.1], for example) will prove useful. For $|x'| = s < 1$, \( x_N \in \mathbb{R}, \ y \in \partial U \)
\begin{equation}
C_1(N)e^{-\alpha |x_N - y_N|} \leq P_U(x, y) \leq C_2(N, s)e^{-\alpha |x_N - y_N|}.
\end{equation}
If $0 < r_1 < s < r_2$, similar estimates hold for $P_{A(r_1, r_2)}$ with $\alpha$ replaced by the square root of the first eigenvalue of $-\Delta$ in $B'_{r_2}\setminus B'_{r_1}$ and constants $C_1, C_2$ depending on $N, r_1, r_2$ and $s$.

**Proposition 2.1.** Assume there exists a positive harmonic function $u$ on $\Omega$ such that $u \geq h_+$ on $U$ and $u$ vanishes on $E$. Then
\begin{equation}
\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < +\infty.
\end{equation}

**Proof.** By (2.2) we have
\begin{align}
\infty &> u(0) \geq \int_{\partial U} u(y)P_U(0, y) \, d\sigma(y) \\
&\geq C(N) \sum_{n=1}^{\infty} \int_{\partial B'\times(a_n, a_{n+1})} u(y)e^{-\alpha y_N} \, d\sigma(y).
\end{align}
We use Harnack's inequalities and (2.1) to see that for $y \in \partial U$ with
\begin{align}
y_N \in \left(a_n + (a_{n+1} - a_n)/4, a_{n+1} - (a_{n+1} - a_n)/4\right)
\end{align}
the following holds
\begin{align}
\begin{aligned}
&\phi(y) \geq C(N)u \left((1 - (a_{n+1} - a_n)/8)y', y_N\right) \\
&\geq C(N)e^{\alpha y_N}\phi \left((1 - (a_{n+1} - a_n)/8)y'\right) \\
&\geq C(N)e^{\alpha y_N}(a_{n+1} - a_n).
\end{aligned}
\end{align}
We deduce from (2.4) and (2.5) that (2.3) holds. \(\square\)

Assume now that $\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < +\infty$. Let $J \in \mathbb{N}$ be large enough so that $a_{n+1} - a_n \leq 1/2$ for $n \geq J$. For ease of exposition we rename the sequence $(a_n)_{n=J}^{\infty}$ as $(b_n)_{n=1}^{\infty}$. We also define $\rho_n = (b_{n+1} - b_n)/2$ for $n \in \mathbb{N}$. We introduce $b_0 = b_1 - 1$ and $\rho_0 = 1/2$. For technical reasons, we will work with
\[ E'' = \bigcup_{n=1}^{\infty} (\mathbb{R}^{N-1}\setminus B') \times \{b_n\} \quad \text{and} \quad E' = (\partial B' \times (-\infty, b_1]) \cup E'', \]
and at the end we will dispense with these additional requirements.

**Lemma 2.1.** There exists a positive constant $c_1$, depending on $N, R$ and $r$, such that for any $x \in \partial U$ we have
\begin{equation}
\mu_{x,T_x\setminus E''}^{T_x}(F_x) \leq \mu_{x,T_x\setminus E''}^{T_x}(\partial T_x) \leq c_1 \mu_{x,T_x\setminus E''}^{T_x}(F_x).
\end{equation}
Proof. Let \( x \in \partial U \). The left hand inequality in (2.6) is obvious since \( F_x \subset \partial T_x \).
Let \( h = H^{+}_{|\chi_{F_x}} \) on \( T_x \) and \( h = \chi_{F_x} \) on \( \partial T_x \). In order to establish the right hand inequality, it is enough to prove that
\[
(2.7) \quad h \leq h(x) \quad \text{on} \quad E'' \cap T_x.
\]
We will borrow an argument from [18, Lemma 2.1]. Using reflection in \( \mathbb{R}^{N-1} \times \{ x_N + 1 \} \) to extend \( h \) to \((\overline{B_R'} \setminus B'_r) \times [x_N - 1, x_N + 3] \), and translation, for \( y \in \partial B' \times (x_N, x_N + 1) \) we obtain
\[
h(y) = H^{+}_{h \chi_{\mathbb{R}^{N-1}}} (y)
= \mu^{T_x}_{x} (\partial B'_r \times [x_N - 1, 2x_N + 1 - y_N]) - \mu^{T_x}_{x} (\partial B'_r \times [2x_N + 1 - y_N, x_N + 1])
\leq \mu^{T_x}_{x} (F_x) = h(x).
\]
By symmetry, \( h(y) \leq h(x) \) for \( y \in \partial D' \times (x_N - 1, x_N + 1) \). Since
\[
h(y) = 0 \leq h(x) \quad \text{for} \quad y \in [\partial B' \times (x_N - 1, x_N + 1)] \cup [(\overline{B_R'} \setminus \overline{B'}) \times \{ x_N - 1, x_N + 1 \}],
\]
using the maximum principle, we see that \( h \leq h(x) \) on \((\overline{B_R'} \setminus \overline{B'}) \times (x_N - 1, x_N + 1)\), which proves (2.7).

We note that Lemma 2.1 holds in a more general context when \( E'' \) is a closed subset of \( \mathbb{R}^N \setminus U \).
To prove Theorem 1.1 we shall need the following estimate.

Lemma 2.2. Let \( \nu > 1 \). Assume that \( (b_n) \) satisfies
\[
(2.8) \quad \frac{1}{\nu} \leq \frac{b_{k+1} - b_k}{b_{j+1} - b_j} \leq \nu
\]
whenever \( |b_k - b_j| < 4 \). Then there exists a constant \( c_2 \), depending only on \( N, r \) and \( \nu \), such that
\[
\mu^{T_x \setminus E''}_{x} (F_x) \leq c_2 (b_{n+1} - b_n)
\]
whenever \( x \in \partial B' \times (b_n, b_{n+1}) \) and \( n \in \mathbb{N} \).

Proof. We suppose that \( x \in \partial B' \times (b_n, b_{n+1}) \) for some \( n \in \mathbb{N} \). We define \( \omega = (\overline{B_R'} \setminus \overline{B'}) \times (b_j, b_{j_0}) \), where \( j_0 = \max \{ j : b_j \leq x_N - 1 \} \) and \( k_0 = \min \{ j : b_j \geq x_N + 1 \} \).
Let \( g = H^\omega_{x(r)} \) on \( \omega \setminus E'' \) and define \( g = \chi_{V(r)} \) elsewhere. Let \( m = \sup_{\partial U} g \). We note that
\[
\mu^{T_x \setminus E''}_{x} (F_x) \leq \mu^{\omega \setminus E''}_{x} (V(r)) \leq m.
\]
We will obtain an upper bound for \( m \) in terms of \( \rho_n \). To do this, we define an open set \( Z \) as follows
\[
Z = \omega \setminus \bigcup_{k=0}^{\infty} \bigcup_{p \in \mathbb{Z}_k} \{ z \in \overline{B}'_s \times \{ p \} : s = (1 - r)(|p - (b_k + \rho_k)| - \rho_k) + 1 \}.
\]
We estimate \( g \) on \( \partial Z \) in terms of \( m \) and \( \rho_n \). Since \( g = 0 \) on \( \partial \omega \setminus U \), we estimate \( g \) on \( \partial Z \cap U \), noting that, for \( y \in \omega \cap U \), we have
\[
(2.9) \quad g(y) = H^\omega_{g U}(y) = H^\omega_{\chi_{V(r)}}(y) + \int_{\partial U \cap \omega} g \, d\mu^\omega_{g U}.
\]
Let \( g_1(y) = H_{\chi_U}(y) \) and \( g_2(y) = \int_{\partial U \cap \omega} g \, d\mu_\omega \) for \( y \in \omega \cap U \). Using the function
\[
f_N(y) = \begin{cases} 
|y|^{3-N} - 1 & (N \geq 4) \\
- \log |y| & (N = 3) \\
1 - |y| & (N = 2)
\end{cases}
\]
and the maximum principle, we find that for \( y \in \partial Z \cap U \)
\[
(2.10) \quad g_1(y) \leq f_N(y)/f_N(rx) \leq C_1(N, r)(1 - |y'|) \leq C_1(N, r, \nu)\rho_n.
\]
We now wish to show that there exists a constant \( C_2(N, \nu) \in (0,1) \) such that
\[
(2.11) \quad g_2 \leq C_2(N, \nu) m \quad \text{on} \quad \partial Z \cap U.
\]

Let \( l = (1-r) \min_{j_0 \leq k \leq k_0-1} \rho_k \) and let \( t = (1, 0, \ldots, 0, t_N) \) with \( t_N \in \{b_k : k = j_0 + 1, \ldots, k_0 - 1\} \). By [5, Lemma 8.5.1], for \( x \in B_{t/2}(1+t, 0, \ldots, 0, t_N) \) we have \( g(x) \leq C(N)(g(p_x) + g(p_-)), \) where \( p_x = (1+t, 0, \ldots, 0, t_N + t/2). \) Using a Harnack chain to cover the longer arc joining \( p_+ \) and \( p_- \) along the circle \( \partial B_{\sqrt{5t}/2}(t) \cap (\mathbb{R} \times \{0\})^{N-2} \times \mathbb{R}, \) we deduce that \( g \leq C_3(N)m \) on that circle. By the invariance of \( g \) under rotations around the \( x_N \)-axis and the maximum principle, this inequality holds on a torus-shaped set enclosing the edge of \((\mathbb{R}^{N-1} \setminus B^t) \times \{t_N\}; \) more precisely on every closed ball centred at a point of \( \partial B^t \times \{t_N\} \) and having radius \( \sqrt{5t}/2. \) When \( t_N = b_{j_0} \) or \( t_N = b_{k_0}, \) this inequality follows directly from [5, Lemma 8.5.1], with a perhaps different constant, \( C_4(N) \) say. In particular, for \( y \in B^t \setminus E^t, \) where \( B^t = B_{\sqrt{5t}/2}(t), E^t = [1, +\infty) \times \mathbb{R}^{N-2} \times \{t_N\} \) and \( t_N \in \{b_k : k = j_0, \ldots, k_0\}, \) we have
\[
(2.12) \quad g(y) \leq H^B_{\chi_U}(y) = \int_{\partial B_t} g \, d\mu^B \leq \max\{C_3(N, C_4(N))\} m H^B_{\chi_{\partial t}}(y).
\]
Since \( t \) is a regular boundary point for \( B^t \setminus E^t, \) there exists \( \delta = \delta(N) > 0 \) such that
\[
(2.13) \quad H^B_{\chi_{\partial t}}(y) \leq \frac{1}{2\max\{C_3(N, C_4(N))\}} \quad (y \in B_{\delta t}(t) \setminus E^t).
\]

Let \( \mathcal{K}_{\delta t} = \bigcup_{k \in j_0} \{y \in \partial U : |y_N - b_k| < \delta t\}. \) In view of (2.12) and (2.13), and the invariance of \( g \) under rotations around the \( x_N \)-axis, we conclude that \( g \leq m/2 \) on \( \mathcal{K}_{\delta t}. \)

Hence, for \( y \in \partial Z \cap U, \) we have
\[
(2.14) \quad g_2(y) \leq \int_{\partial U} g \, d\mu_y \leq \frac{m}{2} \mu^U_y(\mathcal{K}_{\delta t}) + \mu^U_y(\partial U \setminus \mathcal{K}_{\delta t}) \leq m \left( 1 - \frac{1}{2} \mu^U_y(\mathcal{K}_{\delta t}) \right).
\]
We now show that there exists a constant \( C_5(N, \nu) \in (0,1) \) such that \( \mu^U_y(\mathcal{K}_{\delta t}) \geq C_5(N, \nu) \) for \( y \in \partial Z \cap U. \) We first estimate \( \mu^U_y(\mathcal{K}_{\delta t}) \) on some ball centred at \( t \) and then join other points of \( \partial Z \cap U \) by a Harnack chain.

Let \( W_{\delta t} = B^t \times (t_N - \delta t, t_N + \delta t). \) We use a dilation \( \psi(y) = t + (y - t)/(\delta t) \) and note that, by continuity, there exists an absolute positive constant \( \gamma \) such that for \( y \in \psi(W_{\delta t}) \cap B_{\gamma}(t) \) the following inequalities hold
\[
H^W_{\chi_{\partial t}}(\psi^{-1}(y)) = H^\psi_{\chi_U}(W_{\delta t})(y) \geq H_{\chi(1) \times \mathbb{R}^{N-2} \times \{t_N - 1, t_N + 1\}}(y) \geq 1/2.
\]
Hence
\[
\mu^U_y(\mathcal{K}_{\delta t}) \geq \mu^W_y(\mathcal{K}_{\delta t}) \geq 1/2 \quad (y \in B_{\gamma \delta t}(t) \cap U),
\]
and by Harnack’s inequalities
\[ \mu_y^U(K_M) \geq C_5(N, \nu) \quad \text{for all } y \in \partial Z \cap U. \]

Let \( C_2(N, \nu) = 1 - C_5(N, \nu)/2 \). Then (2.11) holds in view of (2.14), and by (2.9) and (2.10) we have
\[ g \leq C_1(N, r, \nu) \rho_n + C_2(N, \nu)m \quad \text{on } \partial Z. \]

By the maximum principle this inequality holds on \( Z \) and implies that
\[ m \leq \frac{C_1(N, r, \nu)}{1 - C_2(N, \nu)} \rho_n. \]

This finishes the proof of lemma. \( \Box \)

We define \( \beta_{E'}(x) \) to be the harmonic measure of \( \partial T_x \) in \( T_x \setminus E' \) evaluated at \( x \). If \( x \in E' \), then \( \beta_{E'}(x) \) is interpreted as 0. We observe that, if \( (b_n) \) satisfies the ratio condition (2.8), then, in view of Lemmas 2.1 and 2.2, we have
\[
\int_{\partial B' \times (b_1, +\infty)} \beta_{E'}(y) \, d\sigma(y) \leq c_1 \int_{\partial B' \times (b_1, +\infty)} \mu_{y}^{T_y \setminus E''}(F_y) \, d\sigma(y)
\]
(2.15)
\[
\leq c_1 c_2 \sigma_{N-1} \sum_{n=1}^{\infty} (b_{n+1} - b_n)^2,
\]
where \( \sigma_{N-1} \) denotes the surface measure of \( \partial B' \) in \( \mathbb{R}^{N-1} \).

Henceforth let \( (b_n) \) satisfy (2.8) and let
\[ \Lambda = \sum_{n=1}^{\infty} (b_{n+1} - b_n)^2 < +\infty. \]

Before we prove the next lemma, we collect together some facts about certain Bessel functions (see [4, Section 4]). Let \( K = K_{(N-3)/2} : (0, \infty) \to (0, \infty) \) denote the Bessel function of the third kind, of order \((N - 3)/2\). Then the function
\[
h_0(x', x_N) = |x'|^{(3-N)/2} K(\pi |x'|) \sin(\pi x_N)
\]
is positive and superharmonic on the strip \( \mathbb{R}^{N-1} \times (0, 1) \), harmonic on \( (\mathbb{R}^{N-1} \setminus \{0'\}) \times (0, 1) \) and vanishes on \( \mathbb{R}^{N-1} \times \{0, 1\} \setminus \{(0', 0), (0', 1)\} \). Moreover, there exists \( c(N) \geq 1 \) such that
\[
c(N)^{-1} \leq (2t/\pi)^{1/2} e^t K(t) \leq c(N) \quad \text{for } t \in [1, +\infty).
\]

We also recall a result of Domar ([13, Theorem 2]). Suppose that \( D \) is a domain in \( \mathbb{R}^N \) and \( F : D \to [0, +\infty] \) is a given upper semicontinuous function on \( D \). Let \( \mathcal{F} \) be the collection of all subharmonic functions \( u \), such that \( u \leq F \) on \( D \). Domar’s result says that if
\[
\int_D [\log^+ F(x)]^{N-1+\varepsilon} \, dx < \infty,
\]
for some \( \varepsilon > 0 \), then the function \( M(x) = \sup_{u \in \mathcal{F}} u(x) \) is bounded on every compact subset of \( D \).

Let \( 0 < r' < \min\{r, 1/2\} \). Define \( V = A(r', \infty) \setminus E' \) and \( U_n = (\mathbb{R}^{N-1} \setminus \overline{B}) \times (b_n, b_{n+1}) \) for \( n \in \mathbb{N} \).
Lemma 2.3. There exists a positive constant \( c_3 \), depending on \( N, R, r \) and \( r' \), such that, for any positive harmonic function \( u \) on \( V \) that is bounded on each \( U_n \) and vanishes on \( E' \),

\[
u(y) \leq c_3 u(rx', x_N)H^T_{\partial T_x}(y) \quad (x \in \partial U, y \in T_x \setminus E').\]

In particular,

\[
u(x', x_N) \leq c_3 \beta_{E'}(x)u(rx', x_N) \quad (x \in \partial U).
\]

Proof. Let \( x \in \partial U, l = (1 + r')/3 \) and \( L = 2R \). Define \( A_x = \{ y : l < |y'| < L, |x_N - y_N| < 2 \} \). We will show that

\[
\frac{u(y)}{C(N, r, r')u(rx', x_N)} \leq F(y) \quad (y \in A_x),
\]

where

\[
F(y) = \begin{cases} 
|1 - |y'||^{1-N}, & |y'| \neq 1, \\
+\infty, & |y'| = 1.
\end{cases}
\]

Step 1. Let \( (y', y_N) \in A_x \cap U \). Harnack’s inequalities yield that

\[
u(y) \leq C(N, r, r')u(rx', x_N)(1 - |y'|)^{1-N}.
\]

Step 2. If \( y \in A_x \cap U \) and \( |y'| - 1 \leq \min\{y_N - b_n, b_{n+1} - y_N \} \), then there is a Harnack chain of fixed length joining \( (y', y_N) \) with \( ((2 - |y'|)y'/|y'|, y_N) \in A_x \cap U \). By Step 1, we have

\[
u(y) \leq C(N)u((2 - |y'|)y'/|y'|, y_N) \leq C(N, r, r')u(rx', x_N)(|y'| - 1)^{1-N}.
\]

Step 3. If \( y \in A_x \cap U_n \) and \( \rho_n \geq |y'| - 1 > \min\{y_N - b_n, b_{n+1} - y_N \} \), we apply [5, Lemma 8.5.1] and Harnack’s inequalities to see that

\[
u(y) \leq C(N)u(y', \tilde{y}_N),
\]

where \( \tilde{y}_N \) is such that \( |\tilde{y}_N - y_N| < |b_n + \rho_n - y_N| \) and \( |y'| - 1 = \min\{\tilde{y}_N - b_n, b_{n+1} - \tilde{y}_N \} \). By Step 2,

\[
u(y) \leq C(N, r, r')u(rx', x_N)(|y'| - 1)^{1-N}.
\]

Step 4. If \( y \in A_x \cap U_n \) and \( |y'| \geq 1 + \rho_n \), let \( V_n = \{ (z', z_N) : 1 + \rho_n < |z'|, z_N \in (b_n, b_{n+1}) \} \). For \( z \in U_n \) we define a function

\[
h_n(z) = h_0((z', z_N - b_n)/(2\rho_n)) \left( \frac{1 + \rho_n}{2\rho_n} \right)^{(N-3)/2}
\]

which is harmonic on \( U_n \) and vanishes on \( \partial U_n \setminus \partial U \). Applying [5, Lemma 8.5.1] and Harnack’s inequalities to \( u \) and \( h_n \), by Step 3, we get

\[
u(z) \leq C(N, r, r')u(rx', x_N)\rho_n^{1-N}h_n(z) \quad \text{for} \quad z \in \partial V_n.
\]

Since \( u \) is bounded on \( V_n \) and \( \infty \) has zero harmonic measure for \( V_n \),

\[
u(y) \leq C(N, r, r')u(rx', x_N)\rho_n^{1-N}h_n(y).
\]

Furthermore, by (2.16) and (2.17)

\[
h_n(y) \leq \left( \frac{1 + \rho_n}{|y'|} \right)^{\frac{N-2}{2}} K \left( \frac{\pi|y'|}{2\rho_n} \right) \left( K \left( \frac{\pi(1 + \rho_n)}{2\rho_n} \right) \right)^{-1}
\]

\[
\leq C(N)e^{-\frac{\pi}{2\rho_n}(|y'| - 1)} \left( \frac{1 + \rho_n}{|y'|} \right)^{\frac{N-2}{2}} \leq C(N)e^{-\frac{\pi}{2\rho_n}(|y'| - 1)} \leq C(N) \left( \frac{|y'| - 1}{\rho_n} \right)^{1-N}.
\]
Hence we see from (2.20) that 
\[ u(y) \leq C(N, r, r')u(rx', x_N)(|y'| - 1)^{1-N}. \]

We conclude that (2.19) follows from Steps 1–4. Since 
\[ \int_{A_x} (\log^+ F(y))^N dy \leq C(N, R), \]
Domar’s result and Harnack’s inequalities (if \( r < l \)) yield 
\[ u(y) \leq C(N, R, r, r')u(rx', x_N) \quad (y \in \overline{T_x}). \]

Therefore 
\[ u(y) = H^{T_x \setminus E'}_u(y) \leq C(N, R, r, r')u(rx', x_N)H^{T_x \setminus E'}_{\chi_{\partial T_x}}(y) \quad (y \in T_x \setminus E'). \]

In particular, 
\[ u(x) \leq C(N, R, r, r')u(rx', x_N)\beta_{E'}(x). \quad \square \]

**Lemma 2.4.** Let \( v: \mathbb{R}^N \cup \{ \infty \} \to [0, +\infty] \) be a Borel measurable function such that \( v(x) \leq e^{\alpha x_N}V(r')(x) \) on \( \mathbb{R}^N \). There exist positive constants \( c_4 \) and \( c_5 \), depending on \( N, R, r \) and \( r' \), such that, if \( \Lambda \leq c_4 \), then \( H^V_v \) exists and 
\[ H^V_v(x) \leq H^{A(r', 1)}_v(x) + c_5\Lambda e^{\alpha x_N} \quad (|x'| = r). \]

**Proof.** Let \( h_n = H^{\min\{v, n\}}_V \) on \( V \) and \( h_n = \min\{v, n\} \) on \( \partial^\infty V \), and let 
\[ m_n = \sup\{e^{-\alpha x_N}h_n(x', x_N): |x'| = r, x_N > -n\}. \]

Then 
\[ (2.21) \quad h_n = H^{A(r', 1)}_{h_n} = H^{A(r', 1)}_{h_n \chi_{\Omega U}} + H^{A(r', 1)}_{h_n \chi_{V(r')}} \quad \text{in} \quad A(r', 1). \]

Let \( \alpha > 0 \) denote the square root of the first eigenvalue of \( -\Delta \) in \( B' \setminus \overline{B'_{r'}} \). Then \( \alpha < \alpha_{r'} \) because the complement of \( B' \setminus \overline{B'_{r'}} \) in \( B' \) is non-polar (see [19, Section 1.3.2]). Since \( d\mu^A_{r'} = P_{A(r', 1)}(x, \cdot) \, d\sigma \) on \( \partial U \), the Poisson kernel estimates yield, for \( |x'| = r \), that 
\[ e^{-\alpha x_N}H^{A(r', 1)}_{h_n \chi_{\Omega U}}(x) \leq C(N, r, r')e^{-\alpha x_N} \int_{\partial U} h_n(y)e^{-\alpha_{r'}|x_N-y_N|} \, d\sigma(y) \]
\[ \leq C(N, r, r') \int_{\partial U} h_n(y)e^{-\alpha y_N} \, d\sigma(y). \]

Noting that \( h_n \) satisfies the hypotheses of Lemma 2.3, we see from (2.15) that, when \( |x'| = r \) we have 
\[ (2.22) \quad e^{-\alpha x_N}H^{A(r', 1)}_{h_n \chi_{\Omega U}}(x) \leq C(N, R, r, r') \int_{\partial U} e^{-\alpha y_N}h_n(r y', y_N)\beta_{E'}(y) \, d\sigma(y) \]
\[ \leq C_1 m_n \Lambda, \]
where \( C_1 \) is a constant depending on \( N, R, r, r' \) and \( \nu \).
Moreover, for $|x'| = r$ we have
\[
    e^{-\alpha x} H_{A(r',1)}^{(r',1)}(x) \leq e^{-\alpha x} \int_{V(r')} e^{\alpha y} dH_{A(r',1)}^{(r',1)}(y)
\]
(2.23)
\[
    \leq C(N, r, r') \int_{V(r')} e^{\alpha |y_N-x_N|} e^{-\alpha |y_N-x_N|} d\sigma(y)
\]
\[
    \leq C(N, r, r') \int_{-\infty}^{+\infty} e^{(\alpha - \alpha r)|y_N-x_N|} dy_N \leq C_2(N, r, r').
\]
By (2.21)–(2.23) we obtain
\[
    e^{-\alpha x} h_n(x) = e^{-\alpha x} H_{A(r')}^{(r',1)}(x) + e^{-\alpha x} H_{A(r',1)}^{(r',1)}(x) \leq C_1 m_n \Lambda + C_2 \quad (|x'| = r).
\]
Taking $c = \max\{C_1, C_2\}$ we arrive at
\[
    m_n \leq c(1 + m_n \Lambda).
\]
We choose $c_4 = (2c)^{-1}$ and suppose that $\Lambda \leq c_4$. Then
\[
    m_n \leq c + m_n c c_4 = c + m_n / 2,
\]
which implies that $m_n \leq 2c$.

It follows from (2.21) and (2.22) that for $|x'| = r$ we have
(2.24)
\[
    e^{-\alpha x} h_n(x) \leq 2c^2 \Lambda + e^{-\alpha x} H_{A(r',1)}^{(r',1)}(x).
\]
We choose $c_5 = 2c^2$ and let $n \to \infty$. By (2.23) the limit of the latter term on the right hand side of (2.24) is finite and so $H_v^{(r',1)}$ exists and satisfies
\[
    H_v^{(r',1)}(x) \leq c_3 \Lambda e^{\alpha x} + H_{A(r',1)}^{(r',1)}(x) \quad (|x'| = r).
\]

**Lemma 2.5.** Let $w: \partial^\infty U \to [0, +\infty)$ be a Borel measurable function such that
(2.25)
\[
    w(y) \leq \beta e^{|y|} (y \in \partial U) \quad \text{and} \quad w(\infty) = 0.
\]
Then, there exists a positive constant $c_6$, depending on $N, R, r$ and $\nu$, such that
\[
    H_{v}^{(r',1)}(x', x_N) \leq c_6 e^{\alpha x N} \Lambda \quad (|x'| = r).
\]

**Proof.** Using (2.2), in view of (2.25) and (2.15), for $|x'| = r$ we have
\[
    H_{v}^{(r',1)}(x', x_N) \leq C(N, r) \int_{\partial U} w(y) e^{-\alpha |y_N-x_N|} d\sigma(y)
\]
\[
    \leq C(N, r) e^{\alpha x N} \int_{\partial U} \beta e^{|y|} d\sigma(y) \leq C(N, R, r, \nu) e^{\alpha x N} \Lambda.
\]

We extend $h_+$ to be 0 outside $U$ and recall that $V$ stands for $A(r', \infty) \setminus E'$. We define inductively a sequence $(s_k)$ as follows
\[
    s_{-2} = s_{-1} = 0, \quad s_0 = h_+,
\]
\[
    s_{2k+1} = \begin{cases} H_{s_{2k}}^{U} & \text{on } V, \\ s_{2k} & \text{on } \mathbb{R}^N \setminus V, \end{cases}, \quad s_{2k+2} = \begin{cases} H_{s_{2k+1}}^{U} + h_+ & \text{on } U, \\ s_{2k+1} & \text{on } \mathbb{R}^N \setminus U. \end{cases}
\]
\[
    \text{We put } s_k(\infty) = 0 \text{ for all } k.
\]

**Lemma 2.6.** There is a positive constant $c_7$, depending on $N, R, r, r'$ and $\nu$, such that, if $\Lambda \leq c_7 \lambda$ for some $\lambda \in (0, 1)$, then:

(a) $(s_k)$ is an increasing sequence of continuous functions on $\mathbb{R}^N$;
(b) each $s_k$ is bounded on $\mathbb{R}^{N-1} \times (-\infty, b_n)$ for each $n \in \mathbb{N}$;
(c) for all $k = 0, 1, \ldots$ we have

$$(s_{2k} - s_{2k-2})(x) \leq \lambda^k e^{\alpha x_N}, \quad |x'| = r.$$  

Proof. We will use ideas from [18, Lemma 3.1]. Suppose that $\Lambda \leq c_7 \lambda$, where $c_7$ is to be determined later. Assume that $s_0 \leq s_1 \leq \ldots \leq s_{2k}$ on $\mathbb{R}^N$ for some $k \geq 0$, that all the functions $s_k'$ are continuous on $\mathbb{R}^N$ for $0 \leq k' \leq 2k$, and that for $0 \leq k' \leq k$

$$(2.26) \quad (s_{2k'} - s_{2k'-2})(x', x_N) \leq \lambda^{k'} e^{\alpha x_N} \quad (|x'| = r).$$

We also fix $n \in \mathbb{N}$ and assume that $s_{2k}$ is bounded on $\mathbb{R}^{N-1} \times (-\infty, b_n)$. Once the terms of $(s_k)$ are seen to be finite, it is clear that the upper PWB solutions appearing in their definitions are actually well defined PWB solutions. The induction hypotheses clearly hold for $k = 0$. We split the proof of Lemma 2.6 into three steps.

Step 1. We show that $s_{2k+1}$ is a finite-valued continuous function on $\mathbb{R}^N$ which is bounded on $\mathbb{R}^{N-1} \times (-\infty, b_n)$. Harnack’s inequalities and (2.26) yield the existence of a constant $c_8 = c_8(N, r, r') > 0$ such that

$$(2.27) \quad (s_{2k} - s_{2k-2})(y) \leq c_8 \lambda^k e^{\alpha y_N} \quad (|y'| = r').$$

Now, for $|x'| = r$, by (2.27) and Lemma 2.4 we have

$$(s_{2k+1} - s_{2k-1})(x) \leq \overline{H}_{s_{2k} - s_{2k-2}}(x) = \overline{H}_{(s_{2k} - s_{2k-2})x_V|_{\partial V_N}}(x)$$

$$\leq c_5 c_8 \lambda^k \Lambda e^{\alpha x_N} + H^A_{(r',1)}(s_{2k} - s_{2k-2})x_V|_{\partial V_N}(x).$$

Since $s_{2k} - s_{2k-1} = 0$ on $\partial U$ and $s_{2k} - s_{2k-1} = s_{2k} - s_{2k-2}$ on $V(r')$, it follows that $s_{2k} - s_{2k-1}$ belongs to the upper class for $H^A_{(r',1)}$. Hence

$$(s_{2k+1} - s_{2k-1})(x) \leq c_5 c_8 \lambda^k \Lambda e^{\alpha x_N} + (s_{2k} - s_{2k-1})(x),$$

and so

$$(2.28) \quad (s_{2k+1} - s_{2k})(x) \leq c_5 c_8 \lambda^k \Lambda e^{\alpha x_N} \quad (|x'| = r).$$

This proves finiteness of $s_{2k+1}$.

A result of Armitage concerning a strong type of regularity for the PWB solution of the Dirichlet problem (see [3, Theorem 2]) implies that $s_{2k+1}$ is continuous at points $\partial V \backslash \bigcup_{n=1}^{\infty} (\partial B' \times \{b_n\})$. Applying Lemma 2.3 to $v_j = H^V_{\min(s_{2k,j})}$ and $x \in \bigcup_{n=1}^{\infty} (\partial B' \times \{b_n\})$ we obtain

$$v_j(y) \leq c_3 v_j(rx', x_N) H^V_{x_0, T_x \backslash E'}(y) \quad (y \in T_x \backslash E')$$

Letting $j \to \infty$ we notice that the same inequality holds for $s_{2k+1}$, and hence the regularity of $x$ for $T_x \backslash E'$ implies that $s_{2k+1}$ vanishes at $x$. We conclude that $s_{2k+1}$ is continuous on $\mathbb{R}^N$.

We also have $s_{2k+1} = H^V_{s_{2k+1}}(\mathbb{R}^{N-1} \times (-\infty, b_n])$ on $V \cap [\mathbb{R}^{N-1} \times (-\infty, b_n)]$. Further, since $s_{2k+1}$ is continuous on $\overline{B} \backslash \{b_n\}$, vanishes on $E$ and is bounded on $\overline{(\mathbb{R}^N \backslash V)} \cap [\mathbb{R}^{N-1} \times (-\infty, b_n)]$ in view of the induction hypothesis, we deduce that $s_{2k+1}$ is bounded above on $\mathbb{R}^{N-1} \times (-\infty, b_n)$.

Step 2. We now prove that $s_{2k} \leq s_{2k+1} \leq s_{2k+2}$ on $\mathbb{R}^N$. We note that $s_{2k} = H^A_{s_{2k}}(r', 1)$ on $A(r', 1)$ (for a simple proof see Step 2 in the proof of [18, Lemma 3.1]).
It follows immediately from the induction hypothesis, that
\[ s_{2k+1} = H^V_{s_{2k}} \geq H^V_{s_{2k-2}} = s_{2k-1} \quad \text{on } V. \]
In particular, this gives \( s_{2k+1} \geq s_{2k} \) on \( \mathbb{R}^N \setminus U \). Hence, \( s_{2k+1} \geq s_{2k} \) on \( \partial U \cup \partial V \).

Using [5, Theorem 6.3.6], we obtain
\[ s_{2k+1} = H^V_{s_{2k}} = H^A_{s_{2k+1}} \geq H^A_{s_{2k}} = s_{2k} \quad \text{on } A(r', 1). \]
Therefore, \( s_{2k+1} \geq s_{2k} \) on \( \mathbb{R}^N \). We now deduce that
\[ s_{2k+2} = H^U_{s_{2k+1}} + h_+ \geq H^U_{s_{2k-1}} + h_+ = s_{2k} = s_{2k+1} \quad \text{on } \mathbb{R}^N \setminus V. \]

We finally note that, if \( s_{2k+2} \) belongs to the upper class for \( \overline{A}_{s_{2k+2}}^{(r', 1)} \), we obtain
\[ s_{2k+2} \geq \overline{A}_{s_{2k+2}}^{(r', 1)} \geq H^A_{s_{2k+1}} = s_{2k+1} \quad \text{on } A(r', 1), \]
and so \( s_{2k+2} \geq s_{2k+1} \) on \( \mathbb{R}^N \). To verify that \( s_{2k+2} \) belongs to the upper class for \( \overline{A}_{s_{2k+2}}^{(r', 1)} \), it is enough to check that \( \liminf_{x \to y} s_{2k+2}(x) \geq s_{2k+2}(y) \) for \( y \in \partial U \). This is clear from regularity and the continuity of \( s_{2k+1} \), as if \( s_{2k+2} \neq +\infty \), then for \( y \in \partial U \) we have
\[ \liminf_{x \to y} s_{2k+2}(x) = \liminf_{x \to y} H^U_{s_{2k+1}}(x) \geq \liminf_{x \to y, x \in \partial U} s_{2k+1}(x) = \limsup_{x \to y} s_{2k+1}(y) = s_{2k+2}(y). \]

Step 3. In the final step we will prove that
\[ (s_{2k+2} - s_{2k})(x) \leq \lambda^{k+1} e^{\alpha x N} \quad (|x'| = r). \]

Then, using [3, Theorem 2], we can conclude that \( s_{2k+2} \) is continuous on \( \mathbb{R}^N \). Further, \( s_{2k+2} - h_+ = H^U_{s_{2k+1}} = H^U_{s_{2k+2} - h_+} \) on \( U \cap [\mathbb{R}^N \setminus (-\infty, b_n)] \). By continuity, \( s_{2k+2} \) is bounded on \( \overline{U} \times \{b_n\} \). On \( \mathbb{R}^N \setminus U \) we have \( s_{2k+2} = s_{2k+1} \), which is bounded on \( (\mathbb{R}^N \setminus U) \cap [\mathbb{R}^N \setminus (-\infty, b_n)] \) by Step 1. Hence \( s_{2k+2} \) is bounded on the whole of \( \mathbb{R}^N \setminus (-\infty, b_n) \).

To prove the desired inequality (2.29), we first recall that
\[ U_m = (\mathbb{R}^N \setminus \overline{B'}) \times (b_m, b_{m+1}) \quad (m \in \mathbb{N}). \]

Noting that
\[ s_{2k+1} = H^V_{s_{2k}} = H^U_{s_{2k+1}} = H^U_{s_{2k+1} \setminus \partial B' \times (b_m, b_{m+1})} \quad \text{on } U_m, \]
and that, by continuity, \( s_{2k+1} \) is bounded on \( \partial B' \times (b_m, b_{m+1}) \), we see that \( s_{2k+1} - s_{2k-1} \) satisfies the hypotheses of Lemma 2.3. Hence, for \( x \in \partial U \), we have
\[ (s_{2k+1} - s_{2k-1})(x) \leq c_3 \beta E(x)(s_{2k+1} - s_{2k-1})(rx', x_N) \]
\[ = c_3 \beta E(x)[(s_{2k+1} - s_{2k})(rx', x_N) + (s_{2k} - s_{2k-1})(rx', x_N)] \]
\[ \leq c_3 \beta E(x)[(s_{2k+1} - s_{2k})(rx', x_N) + (s_{2k} - s_{2k-2})(rx', x_N)]. \]

It follows from (2.28) and our induction hypothesis that
\[ (s_{2k+1} - s_{2k-1})(x) \leq c_3(c_5 c_8 + 1) \lambda^k e^{\alpha x N} \beta E(x) \quad (x \in \partial U). \]

Assuming that \( c_7 \leq 1 \) and letting \( c_0 = c_3(c_5 c_8 + 1) \) we obtain
\[ (s_{2k+1} - s_{2k-1})(x) \leq c_0 \lambda^k e^{\alpha x N} \beta E(x) \quad (x \in \partial U). \]
By Lemma 2.5, for \(|x'| = r\), we have
\[
(s_{2k+2} - s_{2k})(x) \leq \frac{H_U}{s_{2k+1}-s_{2k-1}}(x) \leq c_9 \lambda^k c_6 \Lambda e^{axN} = c_6 c_7 \lambda^{k+1} e^{axN}.
\]
Taking \(c_7 = \min\{1, (c_6 c_9)^{-1}\}\) we find that (2.29) holds, and the proof is complete. \(\square\)

3. Proof of Theorem 1.1

Proposition 2.1 gives the implication \(a \Rightarrow b\). To prove that \((b) \Rightarrow (a)\) we first observe that taking \(J\) large enough when setting \(b_1 = a_J\), we can ensure that \(\Lambda \leq c_7 \lambda\) for some \(\lambda \in (0, 1)\). Let \(\Omega' = \mathbb{R}^N \setminus E'\) and \(u' = \lim_{k \to \infty} s_k\). By Lemma 2.6, for \(|x'| = r\) we obtain
\[
s_{2k}(x) = \sum_{j=0}^{k} (s_{2j} - s_{2j-2})(x) \leq \sum_{j=0}^{k} \lambda^j e^{axN} \leq \frac{1}{1 - \lambda} e^{axN}.
\]
Hence \(u' \neq +\infty\). As a limit of an increasing sequence \((s_{2k})\) of harmonic functions on \(U\), the function \(u'\) is harmonic on \(U\). Since \(u'\) is the limit of an increasing sequence \((s_{2k+1})\) of harmonic functions on \(V\), it is also harmonic on \(V\). Hence \(u'\) is harmonic in \(\Omega'\). It follows from the monotonicity of \((s_k)\) that \(u' \geq h_+\) on \(U\).

For \(x \in E'\) we have \(u'(x) = 0\). By the monotone convergence theorem applied to the equation \(s_{2k+1} = H^V_{s_{2k}}\) we obtain \(u' = H^V_{u'}\) on \(V\). We can follow the reasoning from the second last paragraph of Step 1 in the proof of Lemma 2.6 to see that \(u'\) vanishes continuously on \(E'\).

We next prove that \(u'\) is minimal on \(\Omega'\) using an argument from [18, Theorem 1.1]. As a consequence of the monotone convergence theorem we find that
\[
(3.1) \quad u'(x) = H^U_{u'}(x) + h_+(x) \quad (x \in U).
\]
Let \(\Delta_1\) denote the minimal Martin boundary of \(\Omega'\) and let \(M\) be the Martin kernel of \(\Omega'\) relative to the origin. By the Martin representation theorem (see [5, Theorem 8.4.1]) we have
\[
(3.2) \quad u'(x) = \int_{\Delta_1} M(x, z) \, d\nu_{u'}(z) \quad (x \in \Omega'),
\]
where \(\nu_{u'}\) is uniquely determined by \(u'\).

We define \(T = \{z \in \Delta_1: \Omega' \setminus U\ \text{is minimally thin at}\ z\}\) so that
\[
(3.3) \quad R^{\Omega'\setminus U}_{M(\cdot, z)} = M(\cdot, z) \quad (z \in \Delta_1 \setminus T).
\]
Changing the order of integration, and using (3.1)–(3.3) and [5, Theorem 6.9.1], we obtain
\[
h_+(x) = \int_{\Delta_1} \left( M(x, z) - \int_{\partial U} M(y, z) \, d\mu^U_x(y) \right) \, d\nu_{u'}(z)
= \int_{\Delta_1} \left( M(x, z) - R^{\Omega'\setminus U}_{M(\cdot, z)}(x) \right) \, d\nu_{u'}(z)
= \int_T \left( M(x, z) - R^{\Omega'\setminus U}_{M(\cdot, z)}(x) \right) \, d\nu_{u'}(z) \quad (x \in U).
\]
We now claim that $\nu_u|_T$ is concentrated at a single point. For the sake of contradiction suppose that there are two distinct points $y_1, y_2 \in \Delta_1 \cap \text{supp}(\nu_u|_T)$ and let $N_1, N_2$ be disjoint neigbourhoods of $y_1$ and $y_2$ respectively. We define
\[
h_j(x) = \int_{N_j \cap T} \left( M(x, y) - R^\Omega_{M(x, y)}(x) \right) d\nu_u(y) \quad (x \in \Omega', j = 1, 2),
\]
and note that $h_j \leq h_+ \text{ on } U$. Minimality of $h_+$ on $U$ implies that
(3.4) \[h_j/h_j(0) = h_+ \text{ on } U \quad (j = 1, 2).\]
We now define
\[
v_j(x) = \int_{N_j \cap T} M(x, y) d\nu_u(y) \quad (x \in \Omega', j = 1, 2).
\]
Then $h_j \leq v_j \leq u'$ on $\Omega'$, and by (3.4), $v_j/h_j(0) \geq h_+ \text{ on } \Omega'$ $(j = 1, 2)$. In view of the definition of $s_k$ we have $v_j/h_j(0) \geq s_k$ on $\Omega'$ for all $k \in \mathbb{N}$ and so $v_j/h_j(0) \geq u'$ on $\Omega'$ $(j = 1, 2)$. It follows that $h_1(0)/v_1 \leq v_1$ on $\Omega'$.

This implies that $\nu_u|_{T \cap N_2}$ is minorized by a multiple of $\nu_u|_{T \cap N_2}$, which contradicts the fact that $N_1 \cap N_2 = \emptyset$. Hence $\nu_u|_T = c_0\nu$ for some $c_0 \in T$ and $c > 0$. Furthermore, the minimal harmonic function $\nu = c M(\cdot, t')$ on $\Omega'$ satisfies $u' \geq \nu$ on $\Omega'$ and $\nu \geq h_+$ on $U$. We observe that $\nu \geq s_k$ on $\Omega'$ for all $k \in \mathbb{N}$, and so $\nu \geq u'$. Hence $\nu \equiv u'$ and we conclude that $u'$ is minimal on $\Omega'$.

Let $\Omega'' = \mathbb{R}^N \setminus \Omega''$. We define $g = H^\Omega_{\lambda \Omega''} \text{ and } g = \chi_{\Omega''} \text{ on } \partial^\infty \Omega'$. By [5, Theorem 6.9.1] we have $g = R^{\Omega'' \setminus \Omega'}_1 \Omega''$ (reductions with respect to non-negative superharmonic functions on $\Omega''$). Since $\Omega'' \setminus \Omega'$ is non-thin at each constituent point, it follows from [5, Theorem 7.3.1(i)] that $R^{\Omega'' \setminus \Omega'}_1 = R^{\Omega'' \setminus \Omega'}_1$ on $\Omega''$ and so $g$ is superharmonic there. Let $h$ be a non-negative harmonic minorant of $g$ on $\Omega''$. Then $h$ is bounded on $\Omega''$ and vanishes quasi-everywhere on $\partial \Omega''$. Since a polar subset of $\partial \Omega''$ and $\{\infty\}$ are both negligible for $\Omega''$ (see [5, Theorems 6.5.5 and 7.6.5]), we deduce that $h \equiv 0$. Hence $g$ is a potential on $\Omega''$.

Let $W = [\mathbb{R}^{N-1} \times (-\infty, b_n)] \cap \Omega'$ for some $n > 1$. Since $1 - g$ is positive and continuous on $\overline{B} \times \{b_n\}$, it follows that $1 - g$ is bounded below by a positive constant on this set while $u'$ is bounded from above there. Hence there exists a positive constant $c$ such that $c(1 - g) \geq u'$ on $\overline{B} \times \{b_n\}$, and thus on $\partial W$. By Lemma 2.6(b) each $s_k$ is bounded on $W$ and so it belongs to the lower class for $H^W_{s_k}$. These facts combined with monotonicity of $(s_k)$ lead to the observation that
\[
s_k \leq H^W_{s_k} \leq H^W_{u'} \leq cH^W_{1-g} = c(1 - g) \text{ on } W.
\]
Therefore, $u' \leq c(1 - g)$ on $W$. Since $c(1 - g) - u'$ is a non-negative harmonic function on $W$ which vanishes on $\Omega'' \setminus \Omega'$, we conclude that $c(1 - g) - u'$ is subharmonic on $\Omega''$, so that $u' + cg$ is superharmonic on $\Omega''$.

By the Riesz decomposition,
(3.5) \[u' + cg = u'' + G_{\Omega''} \mu \text{ on } \Omega'',\]
where $u''$ is the greatest harmonic minorant of $u' + cg$ on $\Omega''$ and $G_{\Omega''} \mu$ is the Green potential of the Riesz measure $\mu$ associated with $u' + cg$. Hence $u''$ vanishes on $\Omega'' \setminus (\partial B' \times \{b_1\})$ and for each $n \in \mathbb{N}$ it is bounded on $\mathbb{R}^{N-1} \times (-\infty, b_n)$. It follows from a removable singularity result (see [5, Theorem 5.2.1]) that $u''$ extends to a subharmonic function on $\mathbb{R}^N$. This together with the non-thinness of $E''$ at points
of $\partial B' \times \{b_1\}$ implies that $u''$ vanishes also on $\partial B' \times \{b_1\}$. Since $h_+$ is a subharmonic minorant of $u'+cg$ on $\Omega''$, we deduce that $h_+ \leq u''$ on $\Omega''$.

It remains to show that $u''$ is minimal. Let $h$ be a positive harmonic minorant of $u''$ on $\Omega''$. We notice that $h$ is bounded on $\Omega'' \setminus \Omega'$ and vanishes on $\partial \Omega''$. Hence the greatest harmonic minorant of $R_{h}^{\Omega'' \setminus \Omega'}$ on $\Omega''$ is bounded and vanishes on $\partial \Omega''$, and we see that $R_{h}^{\Omega'' \setminus \Omega'}$ is a potential on $\Omega''$. Since the upper-bounded harmonic function $h - R_{h}^{\Omega'' \setminus \Omega'} - u'$ on $\Omega'$ satisfies

$$\limsup_{x \to y}(h - R_{h}^{\Omega'' \setminus \Omega'} - u')(x) \leq 0 \quad \text{for } y \in \partial \Omega',$$

and $\{\infty\}$ has zero harmonic measure for $\Omega'$, it follows that

$$h - R_{h}^{\Omega'' \setminus \Omega'} - u' \leq 0 \quad \text{on } \Omega'.$$

Now, since $h - R_{h}^{\Omega'' \setminus \Omega'}$ is a positive harmonic minorant of the minimal function $u'$ on $\Omega'$, we conclude that $h - R_{h}^{\Omega'' \setminus \Omega'} = au'$ for some $a \in (0,1]$. Substituting this into (3.5) we obtain

$$h + acg = au'' + aG_{\Omega''} + R_{h}^{\Omega'' \setminus \Omega'} \quad \text{on } \Omega''.$$

Taking the greatest harmonic minorant in $\Omega''$ of both sides we get $h = au''$, which means that $u''$ is minimal.

Let $u = u'' - H_{\Omega''}^{\Omega}$. Since $u'' - h_+ \geq 0$ is superharmonic on $\Omega''$ and equals $u''$ on $\Omega'' \setminus \Omega$, we have

$$u = u'' - R_{u''}^{\Omega'' \setminus \Omega'} = u'' - R_{u'' - h_+}^{\Omega'' \setminus \Omega'} \geq h_+.$$

Since the points of $\partial \Omega$ are regular for $\Omega$ and $u''$ is continuous, it follows that $u$ vanishes on $\partial \Omega$. Further, [5, Theorem 9.5.5] shows that $u$ is minimal.

**Remark.** The proof of the implication $(a) \Rightarrow (b)$ in Theorem 1.1 does not rely on condition (1.1). It is in the proof of the converse that our methods rely on such a condition. However, it is enough to assume merely that $\Omega$ is contained in a comb-like domain $\Omega_0$ for which (1.1) holds. To see this, suppose that $(b)$ holds. Theorem 1.1 applied to $\Omega_0$ yields the existence of a minimal harmonic function $u_0$ on $\Omega_0$ which vanishes on $\partial \Omega_0$ and satisfies $u_0 \geq h_+$. Let $u = u_0 - H_{\Omega_0}^{\Omega}$ on $\Omega$. The argument from the previous paragraph shows that $u$ is as stated in $(a)$.

**References**


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